

# Chapter 1

## Introduction

The theory of dynamical systems has its roots in many disciplines, from mathematics to physics. However, most people regard that the qualitative theory of dynamical systems originated from the work of Henri Poincaré on the three-body problem during the late nineteenth century. In the twentieth century, mathematicians like Birkhoff, Anosov, Smale, Kolmogorov and Ruelle, among others contributed significantly towards the development of this field. With advances in mathematics and technology, the theory of dynamical systems has found its applications in various branches of science and engineering.

A *dynamical system* comprises of three components: (i) a phase space, which describes all possible states of a physical or artificial system, (ii) a set of times, and (iii) a continuous rule, which gives consequent to a state at a given time. The time set is either discrete, taking integer values, or continuous, varying over the real values. When the time set is discrete, we call the system a discrete dynamical system, and if the time set is continuous, the system is called a continuous dynamical system or a flow. Gottschalk and Hedlund unified the views of discrete and continuous time sets in dynamical systems with the general approach of topological groups acting on a space. These ideas and others were described in [23], the first treatise that explicitly studied the topological view of dynamics.

The central idea in the theory of dynamical systems is to study the long-term

behavior of the system and to describe the steady states of the system. However, some systems vary irregularly and randomly, and do not tend to repeat their behavior, even when observed for a long extent. This unpredictability in simple nonlinear deterministic systems was first observed by Poincaré. Later in 1961, Lorenz while running a numerical model discovered that small rounding of errors could lead to significantly different outcomes, and thus motivated the idea of sensitive dependence on initial conditions. Lorenz's work popularized the concept of chaos in the field of science, with the introduction of the term "butterfly effect".

Since then, many researchers have introduced and studied various aspects of chaos. However, the proper characterization of chaotic behavior is still a topic of debate. Most explanations of chaos are based on one of two aspects: random behavior and sensitivity to initial conditions. The former centers on the apparent randomness and unpredictability of chaotic systems, seeking to comprehend chaos in terms of randomness. The latter emphasizes the sensitivity to initial conditions and explores how small changes in the starting conditions can result in significant differences in the long-term behavior of trajectories in phase space. Despite numerous proposed criteria and indicators for identifying chaotic phenomena, none has attained the status of a universally accepted definition.

Some popular notions of chaos in the field of topological dynamics include positive topological entropy, Li-Yorke chaos, Devaney chaos, specification property, and distributional chaos, amongst others. Acknowledging the inherent challenge of formulating a universal definition that fully encompasses the intricate behaviors of a dynamical system, it is essential to explore the connections between the various definitions of chaos. The study of interrelationships between various notions of chaos has been a major focus of researchers in the recent years, primarily within the context of compact metric spaces.

In this thesis, we aim to study the chaotic properties of dynamical systems defined over general topological spaces. Broadly, we address the topological notions of distributional chaos and specification defined for uniformly continuous self-maps

on uniform spaces. In particular, we explore the relation between the notions of topological distributional chaos and topological specification property. Moreover, we study the topological notions of distributional chaos in sequence and weaker forms of specification for uniformly continuous self-maps defined on uniform spaces. We also study the notions of distributional chaos and specification for the case of continuous  $\mathbb{Z}^d$ -actions.

## 1.1 Preliminaries

In this section, we review some basic notations and definitions related to the study of discrete dynamical systems.

Throughout the thesis,  $\mathbb{R}$  denotes the set of real numbers,  $\mathbb{Z}$  denotes the set of integers,  $\mathbb{N}$  denotes the set of natural numbers and the set  $\mathbb{N} \cup \{0\}$  is denoted by  $\mathbb{N}_0$ . By a discrete dynamical system, we mean a pair  $(X, f)$ , where  $X$  is a metric space with metric  $d$  and  $f$  is a continuous self-map defined on  $X$ . For any  $n \in \mathbb{N}$ ,  $f^n$  denotes the  $n$ -fold composition of  $f$  with itself, that is  $f^n = f \circ f \circ \dots \circ f$  ( $n$ -times), and  $f^0$  denotes the identity map. For basics of dynamical systems one can refer [4, 5, 64].

For any  $x \in X$ , the set  $\{f^n(x) \mid n \in \mathbb{N}_0\}$  is called the *orbit of  $x$  under  $f$*  (often referred as forward orbit) and is denoted by  $O_f(x)$ . A point  $x \in X$  is said to be a *periodic point of  $f$*  if  $f^n(x) = x$ , for some  $n \in \mathbb{N}$ . If  $n \in \mathbb{N}$  is such that  $f^n(x) = x$  and  $f^i(x) \neq x$ , for all  $1 \leq i < n$ , then  $x$  is said to be a periodic point of  $f$  with period  $n$ . A periodic point of  $f$  of period 1 is called a *fixed point of  $f$* .

