SUMMARY

of the thesis entitled

Chaotic Properties of Dynamical Systems

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Summary

Title: Chaotic Properties of Dynamical Systems

The study of the chaotic behavior of systems is one of the central ideas in the qualitative theory of dynamical systems. The concept of chaos can be traced back to Poincaré, who first realized the fact that many simple nonlinear deterministic systems can behave in an unpredictable manner. Note that, defining chaos in a way that it fully captures the complex behavior of a dynamical system is difficult. Various researchers have introduced and studied different notions of chaos. It is interesting to explore how these different notions of chaos are related to one another.

The first mathematical definition of chaos was given by Li and Yorke in 1975 [9]. Various extensions of Li-Yorke chaos are studied based on the size and properties of the scrambled set. In 1994, Schweizer and Smítal extended Li and Yorke's approach by considering the probabilistic density of proximity of distinct orbits, and introduced the notion of distributional chaos [17]. The notion of distributional chaos is further generalized into three nonequivalent variants namely $DC1$, $DC2$ and $DC3$ [4].

Specification is another strong notion of chaos introduced by Bowen [5]. Many researchers have studied different versions of the specification property, their properties, and their interrelations with other notions of chaos. The relationship between the notions of specification property and distributional chaos is studied in [21, 14]. In [25], the authors introduced and studied the weaker forms of specification property. It is worth noting that, the study mentioned above is primarily restricted

to discrete dynamical systems defined on compact metric spaces. In recent years, many researchers have focused on the study of chaos for maps defined on general topological spaces (see [3, 22, 19, 18, 1, 20, 24]).

It is desirous to have a theory that can be applied to general topological spaces, which are not necessarily compact and metrizable. With this aim, we extend and study the connections mentioned above, as well as other notions of chaos for dynamical systems defined on uniform spaces. We also examine the notions of distributional chaos and specification property for the case of continuous $\mathbb{Z}^d$-actions.

The thesis entitled "Chaotic Properties of Dynamical Systems" includes the findings of the research work carried out by the author in the aforementioned direction under the guidance of Dr. Sejal Shah. The thesis consists of six chapters, which are organized as follows:

Chapter 1 aims to give an introduction to the subject matter of the thesis. It recalls the preliminaries required for the study of further chapters and gives the historical context of the problems studied. Moreover, it provides an overview of the work done in the thesis.

In Chapter 2, we study the relationship between the topological notions of distributional chaos and specification property defined for uniformly continuous self-maps on uniform spaces. Essentially, we prove the following result:

Theorem 1. *Let $(X, \mathcal{U})$ be a uniformly locally compact Hausdorff uniform space consisting of closed entourages without isolated points having a distal pair and let f be a uniformly continuous map of X onto itself. If f has topological weak specification property, then f is topologically distributionally chaotic of type 1.*

The above results extends the result proved by Oprocha and Štefánková for continuous self-maps on compact metric spaces [14]. As a consequence, we obtain that a uniformly continuous surjective self-map on a uniformly locally compact, totally bounded, Hausdorff uniform space with topological shadowing, topological mixing, and a distal pair is topologically distributionally chaotic of type 1. We

also obtain that a uniformly continuous surjective self-map on a uniformly locally compact totally bounded Hausdorff uniform space with ergodic shadowing and a distal pair is topologically distributionally chaotic of type 1.

Further, we study the invariance of topologically distributionally scrambled set for maps on uniform spaces. In [13], Oprocha studied the invariance of distributionally scrambled sets for interval maps. He stated the following open question: Does every map with the specification property and a fixed point contain an invariant distributionally scrambled set? Doleželová answered this question positively with additional assumptions and proved the existence of dense invariant distributionally scrambled sets for continuous self-maps on compact metric spaces [7]. We strengthen the above results by proving that a uniformly continuous surjective self-map on a uniformly locally compact Hausdorff uniform space with topological weak specification property, a fixed point and countably many periodic points with mutually different periods has an invariant topological distributionally scrambled set of type 1. Furthermore, if the underlying space is second countable, we have obtained the following result.

