Chapter 2

Distributional chaos on uniform spaces

In the last few decades, many researchers have studied the extensions of Li-Yorke chaos based on the properties of scrambled sets, for example, dense chaos, generic chaos, ϵ-Li-Yorke chaos and others. In 1994, Schweizer and Smítal introduced a stronger notion of chaos based on the probabilistic measure of the distance between two trajectories for interval maps, popularly termed as distributional chaos [48]. Since then the notion of distributional chaos has evolved into three nonequivalent variants termed as $DC1$, $DC2$ and $DC3$ (ordered from strongest to weakest) [8].

The relation between distributional chaos and different kinds of specification properties is explored by various authors. Sklar and Smítal proved that for a continuous self-map on a compact metric space, specification property implies distributional chaos of type 3 [58]. Further, Oprocha and Štefánková proved that a continuous self-map on a compact metric space having weak specification property and a distal pair is distributionally chaotic of type 1 and possesses a dense scrambled set [43]. Doleželová studied the invariance of distributionally scrambled set for a continuous self-map on a compact metric space [20].

In this chapter, we consider the topological notions of distributional chaos and specification property defined for uniformly continuous self-maps on uniform spaces

[54, 55]. In Section 2.1, we explore the relation between topological specification property and topological distributional chaos defined for uniformly continuous self-maps on uniform spaces. Essentially, we prove that a uniformly continuous surjective self-map acting on a uniformly locally compact Hausdorff uniform space with topological weak specification property and a pair of distal points is topologically distributionally chaotic of type 1. This extends the result due to Oprocha and Štefánková [43]. In Section 2.2, we study the invariance of topologically distributionally scrambled set for maps on uniform spaces. Results of Section 2.1 are published in International Journal of Bifurcation and Chaos [71] and results of Section 2.2 are communicated.

Throughout this chapter, by a dynamical system we mean a pair (X, f), where $(X,\mathcal{U})$ is a Hausdorff uniform space without isolated points and $f : X \to X$ is a uniformly continuous map. For given dynamical system (X, f), we denote the product map $f \times f$ on $X \times X$ by F.

2.1 Topological distributional chaos and topological specification property

In this section, we study the relation between the topological notions of distributional chaos and specification property defined for uniformly continuous self-maps on uniform spaces. First we recall, the definitions of topological distributional chaos and topological specification property.

Topological distributional chaos

The notion of topological distributional chaos for uniformly continuous self-maps on uniform spaces is introduced by Shah et al. in [55]. For any positive integer n, points x, $y \in X$ and $E \in \mathcal{U}$, define the lower and upper distribution functions

$F_{xy}(E)$ and $F^*_{xy}(E)$ as follows:

$$F_{xy}(E) = \liminf_{n\to\infty} \frac{1}{n} \#\{0 \le i < n \mid F^i(x,y) \in E\},$$
$$F^*_{xy}(E) = \limsup_{n\to\infty} \frac{1}{n} \#\{0 \le i < n \mid F^i(x,y) \in E\},$$

where $\#A$ denotes the cardinality of the set A.

Definition 2.1.1 [55]**.** A pair of points x, $y \in X$ is called

(i) *topologically distributionally chaotic of type 1* (briefly TDC1) if $F_{xy}(E) = 0$, for some $E \in \mathcal{U}$, and $F^*_{xy}(E) = 1$, for all $E \in \mathcal{U}$,

(ii) *topologically distributionally chaotic of type 2* (briefly TDC2) if $F_{xy}(E) < F^*_{xy}(E)$, for some $E \in \mathcal{U}$, and $F^*_{xy}(E) = 1$, for all $E \in \mathcal{U}$,

(iii) *topologically distributionally chaotic of type 3* (briefly TDC3) if $F_{xy}(E) < F^*_{xy}(E)$, for some $E \in \mathcal{U}$.

A set S containing at least two points is called a *topologically distributionally scrambled set of type k* for f if every pair of distinct points in S is topologically distributionally chaotic of type k, where $k \in \{1, 2, 3\}$.

Definition 2.1.2. [55] A map f is said to be *topologically distributionally chaotic of type k*, where $k \in \{1, 2, 3\}$, if there exists an uncountable topologically distributionally scrambled set of type k for f.

Topological specification property

The notion of topological specification property for self-homeomorphisms on uniform spaces is introduced and studied by Shah et al. in [54]. However one can define the notion for continuous maps on uniform spaces as follows:

Definition 2.1.3 [53]**.** A map f is said to have *topological specification property* (briefly TSP) if for every $E \in \mathcal{U}$ there exists a positive integer M such that for any finite sequence $x_1, x_2, \ldots, x_k$ in X, any integers $0 = a_1 \le b_1 < a_2 \le b_2 < \cdots < a_k \le b_k$ with $a_j - b_{j-1} \ge M$ $(2 \le j \le k)$ and $p > M + (b_k - a_1)$, there exists $x \in X$ such that $f^p(x) = x$ and $F^i(x, x_j) \in E$ $(a_j \le i \le b_j,\ 1 \le j \le k)$.