A subset  $A \subset X$  is said to be *invariant* (or *positive invariant*) under the map  $f$  if  $f(A) \subset A$ . In the case that  $f$  is invertible,  $A \subset X$  is said to be *invariant* under the map  $f$  if  $f(A) = A$ . If  $Y$  is a nonempty invariant subset of  $X$ , then  $(Y, f|_Y)$  is also a dynamical system and is called a *subsystem* of  $(X, f)$ , where  $f|_Y$  denotes the restriction of map  $f$  to  $Y$ . A dynamical system  $(X, f)$  is said to be *minimal* if it has no proper subsystem. Equivalently, a dynamical system  $(X, f)$  is *minimal*

if every point  $x \in X$  visit every open set  $V$  in  $X$  under the iterations of map  $f$ , i.e.,  $f^n(x) \in V$ , for some  $n \in \mathbb{N}$ . However, there are real physical systems where the state is never measured exactly, but always up to a certain error. So instead of points one should study small open subsets of the given space and describe how they move in the space. This motivated the study of the systems where every nonempty open subset  $U$  of  $X$  visits every nonempty open subset  $V$  of  $X$  in the following sense:  $f^n(U) \cap V \neq \emptyset$ , for some  $n \in \mathbb{N}$ . If the system  $(X, f)$  has this property, then it is called *topologically transitive*. More precisely,

**Definition 1.1.1.** A continuous map  $f : X \rightarrow X$  is said to be *topologically transitive* (or *transitive*) if for every pair of nonempty open sets  $U$  and  $V$  in  $X$ , there exists  $n \in \mathbb{N}$  such that  $f^n(U) \cap V \neq \emptyset$ .

*Remark 1.1.2.* Transitivity of a map  $f$  guarantees that the phase space cannot be decomposed into two disjoint sets with nonempty interiors, which do not interact under the map  $f$ .

**Definition 1.1.3.** A continuous map  $f : X \rightarrow X$  is said to be *topologically weakly mixing* if the product map  $f \times f$  is transitive. That is, for nonempty open sets  $U_1, U_2, V_1, V_2$  in  $X$ , there exists  $n \in \mathbb{N}$  such that  $f^n(U_1) \cap V_1 \neq \emptyset$  and  $f^n(U_2) \cap V_2 \neq \emptyset$ .

**Definition 1.1.4.** A continuous map  $f : X \rightarrow X$  is said to be *topologically mixing* if for every pair of nonempty open sets  $U$  and  $V$  in  $X$ , there exists an integer  $N > 0$  such that  $f^n(U) \cap V \neq \emptyset$ , for all  $n \geq N$ .

*Remark 1.1.5.*

$$\text{Mixing} \Rightarrow \text{Weakly mixing} \Rightarrow \text{Transitivity}$$

**Definition 1.1.6.** For a dynamical system  $(X, f)$ , a point  $x \in X$  is said to be a *transitive point of  $f$*  if the orbit of  $x$  is dense in  $X$ , i.e.,  $\overline{O_f(x)} = X$ .

*Remark 1.1.7.* If  $X$  is a compact metric space, then the existence of a transitive point is equivalent to topological transitivity.

**Definition 1.1.8.** A continuous map  $f : X \rightarrow X$  is said to be *sensitive* (or to have *sensitive dependence on initial conditions*) if there is a  $\epsilon > 0$  such that for each  $x \in X$  and each neighborhood  $U$  of  $x$ , there exists  $y \in U$  and  $n \in \mathbb{N}$  such that  $d(f^n(x), f^n(y)) > \epsilon$ , where  $\epsilon$  is called a sensitivity constant for  $f$ .

Transitivity, mixing property, and sensitivity are the most commonly used notions to describe chaos. Transitivity and mixing describe the non-separability and asymptotic properties, respectively. Whereas, sensitivity is often used to describe unpredictability. In dynamical systems having sensitive dependence on initial conditions, the orbits of nearby points deviate after finite iterations. It is also popularly called the “Butterfly effect”. Devaney’s definition of chaos is one of the most popular and widely used notions of chaos.

**Definition 1.1.9 [19].** A continuous map  $f : X \rightarrow X$  is said to be *chaotic in the sense of Devaney* (or *Devaney chaotic*) on  $X$ , if the following conditions hold:

- (i)  $f$  is topologically transitive,
- (ii) the set of all periodic points of  $f$  is dense in  $X$  and
- (iii)  $f$  has sensitive dependence on initial conditions.

*Remark 1.1.10.*

- (i) For continuous self-maps defined on infinite metric spaces, sensitive dependence on initial conditions is redundant in the definition of Devaney chaos [9].
- (ii) For continuous self-maps on intervals, transitivity is equivalent to Devaney chaos [63].

Expansivity of maps is another significant property in qualitative theory of dynamical systems. The notion of expansive homeomorphism was introduced by Utz in 1950, by the name unstable homeomorphism [62]. Later on, Gottschalk and

Hedlund termed them as expansive homeomorphisms [23]. Eisenberg studied this concept for continuous semigroups acting on a uniform space and termed it as positively expansive map [22].

**Definition 1.1.11.** A continuous map (respectively, a homeomorphism)  $f : X \rightarrow X$  is said to be *positively expansive* (respectively, *expansive*) if there exist a number  $e > 0$  such that for any pair of distinct points  $x, y \in X$  there is an  $n \in \mathbb{N}$  (respectively,  $n \in \mathbb{Z}$ ) such that  $d(f^n(x), f^n(y)) > e$ . The number  $e$  is called an expansivity constant for  $f$ .

