Chapter 2

Boson Stars

The concept of particles forming stable macroscopic objects has a historical foundation, with John Wheeler proposing "geons" as a solution within General Relativity in 1955 [68]. While geons proved unstable, the consideration of a complex scalar field, governed by the Klein-Gordon equation, led to the discovery of localized and stable solutions known as boson stars (BSs) [69, 70, 71]. The renewed interest in BSs, especially after the confirmation of the Higgs boson's existence, aligns with the acknowledgment of fundamental bosons in nature. Various theoretical extensions of the Standard Model, such as the Peccei-Quinn mechanism [59], inflation models [72, 73], supersymmetry [74], and string theory [75], predict the existence of fundamental complex scalar fields. Studying BSs provides insights into the properties of compact self-gravitating objects and explores the intriguing possibility of BSs serving as dark matter in the universe [76, 77].

Boson stars (BS) represent soliton solutions to the Klein-Gordon equation, describing a classical complex scalar field Φ interacting with gravity. The self-gravitational energy of the boson field arises from spatial gradients and the time derivative of the field, creating the necessary pressure to balance the gravitational field from the self-gravity of the bosons. This wavelike behavior is connected to the Heisenberg uncertainty principle, linking the confined mass of particles within the star to their velocity. In addition to boson stars, solitonic solutions exist for a real scalar field, leading to "oscillitons" [78, 79, 80, 81]. Unlike boson stars, oscillitons have a time-dependent metric and emerge when there is no conserved charge. The rich variety of solitonic solutions includes several other classes, but one particularly well-motivated category is that of "axion stars"[82, 83, 84, 85, 86]. These solutions are specifically associated with axion-like particles and are a subject of focused investigation.

Understanding the formation of axion stars in astrophysical settings involves two observed mechanisms in simulations of the Schrödinger-Poisson equations (SPEs). The dominant mechanism depends on the scale R of gravitational fluctuations compared to the de Broglie wavelength (λ_{dB}):

Direct Collapse: This mechanism occurs when the scale of gravitational fluctuations is comparable to the de Broglie wavelength ($R \approx \lambda_{\mathrm{dB}}$). In this scenario, axion stars form rapidly due to gravitational collapse on the freefall timescale. This process is most relevant in the smallest objects near the cutoff scale of gravitational fluctuations. The resulting axion stars are found in the centers of dark matter halos close to the cutoff scale. The mass of these stars (M_*) is related to the halo mass (M) by a numerical relationship, as described by equation $M_* \propto \left(\frac{M}{M_0}\right)^{\frac{1}{3}} M_0$ in the text. This mechanism is particularly significant for the formation of solitonic cores in dwarf galaxies in the SFDM regime and the formation of axion stars in miniclusters [87].

Kinetic Condensation: This mechanism occurs when the scale of gravitational fluctuations is much larger than the de Broglie wavelength ($R \gg \lambda_{\mathrm{dB}}$). In this scenario, axion stars form through a process called kinetic condensation. This mechanism may lead to the formation of axion stars in larger structures or over larger scales compared to direct collapse [88, 89].

Despite progress in our understanding of the formation and growth of axion stars, at the time of writing their abundance and galactic distribution is not fully understood even in benchmark models. The problem is partly one of scale: we do not know the mass above which the relation $M_* \propto \left(\frac{M}{M_0}\right)^{\frac{1}{3}} M_0$ breaks down and halos have no central soliton, but instead grow many small solitons in the kinetic regime.

Axion stars, with their unique properties, have the potential to manifest several notable phenomenological consequences:

Galactic Cores: Solitons formed by Fuzzy Dark Matter (FDM) with a mass on the order of 10^{-22} eV could provide an explanation for the observed flat central densities in Milky Way dwarf satellite galaxies. Tracer stars residing within the soliton or outside it may contribute to flat central densities or central mass excesses, respectively [87, 90].

Direct Detection: Although the passage of axion stars through Earth is a rare event, it can significantly enhance the signal in direct detection experiments. Coordinated networks of detectors, such as the Global Network of Optical Magnetometers for Exotic physics searches (GNOME) and GPS.DM, could potentially identify these events [91, 57].

Indirect Detection: The high density of axions in axion stars can lead to a more substantial radio signal resulting from the decay and conversion of axions into photons. Cataclysmic signals may also arise if axion stars reach the critical mass for an axion nova or undergo stimulated decay due to interactions.