If periodicity condition is omitted in above definition, we say that f has *topological weak specification property* (TWSP). The notion of topological weak specification property can be reformulated in the following (equivalent) way:

Definition 2.1.4. A map f is said to have *topological weak specification property* (TWSP) if for every $E \in \mathcal{U}$ and any n there exists a positive integer $M = M(E, n)$ such that for any sequence of non-negative integers $0 = T_0 < T_1 < T_2 < \cdots < T_n$ such that $T_{i+1} - T_i > M$ whenever $0 \le i < n$ and any two points $u, v \in X$, there is a point $z \in X$ such that

$$F^j(z, u) \in E, \quad \text{if } T_{2i} \le j \le T_{2i+1} - M \text{ and } 2i + 1 \le n, \text{and}$$
$$F^j(z, v) \in E, \quad \text{if } T_{2i-1} \le j \le T_{2i} - M \text{ whenever } 2i \le n.$$

For any entourage $E \in \mathcal{U}$, if $F^j(x, y) \in E$ for any j with $a \le j < b - M$, where $M = M(E, n)$ is given by TWSP, we say that x *E-traces* y on the interval $[a, b]$. Note that, the definition of TWSP can be easily reformulated using any finite sequence of points $u_1, u_2, \ldots, u_k$ instead of just a pair of two points u, v.

In [55], the authors proved that a uniformly continuous self-map on a uniform locally compact Hausdorff space with topological weak specification property and a distal pair admits a topological distributionally scrambled set of type 3. In the following theorem, we extend this result and prove that such maps are topologically distributionally chaotic of type 1. Recall that, a pair of points $(x, y) \in X \times X$ is said to be *distal with respect to f*, if there exists $U \in \mathcal{U}$, such that $F^i(x, y) \notin U$, for all $i \in \mathbb{N}_0$.

Theorem 2.1.5. *Let $(X, \mathcal{U})$ be a uniformly locally compact Hausdorff uniform space consisting of closed entourages without isolated points having a distal pair and let f be a uniformly continuous map of X onto itself. If f has topological weak specification property, then f is topologically distributionally chaotic of type 1.*

Proof. Let u, v be a distal pair of points in X. Then there exists $U_1 \in \mathcal{U}$ such that $F^i(u, v) \notin U_1$, for all $i \in \mathbb{N}_0$. Also since X is uniformly locally compact, there exists $U_2 \in \mathcal{U}$ such that $U_2[t]$ is compact for each $t \in X$. Let $U = U_1 \cap U_2$ and $U_0 \in \mathcal{U}$ be such that $U_0^4 \subset U$.

Let $V_0 = U_0$, and let V_i be an entourage in $\mathcal{U}$ such that $V_i^2 \subset V_{i-1}$, for any $i \geq 1$. Let $M_i = M(V_i, 4^i)$, as in the definition of TWSP, for all $i \geq 1$. Also, let $0 = T_0 < T_1 < T_2 < \cdots$ be an increasing sequence of integers such that $T_i - T_{i-1} > M_i$, for any $i \geq 1$, and $\lim\limits_{i\to\infty} \frac{T_{i+1}-M_{i+1}}{T_i} = \infty$. Further, we consider a subsequence $\{m(i)\}_{i=1}^{\infty}$ of the sequence of positive integers, so that the number of iterations needed for switching from one piece of orbit to the other (applying TWSP) is very small compared to $|T_{m(i)+1} - T_{m(i)}|$.

Step 1: Since (X, f) has TWSP, for entourage V_2 in $\mathcal{U}$ and the sequences of points g, u, v, u and g, u, v, v, we get integers $m(1), m(2)$ such that $1 < m(1) < m(2)$ and points $x_u, x_v \in X$, respectively, so that the points x_u, x_v: V_2-trace g on $[0, T_1]$, V_2-trace u on $[T_{m(1)}, T_{m(1)+1}]$, V_2-trace v on $[T_{m(2)}, T_{m(2)+1}]$. That is,

$$F^p(x_u, g) \in V_2, F^p(x_v, g) \in V_2, \text{ for } 0 \leq p \leq T_1 - M_2,$$

$$F^p(x_u, u) \in V_2, F^p(x_v, u) \in V_2, \text{ for } T_{m(1)} \leq p \leq T_{m(1)+1} - M_2, \text{ and}$$

$$F^p(x_u, v) \in V_2, F^p(x_v, v) \in V_2, \text{ for } T_{m(2)} \leq p \leq T_{m(2)+1} - M_2.$$

Moreover, x_u V_2-trace u on $[T_{m(2)+1}, T_{m(2)+2}]$ and x_v V_2-trace v on $[T_{m(2)+1}, T_{m(2)+2}]$. Note that, $x_u, x_v \in U[g]$.

Let $W_1 = \bigcap_{i=0}^{T_{m(2)+2}} F^{-i}(V_2) \in \mathcal{U}$. Then $W_1 \subset V_2$, which implies that

$$W_1[x_u] \subset U[g] \subset U_2[g] \text{ and}$$

$$W_1[x_v] \subset U[g] \subset U_2[g].$$

Since $U_2[g]$ is compact, it follows that $W_1[x_u]$ and $W_1[x_v]$ are compact sets. Further, $W_1[x_u] \cap W_1[x_v] = \emptyset$. For if $x \in W_1[x_u] \cap W_1[x_v]$, then $(x, x_u) \in W_1$ and $(x, x_v) \in W_1$, this implies that $F^p(x, x_u) \in V_2$ and $F^p(x, x_v) \in V_2$, for $0 \leq p \leq T_{m(2)+2}$. Also, $F^p(x_u, u) \in V_2$ and $F^p(x_v, v) \in V_2$, for $T_{m(2)+1} \leq p \leq T_{m(2)+2} - M_2$. Thus, $F^p(x, u) \in V_2^2$ and $F^p(x, v) \in V_2^2$, which implies that $F^p(u, v) \in V_2^4 \subset U$, a contradiction. Also, for any point $x \in W_1[x_t]$, x V_2-traces x_t on $[0, T_{m(2)+2}]$ for $t = u$, v. Without loss of generality we can assume that $W_1 \subset V_{j(1)}$, where $j(1) \geq 3$.