In some cases, when calculating the orbit of a specific point using numerical simulation, the computer may calculate an approximate trajectory of the system at that point rather than the actual trajectory. This approximate trajectory is called a pseudo trajectory or a pseudo-orbit. This led to the development of the concept of the shadowing property. Roughly, a system is said to have the shadowing property if these approximate orbits are followed by actual orbits. For any  $\delta > 0$ , a sequence  $\{x_i\}_{i=0}^{\infty}$  of points in  $X$  is said to be a  $\delta$ -pseudo orbit for  $f$  if  $d(f(x_i), x_{i+1}) < \delta$ , for all  $i \in \mathbb{N}_0$ . For a point  $y \in X$  and  $\epsilon > 0$ , a sequence  $\{x_i\}_{i=0}^{\infty}$  in  $X$  is said to be  $\epsilon$ -shadowed (or  $\epsilon$ -traced) by  $y$  if  $d(f^i(y), x_i) < \epsilon$ , for all  $i \in \mathbb{N}_0$ .

**Definition 1.1.12.** A dynamical system  $(X, f)$  is said to have *pseudo-orbit tracing property* (or *shadowing property*) if for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that every  $\delta$ -pseudo orbit for  $f$  is  $\epsilon$ -shadowed by a point of  $X$ .

For dynamical systems  $(X, f)$  and  $(Y, g)$ , the maps  $f$  and  $g$  are said to be *topologically semi-conjugate* (respectively, *topologically conjugate*) if there exists a continuous surjection (respectively, a homeomorphism)  $\varphi : X \rightarrow Y$  such that  $\varphi \circ f = g \circ \varphi$  [4]. Such a map  $\varphi$  is called a *topological conjugacy* between  $f$  and  $g$ , and the following diagram is commutative:

$$\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\varphi \downarrow & \circlearrowleft & \downarrow \varphi \\
Y & \xrightarrow{g} & Y
\end{array}$$

**Definition 1.1.13.** A property of a map which is preserved under topological conjugacy is said to be a *dynamical property*.

*Remark 1.1.14.* The notions of minimality, topological transitivity are dynamical properties. Also, when  $X$  is a compact metric space, the notions of expansivity and shadowing are preserved under topological conjugacy.

## 1.2 Literature survey and motivation

The study of notions describing unpredictable and chaotic behavior has been an intriguing area of research in dynamical systems. Various authors have introduced different notions of chaos based on the properties describing complexity and predictability of the system. We now recall some notions of chaos which motivated our study.

### Positive entropy

The concept of entropy can be traced back to Shannon [56], who introduced the term in the context of information theory to describe the amount of information transmitted. Roughly, if we consider a source producing messages that are communicated to a receiver registering them, then the notion of entropy given by Shannon describes the measure of the receiver's average uncertainty about what message the source produces next. Following the idea, Kolmogorov and Sinai formulated the notion of measure-theoretic entropy in ergodic theory. Measure-theoretic entropy can be interpreted as a measure of the indeterminacy of an invariant process. Note that, in a process with discrete states if the past states almost surely determine its present

state, then the process is said to be deterministic, and equivalently its entropy is zero. Thus positive entropy can be regarded as an important characterization of chaos.

In 1965, Adler et al. introduced the notion of topological entropy for a continuous self-map on a compact topological space as an analog of the measure-theoretic entropy [1]. Let  $X$  be a compact topological space and let  $f : X \rightarrow X$  be a continuous map. Given two open covers  $\alpha$  and  $\beta$ , their join  $\alpha \vee \beta$  is the cover  $\{U \cap V \mid U \in \alpha, V \in \beta\}$ . We define  $N(\alpha)$  as the minimum cardinality of subcovers of  $\alpha$ . The topological entropy of  $f$  with respect to the cover  $\alpha$  is given by

$$h(f, \alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right).$$

**Definition 1.2.1.** The *topological entropy* of a continuous self-map  $f$  defined on a compact topological space  $X$  is given by

$$h(f) = \sup_{\alpha} h(f, \alpha),$$

where  $\alpha$  ranges over all open covers of  $X$ .

Later, Bowen introduced two equivalent definitions of topological entropy for a uniformly continuous self-map defined on a metric space in terms of separated sets and spanning sets [11]. We recall here the definition given by Bowen in terms of separated sets. Let  $f$  be a uniformly continuous self-map defined on a metric space  $(X, d)$ . A set  $E \subset X$  is called  $(n, \epsilon)$ -separated set with respect to  $f$  if for each pair of distinct points  $x, y \in E$ , there exists a nonnegative integer  $0 \leq i < n$  such that  $d(f^i(x), f^i(y)) > \epsilon$ . For a compact set  $K \subset X$ , let  $s_n(\epsilon, K, f)$  denote the largest cardinality of an  $(n, \epsilon)$ -separated set  $E$  contained in  $K$ . For  $\epsilon > 0$  and compact set  $K \subset X$ , let

$$s(\epsilon, K, f) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\epsilon, K, f).$$

**Definition 1.2.2.** The *topological entropy* of a uniformly continuous self-map  $f$  on a metric space  $(X, d)$  is defined as

$$h(f) = \sup_K \lim_{\epsilon \rightarrow 0} s(\epsilon, K, f),$$

where the supremum is taken over all compact subsets  $K$  of  $X$ .