Relativistic Axion Stars: In cases where axion stars are dense enough, they may be detected as "Exotic Compact Objects" in gravitational wave detectors [92] and through multi-messenger astronomy [93]. The relativistic nature of these axion stars could provide additional insights into their properties and interactions.

2.1 Schrödinger–Poisson equations

The UBDM (Ultra-light Bosonic Dark Matter) condensate coupled to general relativity follows the Einstein–Klein–Gordon equations, derived from the fundamental action. In the non-relativistic limit, applicable to all forms of UBDM, such as Axion-Like Particles (ALPs), real, or complex scalars, these equations simplify to the Schrödinger–Poisson equations (SPEs).

Let's begin with the Klein–Gordon equation:

$$\Box\phi - \partial_t^2\phi = 0, \tag{2.1.1}$$

where the D'Alembertian is defined as:

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right), \tag{2.1.2}$$

and the potential is given by:

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\lambda\phi^4. \tag{2.1.3}$$

With $\phi = \theta$ and $a = 1$, the metric is given by:

$$g_{\mu\nu} = \mathrm{diag}\left[-(1+2\theta), 1-2\theta, 1-2\theta, 1-2\theta\right]. \tag{2.1.4}$$

To first order, this gives:

$$g_{\mu\nu}^{-1} = \mathrm{diag}\left[-(1-2\theta), 1+2\theta, 1+2\theta, 1+2\theta\right], \tag{2.1.5}$$

$$\Box = 4\theta\partial_t - (1-2\theta)\partial_t^2 + (1+2\theta)\nabla^2. \tag{2.1.6}$$

The Klein–Gordon equation (Eq. (A.79)) reads:

$$-(1-2\theta)\ddot{\phi} + 4\theta\dot{\phi} + (1+2\theta)\nabla^2\phi - m^2\phi - 2\lambda\phi^3 = 0. \tag{2.1.7}$$

Multiplying this equation by $-(1+2\theta)$ (since this term goes to -1 in the non-relativistic limit), the Klein–Gordon equation becomes:

$$\ddot{\phi} - 4\theta\dot{\phi} - (1+4\theta)\nabla^2\phi + (1+2\theta)m^2\phi + (1+2\theta)^2\lambda\phi^3 = 0. \tag{2.1.8}$$

Next, let's take the ansatz for $\phi = \theta$ and write:

$$\phi = \frac{1}{\sqrt{2m}}\left[\psi e^{imt} + \psi^* e^{imt}\right], \tag{2.1.9}$$

$$\dot{\phi} = \frac{1}{\sqrt{2m}}\left[e^{imt}\dot{\psi} + ime^{imt}\psi^* - e^{-imt}\dot{\psi}^* - ime^{-imt}\psi\right], \tag{2.1.10}$$

$$\ddot{\phi} = \frac{1}{\sqrt{2m}}\left[e^{imt}\ddot{\psi} + 2ime^{imt}\dot{\psi} - m^2 e^{imt}\psi + e^{-imt}(\ldots)\right], \tag{2.1.11}$$

$$\nabla^2\phi = \frac{1}{\sqrt{2m}}\left[\nabla^2\psi e^{imt} + \nabla^2\psi^* e^{-imt}\right], \tag{2.1.12}$$

$$\phi^3 = \frac{1}{(\sqrt{2m})^3}\left[e^{imt}2|\psi|^2\psi + e^{-imt}2|\psi|^2\psi^* + \psi^3 e^{3imt} + \psi^{*3}e^{-3imt}\right]. \tag{2.1.13}$$

Since terms for e^{-imt} are the complex conjugate, we only need to consider the terms with e^{imt}. Substituting these into the Klein–Gordon equation, we obtain:

$$i\dot{\psi} - \frac{1}{2m}\nabla^2\psi + m\theta\psi + \lambda\frac{|\psi|^2}{m}\psi = 0 \quad \text{(A.86)} \tag{2.1.14}$$

This equation represents the non-relativistic limit of the Klein–Gordon equation in the Newtonian approximation.