Step 2: Again by TWSP, for the entourage $V_{j(1)}$, there are integers $m(3)$, $m(4)$ such that $m(3) > m(2) + 2$ and $m(4) > m(3) + 3$, and points $x_{uu}, x_{uv}, x_{vu}, x_{vv} \in X$, which $V_{j(1)}$-trace u on $[T_{m(3)}, T_{m(3)+1}]$ and $V_{j(1)}$-trace v on $[T_{m(4)}, T_{m(4)+1}]$. Further,

$$x_{uu}\, V_{j(1)}\text{-trace } u \text{ on } [T_{m(4)+1}, T_{m(4)+2}] \text{ and } [T_{m(4)+2}, T_{m(4)+3}],$$
$$x_{uv}\, V_{j(1)}\text{-trace } u \text{ on } [T_{m(4)+1}, T_{m(4)+2}] \text{ and } v \text{ on } [T_{m(4)+2}, T_{m(4)+3}],$$
$$x_{vu}\, V_{j(1)}\text{-trace } v \text{ on } [T_{m(4)+1}, T_{m(4)+2}] \text{ and } u \text{ on } [T_{m(4)+2}, T_{m(4)+3}],$$
$$x_{vv}\, V_{j(1)}\text{-trace } v \text{ on } [T_{m(4)+1}, T_{m(4)+2}] \text{ and } [T_{m(4)+2}, T_{m(4)+3}].$$

Let $W_2' = \bigcap_{i=0}^{T_{m(4)+3}} F^{-i}(V_{j(1)}) \in \mathcal{U}$, and let $W_2'' \in \mathcal{U}$ be such that

$$W_2''[x_{uu}] \subset W_1[x_u],\ W_2''[x_{uv}] \subset W_1[x_u] \text{ and}$$
$$W_2''[x_{vu}] \subset W_1[x_v],\ W_2''[x_{vv}] \subset W_1[x_v].$$

Choose $W_2 = W_2' \cap W_2''$. Then $W_2[x_{uu}], W_2[x_{uv}]$ are subsets of $W_1[x_u]$ and $W_2[x_{vu}]$, $W_2[x_{vv}]$ are subsets of $W_1[x_v]$. Since $W_1[x_u]$ and $W_1[x_v]$ are compact sets, it follows that $W_2[x_{uu}]$, $W_2[x_{uv}]$, $W_2[x_{vu}]$ and $W_2[x_{vv}]$ are also compact sets. Note that,

$$W_2[x_{uu}] \cap W_2[x_{uv}] = \emptyset,$$
$$W_2[x_{vu}] \cap W_2[x_{vv}] = \emptyset.$$

Further, for $x \in W_2[x_\alpha]$, x $V_{j(1)}$-traces x_α on $[0, T_{m(4)+3}]$, for any $\alpha \in \{u, v\}^2$. Again without loss of generality, we can assume that $W_2 \subset V_{j(2)}$ where $j(2) \geq j(1) + 1$.

Continuing this way, for each $k \in \mathbb{N}$ and any $\alpha \in \{u, v\}^k$, we get a compact set $W_k[x_\alpha]$, positive integers $m(1) < m(2) < \cdots < m(2k)$, and positive integers $j(1) < j(2) < \cdots < j(k)$, such that if x, y are points in $W_k[x_\alpha]$, then $x\, V_{j(k)-1}$-traces y on $[0, T_{m(2k)+k}]$. Also, if $\alpha, \beta \in \{u, v\}^k$ and $\alpha \neq \beta$, then $W_k[x_\alpha] \cap W_k[x_\beta] = \emptyset$ and $W_{k+1}[x_{\alpha u}] \cup W_{k+1}[x_{\alpha v}] \subset W_k[x_\alpha]$.

Moreover, for any $x \in W_k[x_\alpha]$, where $\alpha = a_1 a_2 \cdots a_k$:

(i) x V_i-traces u on $[T_{m(2i-1)}, T_{m(2i-1)+1}]$, and V_i-traces v on $[T_{m(2i)}, T_{m(2i)+1}]$, for $1 \leq i < k$.

(ii) x V_i-traces a_1 on $[T_{m(2i)+1}, T_{m(2i)+2}]$, for $1 \leq i \leq k$, and V_i-traces a_2 on $[T_{m(2i)+2}, T_{m(2i)+3}]$, for $2 \leq i \leq k$. In general, for any j, $1 \leq j \leq k$, x_α V_i-traces a_j on $[T_{m(2i)+j}, T_{m(2i)+j+1}]$, for $j \leq i \leq k$.

Take $S = \bigcap_{n=1}^{\infty} \bigcup_{\alpha \in \{u,v\}^n} W_n[x_\alpha]$. Note that, $W_n[x_\alpha] \supset W_{n+1}[x_{\alpha'}]$, where $\alpha \in \{u,v\}^n$ and $\alpha' = \alpha a_{n+1}$, $a_{n+1} \in \{u,v\}$. Clearly, each string of characters $\{u,v\}^n$ as $n \to \infty$, gives rise to a decreasing sequence of nonempty compact sets having nonempty intersection. Since each sequence in $\{u,v\}^{\mathbb{N}}$ corresponds to a distinct element in S and vice versa, and the set of all infinite binary sequences is uncountable, it follows that S is uncountable. Moreover, any $s \in S$ can be uniquely determined as $s_\alpha = \bigcap_{n=1}^{\infty} W_n[x_{a_1 a_2 \cdots a_n}]$ where $\alpha = a_1 a_2 \cdots a_n \cdots \in \{u,v\}^{\mathbb{N}}$.

