Synopsis

of the thesis entitled

Chaotic Properties of Dynamical Systems

to be submitted to

The Maharaja Sayajirao University of Baroda

for the award of the degree of

Doctor of Philosophy

in

Mathematics

by

Yadav Naveenkumar S.

under the supervision of

Dr. Sejal K. Shah

Department of Mathematics, Faculty of Science
The Maharaja Sayajirao University of Baroda
Vadodara 390002, Gujarat

February 2024

Synopsis

Title: Chaotic Properties of Dynamical Systems

Chaos is one of the central ideas in the study of dynamical systems. The first mathematical definition of chaos in connection with a map was introduced by Li and Yorke [1]. Since then various authors have introduced and studied the notions of chaos based on the size and properties of scrambled sets, for example, dense chaos, generic chaos, ϵ-Li-Yorke chaos and others. Schweizer and Smítal considered the probabilistic measure of the distance between two trajectories and introduced a stronger form of chaos popularly termed as distributional chaos [2]. The notion of distributional chaos is further classified into three non-equivalent variants, namely $DC1$, $DC2$ and $DC3$ [3].

Specification is another widely studied notion of chaos introduced by Bowen [4]. Many researchers have studied different versions of specification, their properties and interrelations with other known notions of chaos. Sklar and Smítal have studied the relation between specification property and distributional chaos for continuous self-map on a compact metric space [5]. They proved that for a continuous self-map on a compact metric space with no isolated points, specification property implies distributional chaos of type 3. Later, Oprocha and Štefánková obtained that a continuous self-map on a compact metric space with distal pair and weak specification property is distributionally chaotic of type 1, moreover it possesses a dense scrambled set [6].

The above study is mostly restricted to continuous self-maps on compact metric spaces. In recent years, many researchers have studied chaos for systems defined on general topological spaces (see [7, 8, 9, 10, 11, 12]). With aim to have a theory that is applicable to more general topological space, we extend and study the above connections and various other notions of chaos for dynamical systems defined on uniform spaces, which are not necessarily compact metrizable.

The present thesis entitled "Chaotic properties of dynamical systems" incorporates the results of research work done in the aforesaid direction by me under the guidance of Dr. Sejal Shah. Major part of this thesis is devoted to the study of chaos for dynamical systems defined on uniform spaces. In particular, we have studied the notions of topological distributional chaos, topological distributional chaos in a sequence, topological specification property and the pointwise notion of topological specification. Moreover, the interrelation of distributional chaos and specification property is also studied in case of multidimensional actions.

There are six chapters in the thesis. The organization of the thesis is as follows:

Chapter 1, gives an overview of the work done in the thesis. In this chapter, we give the historical context of the problems studied, and recall the preliminaries required for the study of further chapters.

In Chapter 2, we relate the notions of topological distributional chaos and topological specification property defined for uniformly continuous self-maps on uniform spaces. Essentially, we prove the following result:

Theorem 1. *Let $(X,\mathcal{U})$ be a uniformly locally compact Hausdorff uniform space consisting of closed entourages without isolated points having a distal pair and let f be a uniformly continuous map of X onto itself. If f has topological weak specification property, then f is topologically distributionally chaotic of type 1.*

This extends the result due to Oprocha and Štefánková for continuous self-maps on compact metric spaces [6]. As a consequence, we obtain that a uniformly continuous surjective self-map on a uniformly locally compact, totally bounded, Hausdorff uniform space with topological shadowing, topological mixing, and a distal pair is topologically distributionally chaotic of type 1. We also obtain that a uniformly continuous surjective self-map on a uniformly locally compact totally bounded Hausdorff uniform space with ergodic shadowing, and a distal pair is topologically distributionally chaotic of type 1.

We further study the existence of invariant topological distributionally scrambled set for maps on uniform spaces. It is obtained that a uniformly continuous surjective self-map on a uniformly locally compact Hausdorff uniform space with topological weak specification property, a fixed point and countably many periodic points with mutually different periods admits an Mycielski invariant topological distributionally scrambled set of type 1. Using this, we prove the following:

Theorem 2. *Let* $(X,\mathcal{U})$ *be a uniformly locally compact second countable Hausdorff uniform space consisting of closed entourages without isolated points. Let* $f : X \to X$ *be a uniformly continuous surjective map having topological weak specification property, a fixed point and countably many periodic points with mutually different periods. Then* X *has a dense Mycielski invariant topological distributionally scrambled set of type 1.*

As a consequence, we obtain that a self-homeomorphism on a uniformly locally compact second countable Hausdorff uniform space with topological specification property and a fixed point has a dense Mycielski invariant topological distributionally scrambled set of type 1.

It is known that Li-Yorke chaos and Devaney chaos need not imply any version of distributional chaos [2, 13]. In 2007, Wang et al. introduced a generalized version of distributional chaos, popularly known as distributional chaos in a sequence [14]. For continuous self-maps defined on the intervals, Li-Yorke chaos and distributional chaos in a sequence are equivalent.