*Remark 1.2.3.*

- (i) Topological entropy measures the exponential growth rate of the number of distinguishable orbits as time advances.
- (ii) For continuous self-maps on compact metric spaces, Bowen's definition of topological entropy is equivalent to the original definition given by Adler et al. [12].
- (iii) Topological entropy is invariant under topological conjugacy.

## Li-Yorke Chaos

The term chaos in connection with a map was first used by Li and Yorke [36]. They studied pairs of points with the property that their orbits neither approach each other asymptotically nor do they eventually separate from each other by any fixed positive distance. The mathematical definition is as follows: For a dynamical system  $(X, f)$ , a set  $S \subset X$  containing at least two points is called a *Li-Yorke scrambled set* for  $f$ , if for any two distinct points  $x, y$  in  $S$ ,

$$\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) > 0. \quad (1.1)$$

**Definition 1.2.4.** A map  $f$  is said to be *chaotic in the sense of Li-Yorke* (briefly *Li-Yorke chaotic*) if there exists an uncountable Li-Yorke scrambled set for  $f$ .

A pair  $(x, y)$  of points in  $X$  satisfying (1.1) is called a *Li-Yorke scrambled pair*. Also, for points  $x, y \in X$  if  $\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0$  then  $(x, y)$  is said to be a *proximal pair*, and if  $\limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) > 0$  then  $(x, y)$  is said to be a *distal pair*.

In [36], Li and Yorke showed that for interval maps the existence of a periodic point of period 3 implies Li-Yorke chaos. Later, Janková and Smítal studied the relation of Li-Yorke chaos with the notion of topological entropy and proved that the

positive topological entropy of an interval map implies the existence of a Cantor Li-Yorke scrambled set [29]. At the same time examples of zero entropy interval maps chaotic in the sense of Li-Yorke were constructed [59]. Thus, positive topological entropy is a stronger notion of chaos than Li-Yorke chaos.

## Distributional chaos

Schweizer and Smítal extended Li and Yorke's approach by measuring lower and upper densities of the rate of proximity of pairs and introduced the notion of distributional chaos for interval maps [48]. Distributional chaos is considered a very important generalization of Li-Yorke chaos because it is equivalent to positive topological entropy and some other concepts of chaos when restricted to the case of compact intervals. It is important to note that these equivalences do not transfer to the case of general compact metric spaces or higher dimensions. The notion of distributional chaos later evolved into three nonequivalent variants namely *DC1*, *DC2*, and *DC3* [8]. We introduce here the contemporary version of the definition. Let  $(X, f)$  be a dynamical system. For any  $n \in \mathbb{N}$ ,  $x, y \in X$  and  $t \in \mathbb{R}$ , define

$$\Phi_{xy}^{(n)}(t) = \frac{1}{n} \#\{0 \leq i < n \mid d(f^i(x), f^i(y)) < t\},$$

where  $\#A$  denotes the cardinality of the set  $A$ . The above function represents the average number of times the distance between the orbits of  $x$  and  $y$  is less than  $t$  when considering the first  $n$  iterations.

Let

$$\begin{aligned} \Phi_{xy}(t) &= \liminf_{n \rightarrow \infty} \Phi_{xy}^{(n)}(t), \\ \Phi_{xy}^*(t) &= \limsup_{n \rightarrow \infty} \Phi_{xy}^{(n)}(t). \end{aligned}$$

Note that,  $\Phi_{xy}$  and  $\Phi_{xy}^*$  are non-decreasing functions, which are also called the lower distribution functions and the upper distribution functions, respectively. Clearly,  $\Phi_{xy}(t) = \Phi_{xy}^*(t) = 0$ , for  $t \leq 0$ , and  $\Phi_{xy}(t) = \Phi_{xy}^*(t) = 1$ , for  $t > \text{diam } X$ .

**Definition 1.2.5.** A pair of points  $x, y \in X$  is called

- (i) *distributionally chaotic of type 1* (briefly DC1) if  $\Phi_{xy}(s) = 0$ , for some  $s > 0$  and  $\Phi_{xy}^*(t) = 1$ , for all  $t > 0$ .
- (ii) *distributionally chaotic of type 2* (briefly DC2) if  $\Phi_{xy} < \Phi_{xy}^*$  and  $\Phi_{xy}^*(t) = 1$ , for all  $t > 0$ .
- (iii) *distributionally chaotic of type 3* (briefly DC3) if  $\Phi_{xy} < \Phi_{xy}^*$ .

A set containing at least two points is called a *distributionally scrambled set of type  $k$  for  $f$*  if any pair of its distinct points is distributionally chaotic of type  $k$ , where  $k \in \{1, 2, 3\}$ .