We now prove that S is topologically distributional scrambled set of type 1. Let (x,y) be a pair of distinct points in S. For any entourage $E \in \mathcal{U}$, since $V_i^2 \subset V_{i-1}$ there will be a positive integer m such that $V_i^2 \subset E$, for all $i > m$. Also from condition (i), we have that both x, y V_i-traces u on $[T_{m(2i-1)}, T_{m(2i-1)+1}]$, and V_i-traces v on $[T_{m(2i)}, T_{m(2i)+1}]$, for all $i \geq 1$. Thus, for all $i \geq 1$, $F^p(x,y) \in V_i^2$, for $T_{m(2i-1)} \leq p \leq T_{m(2i-1)+1} - M_i$ and $T_{m(2i)} \leq p \leq T_{m(2i)+1} - M_i$, which guarantees that $F^*_{xy}(E) = 1$.

Further, $x, y \in S$ implies that $x = x_\alpha, y = y_\beta$, for some $\alpha = a_1 a_2 \cdots a_n \cdots$ and $\beta = b_1 b_2 \cdots b_n \cdots$ in $\{u,v\}^{\mathbb{N}}$. As $x \neq y$, there exist a positive integer j such that $a_j \neq b_j$, without loss of generality we assume that $a_j = u$ and $b_j = v$. From condition (ii), we have that $x\, V_i$-traces u on $[T_{m(2i)+j}, T_{m(2i)+j+1}]$, for $j \leq i$, and $y\, V_i$-traces v on $[T_{m(2i)+j}, T_{m(2i)+j+1}]$, for $j \leq i$. Also, for $U_0 \in \mathcal{U}$, we can find a positive integer m, such that $V_i^2 \subset U_0$, for all $i > m$. Note that $F^p(x,y) \notin U_0$, for $T_{m(2i)+j} \leq p \leq T_{m(2i)+j+1} - M_i$ and for all $i > j$. For if, $F^p(x,y) \in U_0$ for $T_{m(2i)+j} \leq p \leq T_{m(2i)+j+1} - M_i$, then $F^p(u,x) \in V_i$ and $F^p(y,v) \in V_i$ will imply that $F^p(u,v) \in U_0^3 \subset U$, a contradiction. Therefore, $F_{xy}(U_0) = 0$.

Thus, S is an uncountable topologically distributional scrambled set of type 1 for f, and hence (X, f) is topologically distributionally chaotic of type 1. $\square$

Since closed entourages in any uniformity $\mathcal{U}$ forms a base for the uniform structure [25], without loss of generality we can work with the closed entourages in $\mathcal{U}$. Consequently, we have the following result.

Corollary 2.1.6. *Let $(X,\mathcal{U})$ be a uniformly locally compact Hausdorff uniform space without isolated points having a distal pair and let f be a uniformly continuous map of X onto itself. If f has topological weak specification property, then f is topologically distributionally chaotic of type 1.*

In [15], the authors defined various forms of shadowing and specification for uniformly continuous self-maps on uniform spaces, and studied their equivalences. We recall here, the definitions of topological shadowing property and topological ergodic shadowing property. For any entourage $D \in \mathcal{U}$, a sequence $\{x_i\}_{i=0}^{\infty}$ in X is said to be a *D-pseudo orbit* for f if $(f(x_i), x_{i+1}) \in D$, for all $i \in \mathbb{N}_0$. Let $y \in X$ and $E \in \mathcal{U}$, then the sequence $\{x_i\}_{i=0}^{\infty}$ in X is said to be *E-shadowed* by y if $(f^i(y), x_i) \in E$, for all $i \in \mathbb{N}_0$.

Definition 2.1.7 [15]. A dynamical system (X, f) is said to have *topological shadowing property* if for every entourage $E \in \mathcal{U}$ there is an entourage $D \in \mathcal{U}$ such that every D-pseudo orbit is E-shadowed by a point of X.

For entourages $D, E \in \mathcal{U}$ of X, the sequence $\{x_i\}_{i=0}^{\infty}$ in X is said to be *D-ergodic pseudo orbit* for f if

$$\lim_{n\to\infty} \frac{1}{n} \#\{0 \le i < n \mid (f(x_i), x_{i+1}) \notin D\} = 0$$

and the sequence $\{x_i\}_{i=0}^{\infty}$ is said to be *E-ergodic shadowed* by $y \in X$ if

$$\lim_{n\to\infty} \frac{1}{n} \#\{0 \le i < n \mid (f^i(y), x_i) \notin E\} = 0.$$

Definition 2.1.8 [15]. A dynamical system (X, f) is said to have *topological ergodic shadowing property* if for every entourage $E \in \mathcal{U}$ there is an entourage $D \in \mathcal{U}$ such that every D-ergodic pseudo orbit is E-ergodic shadowed by a point of X.

Recall that, a map f is said to be topologically mixing if for any two nonempty open sets U and V of X there exists $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$, for all $n \geq N$. In [15], the authors proved that for a uniformly continuous surjective self-maps on uniform spaces having topological shadowing property, topologically mixing, topological ergodic shadowing property, topological weak specification property are all equivalent. As a consequence, we have the following corollary.