The UBDM (Ultra-light Bosonic Dark Matter) condensate coupled to general relativity is governed by the Einstein–Klein–Gordon equations, which are derived from the variation of the fundamental action. In the non-relativistic limit, often referred to as the Newtonian approximation, these equations for all forms of UBDM (including Axion-Like Particles, real, or complex scalars) simplify to the Schrödinger–Poisson equations (SPEs):

$$i\psi_t + \frac{1}{2m}\nabla^2\psi - m\theta\psi + \lambda\frac{|\psi|^2}{GP}\psi = 0, \tag{2.1.15}$$

$$\nabla^2\theta = 4\pi G_N m^2(|\psi|^2 - \langle\rho_m\rangle\langle|\psi|^2\rangle), \tag{2.1.16}$$

Here, the Newtonian potential is dimensionless, and the field ψ has a canonical mass dimension one, ensuring that the average number density is given by

$$\langle n_{\text{avg}} \rangle = m \int |\psi|^2 \, d^3x. \tag{2.1.17}$$

The Poisson equation includes a subtraction of the background density, following the background-perturbation split of the Einstein equations on the Friedmann background.

Equations (2.1.15) and (2.1.16) collectively form a nonlinear Schrödinger equation for the UBDM condensate, incorporating the Gross–Pitaevski self-coupling term λ_{GP}. This coupling parameter can be computed from the relativistic self-interaction potential V. These Schrödinger–Poisson equations provide a comprehensive description of the nonlinear, non-relativistic structure formation in various astrophysical environments at low redshifts, where the gravitational structure of UBDM is significant at the coherence scale.

Despite the name "Schrödinger," it's crucial to note that these equations lack quantum aspects—ψ does not represent a probability density, and there are no issues related to measurement or wavefunction collapse. Instead, these SPEs are a classical field equation's non-relativistic limit and apply effectively when dealing with a large particle number, analogous to Maxwell's equations for electromagnetic fields.

For a kinetic description of the SPEs, the field ψ can be expressed using the Wigner distribution, describing the occupation probability of modes k. This distribution follows a collisional Boltzmann equation with a scattering timescale τ_{gr} given by [88]:

$$\tau_{gr} \approx \frac{\sqrt{2}}{12\pi^3} \frac{m v^6}{G_N \tilde{n}^2 \log\left(\frac{r_{\max}}{r_{\min}}\right)}, \tag{2.1.18}$$

Here, v represents the typical speed in the system (virial velocity), and $r_{\max}, r_{\min}$ are the maximum and minimum length scales in the problem, respectively. This gravitational scattering timescale τ_{gr} governs the duration over which wave-like effects cause UBDM to dynamically deviate from Cold Dark Matter (CDM).

2.2 Numerical Simuation Technique

The numerical studies are conducted using a 6th-order pseudospectral operator-splitting method [94], known for its unitary, stable, T-symmetric, and symplectic properties, making it well-suited for extended statistical simulations. Here, we outline the method, showcase its characteristics, and estimate the associated numerical errors.

The general solution to the Schrödinger equation (2) takes the form $\psi(t+\Delta t) = \hat{U}\psi(t)$, where $\hat{U} = T\exp\left(-i\int_{t+\Delta t}^{t} dt_0 \left[\frac{\hat{p}^2}{2m} + mU(t_0, x)\right]\right)$ is the quantum propagator, and $\hat{p} \equiv -i\nabla_x$. The discretized form, as proposed in [**?**], replaces this propagator with the following discrete formula:

$$\hat{U} = \prod_{\alpha=1}^{8} e^{-imd_\alpha \Delta t U_\alpha(x) e^{-ic_\alpha \Delta t \hat{p}^2/(2m)}} + O(\Delta t^7) \tag{2.2.19}$$

Here, the product is ordered right-to-left, and the parameters c_α, d_α are given in Table (2.1). The potentials U_β are computed using the "current" field, i.e., $\psi(t)$ multiplied by all operators with $\alpha \leq \beta$. In this method, the time interval Δt is partitioned into two sets of sub-intervals $\{c_\alpha \Delta t, d_\alpha \Delta t\}$, where each set is symmetric with respect to the central point $t + \Delta t/2$. "Kinetic" and "potential" propagators are employed for the c- and d-sub-intervals, respectively.

It's essential to note that Eq. (2.2.19) is valid for the time-dependent potential $U(t, x)$. The numerical application involves introducing a cubic uniform lattice with N_x^3 sites at $x = n_x \Delta x$, where $\Delta x \equiv L/N_x$, and $0 \leq n_i^x < N_x$. Commonly, N_x is set to 128 or 256, and occasionally switched to $N_x = 512$ for resolution tests. The values of the fields $\psi(x)$ and $U(x)$ are stored at the lattice sites.