**Definition 1.2.6.** A map  $f$  is said to be *distributionally chaotic of type  $k \in \{1, 2, 3\}$*  if there exists an uncountable distributionally scrambled set of type  $k$  for  $f$ .

*Remark 1.2.7.* Observe that DC1 is the distributional chaos as defined in [48]. From the definitions it follows that

$$\text{DC1} \implies \text{DC2} \implies \text{DC3}$$

The above implications are not reversible in general.

From the definitions of DC1 and DC2, it follows that each of DC1 and DC2 implies Li-Yorke chaos. There is a continuous map on the interval which is Li-Yorke chaotic but has no DC3 pair [48]. This shows that Li-Yorke chaos need not imply any of the three versions of distributional chaos.

*Remark 1.2.8.* For continuous self-maps defined on compact metric spaces,

- (i) DC1 and DC2 are dynamical properties [60].
- (ii) DC3 is not preserved under topological conjugacy [8].

## Specification property

Specification property is another widely studied notion of chaos in the theory of dynamical systems. It highlights the ability of a chaotic system to assert patterns and structure amid its randomness. The notion of specification property was first introduced by Bowen in 1971 while studying the distribution of periodic points for Axiom A diffeomorphisms [12]. Informally, this means that it is possible to approximate any finite number of distinct orbit segments, spaced sufficiently apart in time, by a single periodic orbit. The formal definition of specification property is as follows:

**Definition 1.2.9.** A continuous self-map  $f$  defined on a compact metric space  $(X, d)$  is said to have *specification property* if for every  $\epsilon > 0$  there exists a positive integer  $M$  such that for any finite sequence of points  $x_1, x_2, \dots, x_k$  in  $X$ , any sequence of non-negative integers  $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$  with  $a_j - b_{j-1} \geq M$  ( $2 \leq j \leq k$ ) and  $p > M + (b_k - a_1)$ , there exists  $x \in X$  such that  $f^p(x) = x$  and  $d(f^i(x), f^i(x_j)) < \epsilon$ , for  $a_j \leq i \leq b_j$ ,  $1 \leq j \leq k$ .

Because of the periodicity conditions, the above notion is sometimes termed as *periodic specification property* (briefly PSP). If  $f$  fulfills conditions of the definition of specification property for the special case  $k = 2$  only, then  $f$  is said to have *weak specification property* (briefly WSP). Note that, the notion of specification property is preserved under topological conjugacy and finite product. Refer [57], for examples and some basic properties of maps having specification property.

*Remark 1.2.10.* If  $f : X \rightarrow X$  is a surjective self-map defined on a compact metric space having specification property, then

- (i)  $f$  is topologically mixing,
- (ii) the set of periodic points of  $f$  is dense in  $X$  and
- (iii)  $f$  has positive topological entropy.

For continuous maps on compact intervals, specification property is equivalent to topologically mixing [10]. The relationship between the various notions of specification property and distributional chaos has been studied extensively in the past few decades. In [48], Sklar and Smítal proved that a continuous map having two disjoint cycles and specification property has a pair admitting distributional chaos of type 3. Their main result is the following.

**Theorem 1.2.11** [48]. *Let  $(X, d)$  be a compact metric space with no isolated points and let  $f : X \rightarrow X$  be a continuous map with the specification property then  $f$  is distributionally chaotic of type 3.*

Further, Oprocha and Štefánková extended the above result and proved that a continuous map  $f$  on a compact metric space  $X$  with weak specification property and a pair of distal points is distributionally chaotic in a very strong sense [43]. They have obtained the following result.

**Theorem 1.2.12** [43]. *Let  $f$  be continuous self-map of a compact metric space  $(X, d)$  with weak specification property such that  $X$  contains a pair of distal points  $u, v$ . Then  $f$  is distributionally chaotic of type 1 and possesses a dense scrambled set  $S$  which is an at most countable sum of pairwise disjoint Cantor sets and such that, for any distinct  $x, y$  in  $S$ ,  $\Phi_{xy}^* \equiv 1$  and  $\Phi_{xy}(\epsilon) = 0$ , where  $\epsilon = \liminf_{n \rightarrow \infty} d(f^n(u), f^n(v)) > 0$ .*

## Distributional chaos in a sequence

Recently in [65], to study links between distributional chaos and Li-Yorke chaos the authors introduced a generalized version of distributional chaos, popularly known as distributional chaos in a sequence.

For a dynamical system  $(X, f)$ , let  $x, y \in X$  and let  $\{p_i\}$  be an increasing sequence of positive integers. For  $\delta > 0$ , define

$$F_{xy}(\delta, \{p_i\}) = \liminf_{n \rightarrow \infty} \frac{1}{n} \#\{0 \leq i < n \mid d(f^{p_i}(x), f^{p_i}(y)) < \delta\},$$

$$F_{xy}^*(\delta, \{p_i\}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \#\{0 \leq i < n \mid d(f^{p_i}(x), f^{p_i}(y)) < \delta\},$$

where  $\#A$  denotes the cardinality of the set  $A$ .