Corollary 2.1.9. *Let $(X,\mathcal{U})$ be a uniformly locally compact, totally bounded, Hausdorff uniform space and let f be a uniformly continuous map from X onto itself. If f is topologically mixing and has topological shadowing property with a distal pair, then f is topologically distributionally chaotic of type 1.*

Proof. From [15, Main Theorem], it follows that f has topological weak specification property. That f is topologically distributionally chaotic of type 1, follows from Theorem 2.1.5. □

The topological notions of shadowing property and ergodic shadowing property and their equivalences for uniformly continuous self-maps on uniform spaces are also studied in [2, 68]. If f has topological ergodic shadowing property, then from [2, Proposition 3.4], it follows that f has topological shadowing property, and from [68, Theorem 1] it follows that f is topologically mixing. Using Corollary 2.1.9, we obtain the following result.

Corollary 2.1.10. *Let $(X,\mathcal{U})$ be a uniformly locally compact, totally bounded, Hausdorff uniform space and let f be a uniformly continuous map from X onto itself. If f has topological ergodic shadowing property and a distal pair, then f is topologically distributionally chaotic of type 1.*

2.2 Invariance of topologically distributionally scrambled set

In [42], Oprocha studied the invariance of distributionally scrambled sets for interval maps. He posed the following open question: Does every map with the specification property and a fixed point contain an invariant distributionally scrambled set? Doleželová answered this question positively for the case of continuous self-maps on compact metric spaces with additional assumptions and proved the existence of dense invariant distributionally scrambled sets [20]. In this section, we study the invariance of topologically distributionally scrambled set for maps on uniform spaces. The results proved here extends the work done by Doleželová, and strengthens the result proved in Section 2.1.

Theorem 2.2.1. *Let $(X,\mathcal{U})$ be a uniformly locally compact Hausdorff uniform space without isolated points. If $f : X \to X$ is a uniformly continuous surjective map having topological weak specification property, a fixed point, and countably many periodic points with mutually different periods, then there is a point $x \in X$ whose forward orbit is a topologically distributionally scrambled set of type* 1*.*

Proof. Since closed entourages forms a base for the uniform structure, so without loss of generality we can work with the closed entourages in $\mathcal{U}$.

Let $p \in X$ be a fixed point of f and let $\{q_i\}_{i=1}^{\infty}$ be the set of periodic points of f with mutually different periods. Consider the sequence

$$p, q_1, p, q_1, q_2, p, q_1, q_2, q_3, p, q_1, q_2, q_3, q_4, p, \ldots,$$

which we denote by $\{u_j\}_{j=1}^{\infty}$. Since X is uniformly locally compact, there exists an entourage $U \in \mathcal{U}$ such that $U[t]$ is compact for each $t \in X$. Let $V_0 = U$ and let V_n's be closed entourages in $\mathcal{U}$ such that $V_n^2 \subset V_{n-1}$, for any $n \in \mathbb{N}$. Let $M_n = M(V_n)$, for $n \geq 1$, as in the definition of TWSP. Let $0 = a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < \cdots$, be an increasing sequence of integers such that $a_{n+1} - b_n > M_n$, for any $n \geq 1$ and $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = \infty$.

Step 1: Let $g \in X$ be an arbitrary point. Since (X, f) has TWSP, for $V_1 \in \mathcal{U}$, points g, u_1 in X, and integers a_1, b_1, a_2, b_2 there exists a point $x_1 \in X$ such that

$$F^i(x_1, g) \in V_1, \text{ for } a_1 \leq i \leq b_1,$$
$$F^i(x_1, u_1) \in V_1, \text{ for } a_2 \leq i \leq b_2.$$

Let $W_1 = \bigcap_{i=0}^{b_2} F^{-i}(V_1) \in \mathcal{U}$. Then $W_1 \subset V_1$ implies that $W_1[x_1] \subset V_1[x_1] \subset U[x_1]$. Since $U[x_1]$ is compact, it follows that $W_1[x_1]$ is compact. Further, for any $x \in W_1[x_1]$, we have $(x, x_1) \in W_1$, which implies that $F^i(x, x_1) \in V_1$, for all $a_2 \leq i \leq b_2$. Also, $F^i(x_1, u_1) \in V_1$, for $a_2 \leq i \leq b_2$, which implies that $F^i(x, u_1) \in V_1^2$, for $a_2 \leq i \leq b_2$. Thus, for any $x \in W_1[x_1]$,

$$F^i(x, u_1) \in V_1^2, \text{ for } a_2 \leq i \leq b_2.$$

Without loss of generality we can assume that $W_1 \subset V_2$ (if W_1 is not contained in V_2, we can always work with $W_1' = W_1 \cap V_2 \in \mathcal{U}$).

Step 2: For $V_2 \in \mathcal{U}$, points x_1, u_2 in X, and integers a_1, b_2, a_3, b_3 by TWSP there exists a point $x_2 \in X$ such that

$$F^i(x_2, x_1) \in V_2, \text{ for } a_1 \leq i \leq b_2,$$
$$F^i(x_2, u_2) \in V_2, \text{ for } a_3 \leq i \leq b_3.$$

Let $W_2' = \bigcap_{i=0}^{b_3} F^{-i}(V_2) \in \mathcal{U}$, and W_2'' be a closed entourage in $\mathcal{U}$ such that $W_2''[x_2] \subset W_1[x_1]$. Let $W_2 = W_2' \cap W_2''$. Then $W_2[x_2] \subset W_1[x_1]$. Since $W_1[x_1]$ is compact, it follows that $W_2[x_2]$ is compact. Note that for any $x \in W_2[x_2]$,

$$F^i(x, u_2) \in V_2^2, \text{ for } a_3 \leq i \leq b_3.$$

Again without loss of generality we can assume that $W_2 \subset V_3$.