In Chapter 3, we consider the topological notions of Li-Yorke chaos and distributional chaos in a sequence defined for uniformly continuous self-maps defined on uniform Hausdorff spaces. The notion of Li-Yorke chaos for a uniformly continuous self-maps defined on a uniform space was introduced by Arai [15]. We describe the notion of Li-Yorke chaos for a uniformly continuous self-maps defined on a uniform space in an alternate manner. It is proved that for uniformly continuous self-map defined on a second countable Baire uniform Hausdorff space without isolated points, weakly mixing implies Li-Yorke chaos.

Further, we introduce and study the notion of topological distributional chaos in a sequence for uniformly continuous self-maps defined on uniform Hausdorff spaces. We prove the following result:

Theorem 3. *Let* $(X,\mathcal{U})$ *be a second countable Baire uniform Hausdorff space without isolated points and let* f *be a uniformly continuous self-map defined on* X*. Then* f *is chaotic in the sense of Li-Yorke if and only if* f *is topologically distributionally chaotic in a sequence.*

In [15], the author proved that for a continuous group action on a second countable Baire uniform Hausdorff space without isolated points Devaney chaos implies Li-Yorks chaos. As a consequence, we get that Devaney chaos implies topological distributional chaos in a sequence. Thus, for a uniform self homeomorphism f defined on a second countable Baire uniform Hausdorff space X without isolated points, we have the following implications:

$$\begin{array}{ccl} \text{Weakly mixing} & \Rightarrow & \text{Li-Yorke chaos} \\ & & \Updownarrow \\ \text{Devaney chaos} & \Rightarrow & \text{Topological distributional chaos in a sequence} \end{array}$$

Chapter 4 is devoted to the study of different forms of specification for uniformly continuous surjective self-maps on uniform spaces. Most definitions of chaos present in the literature are based on the global behavior of dynamical systems. In recent years, many authors have shifted their attention to study the impact of local behavior of a system on the global behavior of a system (see [16, 17, 18, 19]). Following the intent, we introduce and study the notion of a topological periodic specification point and topological specification point for uniformly continuous surjective self-maps on uniform spaces. It is proved that the notion of topological periodic specification point is preserved under finite product and conjugacy. For homeomorphisms on uniform space, it is proved that the existence of a topological periodic specification point alone guarantees that the map has topological periodic specification property. This is proved in the following theorem:

Theorem 4. *If $f : X \to X$ is a homeomorphism on a uniform space $(X,\mathcal{U})$ and f has a topological periodic specification point, then f has topological periodic specification property.*

We also study the connection between pointwise topological specification property and other notions of chaos. It is proved that for uniformly continuous surjective self-maps on uniform Hausdorff spaces, every topological periodic specification point is a Devaney chaotic point. As a consequence, we obtain that for uniformly continuous surjective self-maps on uniform Hausdorff space pointwise topological periodic specification property implies Devaney chaos. We also prove that for a mixing uniformly continuous surjective self-map on a totally bounded uniform space, a topological expansive and topological shadowable point is a topological periodic specification point. Further, it is proved that a uniformly continuous surjective self-map on uniform space with two distinct topological specification points has positive uniform entropy. Moreover, the limiting behavior of a topological specification point under orbital convergence of maps is also explored. We also introduce and study some weaker forms of specification for uniformly continuous surjective self-maps on uniform spaces, namely topological quasi-weak specification property, topological semi-weak specification property and topological periodic quasi-weak specification property.

In Chapter 5, we study the notions of distributional chaos and specification property for a continuous $\mathbb{Z}^d$-action defined on a compact metric space. We prove that a continuous $\mathbb{Z}^d$-action on a compact metric space with weak specification property and with a pair of distal points is distributionally chaotic of type 1. Moreover, we prove the following:

Theorem 5. *Let (X, ρ) be a compact metric space without isolated points and T be a continuous $\mathbb{Z}^d$-action on X, $d \in \mathbb{N}$, $d > 1$, having a distal pair. If T has weak specification property, then X has a dense $DC1$ scrambled set.*

In Chapter 6, we introduce and study the notion of k-type distributional chaos in a sequence for a continuous $\mathbb{Z}^d$-action on a compact metric space. It is proved that for a continuous $\mathbb{Z}^d$-action on a compact metric space k-type weakly mixing implies k-type Li-Yorke chaos. We further study the relation between the notion of k-type Li-Yorke chaos and k-type distributional chaos in a sequence for a continuous $\mathbb{Z}^d$-action on a compact metric space.

The thesis is based on the following published/communicated articles:

1. N. Yadav and S. Shah, "Topological weak specification and distributional chaos on noncompact spaces", International Journal of Bifurcation and Chaos, 32(4), 2250048 (2022). (Indexed in **SCOPUS**, **SCIE**, Impact Factor (2022): **2.2**)

2. N. Yadav and S. Shah, "Li-Yorke chaos and topological distributional chaos in a sequence", Turkish Journal of Mathematics, 46(4), 1360–1368 (2022). (Indexed in **SCOPUS**, **SCIE**, Impact Factor (2022): **1.0**)

3. N. Yadav, "Weaker forms of specification for maps on uniform spaces", Dynamical Systems, 39(1), 150–165 (2024). (Indexed in **SCOPUS**, **SCIE**, Impact Factor (2022): **0.5**)

4. N. Yadav and S. Shah, "A note on distributionally scrambled sets", Communicated.

5. N. Yadav, "Distributional chaos via specification for $\mathbb{Z}^d$-actions", Communicated.

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