**Definition 1.2.13.** A map  $f$  is said to be *distributionally chaotic in a sequence* if there exists a sequence  $\{p_i\}$  of positive integers and an uncountable set  $D \subset X$  such that for any  $x, y \in D$  with  $x \neq y$  we have

(i)  $F_{xy}(\epsilon, \{p_i\}) = 0$ , for some  $\epsilon > 0$ , and

(ii)  $F_{xy}^*(\delta, \{p_i\}) = 1$ , for all  $\delta > 0$ .

*Remark 1.2.14.* If sequence  $\{p_i\} = \{i\}$ , the natural sequence of positive integers, the above definition coincides with the notion of distributional chaos of type 1.

For a continuous map of an interval, Li-Yorke chaos is equivalent to distributional chaos in a sequence [65]. The relation between Li-Yorke chaos and distributional chaos in a sequence for a continuous map of a compact metric space is studied in [35]. Moreover, Devaney chaos and weak mixing both imply distributional chaos in a sequence for continuous self-maps defined on metric spaces [37, 38].

It is important to note that most results dealing with chaotic properties of dynamical systems are obtained under the assumption that the phase space is a compact metric space. However, the results that are proved for the maps defined on compact metric spaces need not hold for the case of non-compact metric spaces. For instance, it is known that expansivity is a dynamical property for homeomorphisms on compact spaces, but this is not true for noncompact spaces (see [18, Example 6]).

In recent years, many authors have given their attention to the study of chaos for general topological spaces which are not necessarily compact, and metrizable. Before discussing the details of studies in this direction, we introduce the definitions and terminologies related to uniform spaces.

For general topological spaces size or distance dependent notions, such as uniform continuity, completeness, and boundedness, cannot be defined unless we have somewhat more structure than the topology provides. Weil introduced uniform

spaces as a generalization of metric spaces and topological groups to study the topological significance of these non-topological properties, such as completeness and uniform continuity [67]. Uniform spaces lie between pseudometric spaces and topological spaces in the sense that any pseudometric space generates a uniform space, and any uniform space generates a topological space.

Let  $X$  be a nonempty set. The set  $\Delta_X = \{(x, x) \mid x \in X\}$  is called the diagonal of the Cartesian product  $X \times X$ . A set  $U \subset X \times X$  is said to be symmetric if  $U = U^T$ , where  $U^T = \{(y, x) \mid (x, y) \in U\}$ . If  $U$  and  $V$  are subsets of  $X \times X$ , then by the composite  $U \circ V$  of  $U$  and  $V$  we mean the set of all pairs  $(x, y)$  such that for some point  $z \in X$  we have that  $(x, z) \in U$  and  $(z, y) \in V$ .

**Definition 1.2.15.** A *uniform structure* (or *uniformity*) on a set  $X$  is a nonempty collection  $\mathcal{U}$  of subsets of  $X \times X$  satisfying the following conditions:

- (i) if  $U \in \mathcal{U}$ , then  $\Delta_X \subset U$ ,
- (ii) if  $U \in \mathcal{U}$ , then  $U^T \in \mathcal{U}$ ,
- (iii) if  $U \in \mathcal{U}$ , then there exists  $V \in \mathcal{U}$  such that  $V \circ V \subset U$ ,
- (iv) if  $U \in \mathcal{U}$  and  $V \in \mathcal{U}$ , then  $U \cap V \in \mathcal{U}$ , and
- (v) if  $U \in \mathcal{U}$  and  $U \subset V \subset X \times X$ , then  $V \in \mathcal{U}$ .

The elements of  $\mathcal{U}$  are called the entourages of the uniform structure and the pair  $(X, \mathcal{U})$  is called a *uniform space*.

If  $U$  is an entourage, then  $U \cap U^T$  is a symmetric entourage, hence we can work with symmetric entourages without loss of generality. For any subset  $A$  of  $X$  and  $U \in \mathcal{U}$ ,  $U[A]$  denotes the set of all points  $y$  in  $X$  such that  $(x, y) \in U$ , for some point  $x$  in  $A$ . For a point  $x \in X$ , the set  $U[x] = \{y \in X \mid (x, y) \in U\}$  is called the cross-section of  $U$  at  $x$ .

*Remark 1.2.16.*

- (i) Every metric space is a uniform space with the uniform structure  $\mathcal{U}$  given by the family  $\{d^{-1}[0, \epsilon) \mid \epsilon > 0\}$ .
- (ii) The uniform structure  $\mathcal{U}$  on  $X$  induces a topology (also called uniform topology) on  $X$  defined as follows: a subset  $G$  of  $X$  is open in  $X$  if for each point  $x$  in  $G$  there is an element  $U$  in  $\mathcal{U}$  such that  $U[x] \subset G$ .
- (iii) The fact that the pair  $(x, y)$  is close to the diagonal  $\Delta_X$  describes the idea of closeness of points  $x$  and  $y$  in  $X$ . This approach can be used to extend various notions defined for metric spaces to uniform spaces.