Continuing like this, for each $n \in \mathbb{N}$, there is a compact set $W_n[x_n]$ such that $W_n[x_n] \subset W_{n-1}[x_{n-1}]$ and for each $x \in W_n[x_n]$,

$$F^i(x, u_n) \in V_n^2, \text{ for } a_{n+1} \leq i \leq b_{n+1}.$$

This gives rise to a nested sequence $W_n[x_n]$ of compact sets and hence there is at least one point $x \in \bigcap_{n=1}^{\infty} W_n[x_n]$ such that for every $n \in \mathbb{N}$,

$$F^i(x, u_n) \in V_n^2, \text{ for } a_{n+1} \leq i \leq b_{n+1}.$$

We now prove that the forward orbit of x is topologically distributionally scrambled set of type 1. Let $y = f^l(x)$ and $z = f^m(x)$, for some non-negative integers l, m with $l > m$. For every $n \geq 1$, consider $s_n = \sum_{j=1}^{n} j$. Then $u_{s_n} = p$, for every $n \geq 1$. From our choice of s_n, it follows that $F^i(x,p) = (f^i(x), p) \in V_{s_n}^2$, for all $a_{s_n+1} \leq i \leq b_{s_n+1}$. This implies that $(f^i(y), p) \in V_{s_n}^2$, for $a_{s_n+1} - l \leq i \leq b_{s_n+1} - l$, and $(f^i(z), p) \in V_{s_n}^2$, for $a_{s_n+1} - m \leq i \leq b_{s_n+1} - m$. As $l > m$, we have $F^i(y,z) \in V_{s_n}^4$, for $a_{s_n+1} - m \leq i \leq b_{s_n+1} - l$. Thus,

$$\#\{0 \leq i \leq b_{s_n+1} | F^i(y,z) \in V_{s_n}^4\} \geq b_{s_n+1} - a_{s_n+1} - (l-m).$$

Now for any entourage $E \in \mathcal{U}$, since $V_n^2 \subset V_{n-1}$, there will be a positive integer N such that $V_n^4 \subset E$, for all $n > N$. Thus, $F_{yz}^*(E) = 1$, for all $E \in \mathcal{U}$.

Next we show that $F_{yz}(E) = 0$, for some $E \in \mathcal{U}$. Choose a periodic point q_r with period $k > l - m$. Then $f^i(q_r) \neq f^j(q_r)$, for $i \neq j$, $i, j \in \{0, 1, 2, \ldots, k-1\}$. Since X is Hausdorff, we can find an entourage $E_r \in \mathcal{U}$ such that $(f^i(q_r), f^j(q_r)) \notin E_r$, for all $i \neq j$, $i, j \in \{0, 1, 2, \ldots, k-1\}$. Let $E \in \mathcal{U}$ be such that $E \circ E \circ E \subset E_r$. Since $V_n^2 \subset V_{n-1}$, there will be a positive integer N such that $V_n^2 \subset E$, for all $n > N$. Note that, $u_{t_n} = q_r$, where $t_n = r + \sum_{j=1}^{n} j$, for every $n \geq r$. Thus, $F^i(x, q_r) \in V_{t_n}^2$, for $a_{t_n+1} \leq i \leq b_{t_n+1}$. This implies that $F^i(y, f^l(q_r)) \in V_{t_n}^2$ and $F^i(z, f^m(q_r)) \in V_{t_n}^2$, for $a_{t_n+1} - m \leq i \leq b_{t_n+1} - l$. Observe that $F^i(y,z) \notin V_{t_n}^2$, for $a_{t_n+1} - m \leq i \leq b_{t_n+1} - l$ and $t_n > N$. For if, $F^i(y,z) \in V_{t_n}^2$, for some $a_{t_n+1} - m \leq i \leq b_{t_n+1} - l$, then $F^i(y, f^l(q_r)) \in V_{t_n}^2$ and $F^i(z, f^m(q_r)) \in V_{t_n}^2$ imply that $F^i(f^l(q_r), f^m(q_r)) \in E \circ E \circ E \subset E_r$, a contradiction. Therefore,

$$\#\{0 \leq i \leq b_{t_n+1} | F^i(y,z) \in E\} \leq a_{t_n+1} + (l-m), \text{ for } t_n > N.$$

Hence, $F_{yz}(E) = 0$. □

Next, we state a lemma proved in [20].

Lemma 2.2.2 [20]. *There is a Cantor set $B \subset \{0,1\}^{\mathbb{N}}$ such that for any distinct $\alpha = \{\alpha(i)\}_{i=1}^{\infty}$ and $\beta = \{\beta(i)\}_{i=1}^{\infty}$ in B, the set $\{j \in \mathbb{N} | \alpha(j) \neq \beta(j)\}$ is infinite.*

Recall that, a set $A \subset X$ is said to be *invariant* if $f(A) \subset A$. A set which is a countable union of Cantor sets is said to be a *Mycielski set*.

Theorem 2.2.3. *Let $(X, \mathcal{U})$ be a uniformly locally compact Hausdorff uniform space without isolated points. If $f : X \to X$ is a uniformly continuous surjective map having topological weak specification property, a fixed point, and countably many periodic points with mutually different periods, then X has a Mycielski invariant topologically distributionally scrambled set of type 1.*

Moreover, each point $g \in X$ has a neighborhood which contains a Cantor topologically distributionally scrambled set of type 1.