The time evolution of $\psi(x)$ is calculated by sequentially applying operators in Eq. (2.2.19). The process begins with the Fourier transform of $\psi(x)$, followed by the solution of the Poisson equation (2) in the momentum space. This sequence is repeated until $\psi(t+\Delta t)$ is obtained.

This numerical approach allows for efficient simulations, leveraging the properties of the pseudospectral operator-splitting method for accurate and stable results.

α	1	2	3	4
c_α	$\frac{w_3}{2}$	$\frac{w_2+w_3}{2}$	$\frac{w_1+w_2}{2}$	$\frac{w_0+w_1}{2}$
d_α	w_3	w_2	w_1	w_0

Table 2.1: Values for α, c_α, and d_α. Combined with equations below:
$c_{9-\alpha} = c_\alpha$, $d_{8-\alpha} = d_\alpha$, $d_8 = 0$, $\sum c_\alpha = \sum d_\alpha = 1$
$w_0 = 1 - 2(w_1 + w_2 + w_3)$, $w_1 = -1.17767998417887$, $w_2 = 0.235573213359359$, $w_3 = 0.784513610477560$

2.3 Axion-Photon Interaction

Let us begin by considering one of the most widely studied UBDM interactions, the axion-photon coupling. The axion-photon coupling is used to convert axions or ALPs into photons in the presence of strong magnetic fields. This is the technique at the heart of Chap. 4. The Faraday tensor $F^{\mu\nu}$ is given by [55]

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{2.3.20}$$

where A^μ is the four-potential and E_i and B_i are the electric and magnetic field components in the Cartesian basis. The dual field tensor is given by

$$\tilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}F_{\mu\nu}, \tag{2.3.21}$$

where $\varepsilon^{\alpha\beta\mu\nu}$ is the Levi-Civita totally antisymmetric tensor. The general structure of the operator for the axion-photon interaction involves one factor of the ultra-light bosonic field a and two factors of the photon field. This structure can be seen more clearly by writing the operator in terms of the four-potential A^μ:

$$aF^{\mu\nu}\tilde{F}^{\mu\nu} = a^{\mu\nu\alpha\beta}\partial_\mu A_\nu \partial_\alpha A_\beta, \tag{2.3.22}$$

showing that indeed this term represents an interaction between an axion and two photons.

The term in the Lagrangian describing the axion-photon interaction is

$$\mathcal{L}_{a\gamma\gamma} = \frac{g_\gamma}{4\pi}\frac{\alpha}{\pi}\frac{a}{f_a}F^{\mu\nu}\tilde{F}^{\mu\nu} = g_{a\gamma\gamma}\frac{1}{4}aF^{\mu\nu}\tilde{F}^{\mu\nu}, \tag{2.3.23}$$

where g_γ is a dimensionless model-dependent coupling factor, α is the fine structure constant, f_a is the spontaneous symmetry breaking scale for the axion/ALP field, and $g_{a\gamma\gamma} = g_\gamma\alpha/(\pi f_a)$ is the axion-photon coupling constant. The form of the Lagrangian in terms of the electric field E and magnetic field B is

$$\mathcal{L}_{a\gamma\gamma} = \frac{g_\gamma \alpha}{\pi f_a}aE \cdot B \approx g_{a\gamma\gamma}\frac{1}{4}aE \cdot B. \tag{2.3.24}$$

One method to calculate the observable physical consequences resulting from the axion-photon interaction is to apply the Euler–Lagrange equation to the Lagrangian describing electromagnetism plus the axion-photon Lagrangian of Eq. (2.3.24), namely

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu + g_{a\gamma\gamma}\frac{1}{4}aF^{\mu\nu}\tilde{F}^{\mu\nu}, \tag{2.3.25}$$

where J^μ is the electromagnetic current and A^μ is the gauge potential. The Euler–Lagrange equation in this case produces a version of Maxwell's equations that includes the effects of an axion field, as discussed in Refs. [95, 96, 97, 98]:

$$\nabla \cdot E = \rho + g_{a\gamma\gamma}B \cdot \nabla a, \tag{2.3.26}$$

$$\nabla \cdot B = 0, \tag{2.3.27}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}, \tag{2.3.28}$$

$$\nabla \times B = \frac{\partial E}{\partial t} + J + g_{a\gamma\gamma}E \times \nabla a - \frac{\partial a}{\partial t}B, \tag{2.3.29}$$

where ρ is the charge density and J is the electric current density.

The consequnces of above equations will be discussed in detail in Chapter 4.