**Definition 1.2.17** [31]. A uniform space  $(X, \mathcal{U})$  is said to be totally bounded if for each  $U \in \mathcal{U}$  there is a finite subset  $F$  of  $X$  such that  $X = U[F]$ , that is  $X = \bigcup_{x \in F} U[x]$ .

**Definition 1.2.18** [25]. A uniform space  $(X, \mathcal{U})$  is said to be *uniformly locally compact* if there exists  $U \in \mathcal{U}$  such that  $U[x]$  is compact for each  $x \in X$ .

A uniform space  $(X, \mathcal{U})$  is said to be *Hausdorff* if  $\bigcap \{U \mid U \in \mathcal{U}\} = \Delta_X$ . For uniform spaces  $(X, \mathcal{U})$  and  $(Y, \mathcal{V})$ , a map  $f : X \rightarrow Y$  is said to be *uniformly continuous* if for every  $E \in \mathcal{V}$ ,  $(f \times f)^{-1}(E) \in \mathcal{U}$ . For more details on uniform spaces, one can refer [31, Chapter 6].

Smale's spectral decomposition theorem is considered to be one of the remarkable results in dynamical systems. It was obtained for Axiom A diffeomorphisms defined on a compact  $C^\infty$  Riemannian manifolds [13]. Spectral decomposition theorem is been extended to homeomorphisms on compact metric spaces [5]. In [18], authors have extended Smale's spectral decomposition theorem to the spaces which are not necessarily metrizable and not necessarily compact. In fact they have obtain the following result:

**Theorem 1.2.19** [18]. *Let  $X$  be a first countable, locally compact, paracompact, Hausdorff space and  $f : X \rightarrow X$  an expansive homeomorphism with the shadowing*

property. Then the non-wandering set  $\Omega(f)$  can be written as a union of disjoint closed invariant sets (called as basic sets for  $f$ ) on which  $f$  is topologically transitive. If  $X$  is compact then this decomposition is finite.

Bowen proved that the basic sets obtained in the Smale's spectral decomposition theorem are further decomposed into sub-basic sets. Bowen's decomposition theorem is also extended to the spaces which are not necessarily metrizable and not necessarily compact.

**Theorem 1.2.20** [17]. *Let  $X$  be a first countable, locally compact, paracompact, Hausdorff space and  $f : X \rightarrow X$  be an expansive homeomorphism with the shadowing property. Then there exists a subset  $S$  of a basic set  $R$  and  $k > 0$  such that  $f^k(S) = S$ ,  $S \cap f^j(S) = \emptyset$  ( $0 < j < k$ ),  $f^k|_S$  is topologically mixing and  $R = \cup_{j=0}^{k-1} f^j(S)$ .*

In [54], Shah et al. introduced and studied the notion of topological specification property for homeomorphism on uniform spaces. It is proved that for self-homeomorphism on uniform space, topological specification property implies Devaney chaos. They also proved the following result.

**Theorem 1.2.21** [54]. *Let  $f$  be a self-homeomorphism on a totally bounded uniform space  $X$ . If  $f$  is topologically mixing, topologically expansive and topologically shadowing, then  $f$  has topological specification property.*

In [53], the authors studied the relationship between the notions of topological specification property and topological entropy for uniformly continuous self-map defined on uniform space.

**Theorem 1.2.22** [53]. *Let  $(X, \mathcal{U})$  be a uniform space and let  $f : X \rightarrow X$  be uniformly continuous self-map having topological specification property then the topological entropy of  $f$  is positive.*

Recently in [55], Shah et al. introduced the notion of distributional chaos for uniformly continuous self-maps on uniform spaces, wherein it is termed as topological distributional chaos. They also studied the relation between topological notions

of weak specification property and distributional chaos for a uniformly continuous self-map on uniformly locally compact Hausdorff uniform space and proved the following result.

**Theorem 1.2.23** [55]. *Let  $(X, \mathcal{U})$  consisting of closed entourages be a uniformly locally compact Hausdorff space without isolated points having a distal pair and let  $f$  be a uniformly continuous map of  $X$  onto itself having topological weak specification property then  $f$  admits a topologically distributionally scrambled set of type 3.*

Let  $X$  be a first countable, locally compact, paracompact, Hausdorff space and  $f : X \rightarrow X$  be an expansive homeomorphism with the shadowing property. Suppose  $(\Omega(f), f)$  is topologically transitive and has a fixed point. Then  $(\Omega(f), f)$  is mixing [55]. If in addition  $X$  is totally bounded and  $f|_{\Omega(f)}$  has shadowing property then  $f|_{\Omega(f)}$  has topological specification property [54]. Hence  $f|_{\Omega(f)}$  admits a topologically distributionally scrambled set of type 3.

The notion of topological entropy for maps on uniform spaces was introduced and studied by Hood in [26]. The notion of sensitivity and Devaney chaos are studied for continuous group actions on uniform spaces in [14]. It is proved that for a continuous group action on an infinite Hausdorff uniform space if the action is topologically transitive and the space admits a dense set of periodic points, then the action is sensitive. This extends the result proved in [9], for the case of a continuous map defined on a metric space.