Proof. Let $\alpha = \{\alpha(i)\}_{i=1}^{\infty} \in B$, where B is the set as in Lemma 2.2.2. Let $p \in X$ be a fixed point of f and let $\{q_i\}_{i=1}^{\infty}$ be the set of periodic points of f with mutually different periods. Define the sequence $\{w_j^{(\alpha)}\}_{j=1}^{\infty}$ in the following manner:

$$w_j^{(\alpha)} = \begin{cases} p, & \text{if } \alpha(j) = 0; \\ q_1, & \text{if } \alpha(j) = 1. \end{cases}$$

Consider the sequence

$$p, q_1, w_1^{(\alpha)}, p, q_1, q_2, w_2^{(\alpha)}, p, q_1, q_2, q_3, w_3^{(\alpha)}, p, q_1, q_2, q_3, q_4, w_4^{(\alpha)}, p, \ldots,$$

which we denote by $\{u_j^{(\alpha)}\}_{j=1}^{\infty}$. Following the recursive steps as discussed in Theorem 2.2.1, we get a point $x^{(\alpha)}$ such that for every $n \geq 1$, $F^i(x^{(\alpha)}, u_n^{(\alpha)}) \in V_n^2$, for $a_{n+1} \leq i \leq b_{n+1}$.

Let $C = \{x^{(\alpha)} | \alpha \in B\}$ and $\hat{C} = \bigcup_{i=0}^{\infty} f^i(C)$. Note that, $\hat{C}$ is an invariant set. We now prove that $\hat{C}$ is a topological distributionally scrambled set of type 1. For $y, z \in \hat{C}$, let $y = f^l(x^{(\alpha)})$ and $z = f^m(x^{(\beta)})$, for some non-negative integers l, m and $\alpha, \beta \in B$.

Case 1: If $\alpha = \beta$, $l \neq m$, then from the proof of Theorem 2.2.1, it follows that (y, z) is a TDC1 pair.

Case 2: Suppose $\alpha \neq \beta$ and $l \geq m$. Then choose a sequence $\{s_n\}_{n=1}^{\infty}$ such that $u_{s_n} = p$, for every $n \geq 1$. Note that, $(f^i(y), p) \in V_{s_n}^2$ and $(f^i(z), p) \in V_{s_n}^2$, for $a_{s_n+1} - m \leq i \leq b_{s_n+1} - l$. Thus, we get $F^i(y, z) \in V_{s_n}^4$, for $a_{s_n+1} - m \leq i \leq b_{s_n+1} - l$ and for every $n \geq 1$. Hence, $F_{yz}^* \equiv 1$.

Since $\alpha, \beta \in B$ and $\alpha \neq \beta$, there is a subsequence $\{w_{j_k}\}_{k=1}^{\infty}$ of $\{w_j\}_{j=1}^{\infty}$ such that $w_{j_k}^{(\alpha)} \neq w_{j_k}^{(\beta)}$, for every $k \geq 1$ (That is, $w_{j_k}^{(\alpha)} = p$ and $w_{j_k}^{(\beta)} = q_1$ or $w_{j_k}^{(\alpha)} = q_1$ and $w_{j_k}^{(\beta)} = p$). We choose a sequence $\{t_n\}_{n=1}^{\infty}$ such that $u_{t_n} = w_{j_k}$, for every $n \geq 1$. Then $F^i(y, f^l(w_{j_k}^{(\alpha)})) \in V_{t_n}^2$ and $F^i(z, f^m(w_{j_k}^{(\beta)})) \in V_{t_n}^2$, for $a_{t_n+1} - m \leq i \leq b_{t_n+1} - l$. Let q_1 be a periodic point of period k. Since X is Hausdorff, we can find an entourage $E_1 \in \mathcal{U}$ such that $(p, f^i(q_1)) \notin E_1$, for all $i \in \{0, 1, 2, \ldots, k-1\}$, and let $E \in \mathcal{U}$ be such that $E \circ E \circ E \subset E_1$. Using arguments similar to that given in proof of Theorem 2.2.1, we get that $F_{yz}(E) = 0$.

Consider the bijective map $h : B \to C$, given by $h(\alpha) = x^{(\alpha)}$, for all $\alpha \in B$. To show that h is a homeomorphism, it is sufficient to show that h is continuous. Let $\{\alpha_m\}_{m=1}^{\infty}$ be a convergent sequence in B such that $\alpha_m \to \alpha$. Then for any $k \geq 1$, there is an m_0 such that the first k members of sequences α_m and α are equal, for all $m > m_0$. Therefore, the first k members of $\{u_j^{(\alpha_m)}\}_{j=1}^{\infty}$ and $\{u_j^{(\alpha)}\}_{j=1}^{\infty}$ are also equal, and hence the corresponding $x^{(\alpha_m)}$ and $x^{(\alpha)}$ belong to the same $W_k[x_k]$. Thus, as $m \to \infty$, $\alpha_m \to \alpha$ implies that $x^{(\alpha_m)} \to x^{(\alpha)}$. Hence, h is a homeomorphism and C is a Cantor set. Since $\hat{C}$ is a topologically distributionally scrambled set of type 1, the mapping $f^i|_C : C \to f^i(C)$ is also a bijective mapping. Continuity of $f^i|_C$ implies that $f^i|_C$ is a homeomorphism, for every $i \geq 1$. Thus, $f^i(C)$ is a Cantor set, for each $i \geq 1$. Hence, $\hat{C}$ is a Mycielski invariant topologically distributionally scrambled set of type 1.