In [2], Ahmadi defined and studied the topological concepts of ergodic shadowing, chain transitivity and topological ergodicity for dynamical systems on uniform spaces. In [15], authors have defined various forms of shadowing and specification for uniformly continuous self-maps on uniform spaces and studied their equivalences. The concepts of topological shadowing and topological chain transitivity are further explored in [3]. The topological notion of average shadowing property for continuous self-maps on compact uniform spaces is defined and studied in [45]. In [6], Arai has introduced and studied the notion of Li-Yorke chaos for group actions on uniform spaces.

**Theorem 1.2.24** [6]. *For the continuous actions of an abelian group on a second countable Baire Hausdorff uniform space without isolated points, Devaney chaos implies Li-Yorke chaos.*

The traditional study of dynamical systems focuses on the iterations of a single transformation, modelling the time evolution of a physical system. This involves the study  $\mathbb{Z}$ -actions or  $\mathbb{R}$ -actions induced by a transformation. However, many systems exhibit additional symmetries, requiring the collective study of multiple commuting transformations. The study of continuous  $\mathbb{Z}^d$ -actions,  $d > 1$ , is an important step in this direction and is much more than a routine generalization of a  $\mathbb{Z}$ -actions. Recently, many authors have started the study of higher dimensional actions, i.e.,  $\mathbb{Z}^d$ -actions or  $\mathbb{R}^d$ -actions, for  $d > 1$ . Historically, the interest in  $\mathbb{Z}^d$ -actions arose from the study of classical lattice gas models. The simplest case where  $d = 1$  led to the development of the thermodynamic approach to the ergodic theory of Anosov and Axiom A systems.

A system comprising of a  $\mathbb{Z}^d$ -action defined on a phase space  $X$  is also known as a multidimensional discrete time dynamical system. Various authors have studied the dynamics for such systems. The notion of shadowing property for  $\mathbb{Z}^d$ -actions is studied by Pilyugin and Tikhomirov [44]. Oprocha proved the Spectral Decomposition Theorem for multidimensional discrete time dynamical system, which is a generalization of similar results obtained by Smale and Bowen [41]. In [33], Kim and Lee have proved general version of the Spectral Decomposition Theorem for the set of  $k$ -type nonwandering points of  $\mathbb{Z}^2$ -actions. In [30], the authors considered a dynamical systems  $(X, f)$  and have studied the dynamics of the induced  $\mathbb{Z}^d$ -action on  $X$ .

### 1.3 Overview of thesis

Since the inception of the chaos theory in dynamical systems, numerous alternative definitions of chaos have been proposed and studied in detail. Over the past

few decades, there has been extensive research into the relationships among these alternative definitions and the conditions for their occurrence. However, it is worth noting that the scope of these studies was primarily limited to compact metric spaces. While these spaces are important, it is advantageous to develop a theory that applies to more general topological spaces, in particular to noncompact nonmetrizable spaces.

The primary motivation behind our study is to extend the previous research and address the need for a theory that can be applied to general topological spaces, which need not be compact and metrizable.

We now provide an outline of the work done in the subsequent chapters of the thesis. In [55], Shah et al. proved that for a uniformly continuous self-map on a uniformly locally compact Hausdorff uniform space with a distal pair, topological weak specification property implies topological distributional chaos of type 3. In Chapter 2, we extend the above result and prove that such maps are topologically distributionally chaotic of type 1. This also extends the result due to Oprocha and Štefánková for continuous self-maps on compact metric spaces [43]. Further, we study the existence of invariant topologically distributionally scrambled sets for maps on uniform spaces.

In Chapter 3, we introduce the topological notion of distributional chaos in a sequence for uniformly continuous self-maps defined on Hausdorff uniform spaces, and term it topological distributional chaos in a sequence. We also discuss an alternate definition of Li-Yorke scrambled pair and prove that Li-Yorke chaos is equivalent to topological distributional chaos in a sequence for uniformly continuous self-maps defined on second countable Baire Hausdorff uniform space without isolated points.

In Chapter 4, we study various forms of specification for uniformly continuous surjective self-maps defined on uniform spaces. We introduce and study the notion of a topological periodic specification point and topological specification point for uniformly continuous surjective self-maps on uniform spaces. We also study the connections between pointwise topological specification property and other notions

of chaos. Moreover, the limiting behavior of a topological specification point under orbital convergence of maps is also explored. We also introduce and study some weaker forms of specification property for uniformly continuous surjective self-maps on uniform spaces, namely topological quasi-weak specification property, topological semi-weak specification property, and topological periodic quasi-weak specification property.

In Chapter 5, we study the notion of distributional chaos for a continuous  $\mathbb{Z}^d$ -action defined on a compact metric space. Also, the relationship between the notions of Distributional chaos and specification property is explored for continuous  $\mathbb{Z}^d$ -actions.

In Chapter 6, we introduce and study the notion of  $k$ -type distributional chaos in a sequence for a continuous  $\mathbb{Z}^d$ -action. We also prove that the notion of  $k$ -type Li-Yorke chaos is equivalent to  $k$ -type distributional chaos in a sequence for a continuous  $\mathbb{Z}^d$ -action.