Moreover, for every $\alpha \in B$, the corresponding $x^{(\alpha)} \in U[g]$, and hence $C \subset U[g]$. Since $g \in X$ is arbitrary, it follows that each point in X has a neighborhood which contains a Cantor topologically distributionally scrambled set of type 1. $\square$

Theorem 2.2.4. *Let $(X,\mathcal{U})$ be a uniformly locally compact second countable Hausdorff uniform space without isolated points. If $f : X \to X$ is a uniformly continuous surjective map having topological weak specification property, a fixed point and countably many periodic points with mutually different periods, then X has a dense Mycielski invariant topologically distributionally scrambled set of type 1.*

Proof. Let $\{\dot{F}_i\}_{i=1}^{\infty}$ denote the countable closed base for the uniform topology on X induced by $(X,\mathcal{U})$, where $\dot{F}_i = U_i[g_i]$ for some $U_i \in \mathcal{U}$, $g_i \in X$. Since X is uniformly locally compact, there exists an entourage $U \in \mathcal{U}$ such that $U[t]$ is compact for each $t \in X$. Let $V_0 = U$ and let V_n be a closed entourage in $\mathcal{U}$ such that $V_n^2 \subset V_{n-1}$, for all $n \in \mathbb{N}$. For each $g_i \in X$, using the arguments given in proof of Theorem 2.2.3, for the sequence

$$\begin{aligned}\{u_j^{(\alpha,i)}\}_{j=1}^{\infty} = \quad & \{p, q_1, w_1^{(\alpha)}, \mathbf{q_i}, p, q_1, q_2, w_2^{(\alpha)}, \mathbf{q_i}, p, q_1, q_2, q_3, \\ & w_3^{(\alpha)}, \mathbf{q_i}, p, q_1, q_2, q_3, q_4, w_4^{(\alpha)}, \mathbf{q_i}, p, \ldots\},\end{aligned}$$

$\alpha \in B$, there exists a Cantor set $C_i = \{x^{(\alpha,i)} | \alpha \in B\} \subset \dot{F}_i$, which is topologically distributionally scrambled set of type 1. Let $M = \bigcup_{i=1}^{\infty} \hat{M}_i$, where $\hat{M}_i = \bigcup_{r=0}^{\infty} f^r(C_i)$. Then, M is a dense Mycielski invariant set in X.

We now prove that M is a topologically distributionally scrambled set of type 1. Clearly, $\hat{M}_i$ is a topologically distributionally scrambled set of type 1, for each $i \geq 1$. Consider $y, z \in M$ such that $y \in \hat{M}_i$ and $z \in \hat{M}_j$, $i \neq j$. Then $y = f^l(x^{(\alpha,i)})$ and $z = f^m(x^{(\beta,j)})$, for some non-negative integers l, m and $\alpha, \beta \in B$. We assume $l \geq m$. Choose a sequence $\{s_n\}_{n=1}^{\infty}$ such that $u_{s_n}^{(\alpha,i)} = u_{s_n}^{(\beta,j)} = p$, for every $n \geq 1$. Then,

$$(f^k(y), p) \in V_{s_n}^2 \text{ and } (f^k(z), p) \in V_{s_n}^2, \text{ for } a_{s_n+1} - m \leq k \leq b_{s_n+1} - l.$$

Thus, we get $F^k(y, z) \in V_{s_n}^4$, for $a_{s_n+1} - m \leq k \leq b_{s_n+1} - l$ and for every $n \geq 1$. Hence, $F_{yz}^* \equiv 1$.

Next, choose a sequence $\{t_n\}_{n=1}^{\infty}$ such that $u_{t_n}^{(\alpha,i)} = q_i$ and $u_{t_n}^{(\beta,j)} = q_j$, for every $n \geq 1$. Then,

$$F^k(y, f^l(q_i)) \in V_{t_n}^2 \text{ and } F^k(z, f^m(q_j)) \in V_{t_n}^2, \text{ for } a_{t_n+1} - m \leq k \leq b_{t_n+1} - l.$$

If q_i is a periodic point of period λ and q_j is a periodic point of period μ, consider the entourage $E \in \mathcal{U}$ such that $(f^{k_1}(q_i), f^{k_2}(q_j)) \notin E$, for all $k_1 \in \{0, 1, 2, \ldots, \lambda - 1\}$, $k_2 \in \{0, 1, 2, \ldots, \mu - 1\}$. Let $E_0 \in \mathcal{U}$ be such that $E_0 \circ E_0 \circ E_0 \subset E$. Further, since $V_n^2 \subset V_{n-1}$, there will be a positive integer N such that $V_n^2 \subset E$, for all $n > N$. For $t_n > N$, if $F^k(y, z) \in V_{t_n}^2$, for some $a_{t_n+1} - m \leq k \leq b_{t_n+1} - l$, then $F^k(y, f^l(q_i)) \in V_{t_n}^2$ and $F^k(z, f^m(q_j)) \in V_{t_n}^2$ imply that $F^k(f^l(q_i), f^m(q_j)) \in E_0 \circ E_0 \circ E_0 \subset E$, a contradiction. Thus,

$$\#\{0 \leq k \leq b_{t_n+1} | F^k(y, z) \in E\} \leq a_{t_n+1} + (l - m), \text{ for } t_n > N.$$

Therefore, $F_{yz}(E) = 0$. Hence, M is a topologically distributionally scrambled set of type 1. □

In [54], the authors proved that for self-homeomorphisms on uniform spaces with topological specification property, the map has a dense set of periodic points. The construction suggests that such a map can have infinitely many periodic points with mutually different periods. As a consequence, we have the following result:

Corollary 2.2.5. *Let $(X, \mathcal{U})$ be a uniformly locally compact second countable Hausdorff uniform space without isolated points. If $f : X \to X$ is a homeomorphism having topological specification property and a fixed point, then X has a dense Mycielski invariant topologically distributionally scrambled set of type* 1.