Theorem 2. *Let $(X, \mathcal{U})$ be a uniformly locally compact second countable Hausdorff uniform space without isolated points. If $f : X \to X$ is a uniformly continuous surjective map having topological weak specification property, a fixed point and countably many periodic points with mutually different periods, then X has a dense Mycielski invariant topologically distributionally scrambled set of type 1.*

In 2007, Wang et al. introduced a generalized version of distributional chaos by considering the distribution function of distance between trajectories in terms of a given sequence of time, popularly known as distributional chaos in a sequence [23]. It is proved that both Devaney chaos and weakly mixing imply distributional chaos in sequence for self-maps defined on metric spaces [10, 11]. Moreover, for continuous self-maps defined on the intervals, it is proved that Li-Yorke chaos and distributional chaos in a sequence are equivalent.

In Chapter 3, we introduce and study the notion of topological distributional chaos in a sequence defined for a uniformly continuous self-map on Hausdorff uniform space. Arai introduced the notion of Li-Yorke chaos for group actions on uniform spaces [2]. We reformulate the notion of Li-Yorke chaos for a uniformly continuous self-maps defined on a uniform space and study the relation between notions of weakly mixing and Li-Yorke chaos for a uniformly continuous self-map defined on a second countable Baire Hausdorff uniform space without isolated points.

Theorem 3. *Let* $(X,\mathcal{U})$ *be a second countable Baire Hausdorff uniform space without isolated points and let* f *be a uniformly continuous self-map defined on* X. *If* f *is weakly mixing, then* f *is Li-Yorke chaotic.*

Further, we study the relationship between the notions of Li-Yorke chaos and topological distributional chaos in a sequence for a uniformly continuous self-map defined on a Hausdorff uniform space. We have obtained the following result:

Theorem 4. *Let* $(X,\mathcal{U})$ *be a second countable Baire uniform Hausdorff space without isolated points and let* f *be a uniformly continuous self-map defined on* X. *Then* f *is chaotic in the sense of Li-Yorke if and only if* f *is topologically distributionally chaotic in a sequence.*

Arai proved that for a continuous group action on a second countable Baire uniform Hausdorff space without isolated points Devaney chaos implies Li-Yorks chaos [2]. As a consequence, we get that Devaney chaos implies topological distributional chaos in a sequence. Thus, for a uniform self homeomorphism f defined on a second countable Baire uniform Hausdorff space X without isolated points, we have the following implications:

$$\text{Weakly mixing} \implies \text{Li-Yorke chaos}$$
$$\Updownarrow$$
$$\text{Devaney chaos} \implies \text{Topological distributional chaos in a sequence}$$

Chapter 4 deals with the study of different forms of specification property for uniformly continuous surjective self-maps on uniform spaces. In recent years, many authors have focused on studying how the local behavior of a system affects its global behavior (see [15, 26, 12, 6]). In line with this, we introduce and examine the concept of a topological periodic specification point and topological specification point for a uniformly continuous surjective self-maps on uniform spaces. We obtain that the notion of a topological periodic specification point is preserved under finite product and conjugacy. For homeomorphisms on a uniform space, the existence of a topological periodic specification point guarantees that the map has topological periodic specification property. This is proved in the following theorem:

Theorem 5. *If $f : X \rightarrow X$ is a homeomorphism on a uniform space $(X, \mathcal{U})$ and f has a topological periodic specification point, then f has topological periodic specification property.*

We also study the connection between the notion of pointwise topological specification property and other notions of chaos. We prove that for uniformly continuous surjective self-maps on a uniform Hausdorff space, every topological periodic specification point is a Devaney chaotic point. As a consequence, we obtain the following result:

Theorem 6. *Let f be a uniformly continuous surjective self-map on a uniform Hausdorff space $(X, \mathcal{U})$. If f has pointwise topological periodic specification property, then f is Devaney chaotic.*

Also, we prove that for a mixing uniformly continuous surjective self-map on a totally bounded uniform space, a topological expansive and a topological shadowable point is a topological periodic specification point. Further, we examine the relation between topological specification point and uniform entropy. We have proved the following result:

Theorem 7. *Let $f : X \rightarrow X$ be a uniformly continuous surjective self-map on a*

uniform space $(X, \mathcal{U})$. If f has two distinct topological specification points, then f has positive uniform entropy.

Moreover, the limiting behavior of a topological specification point under orbital convergence of maps is also explored. We also introduce and study some weaker forms of specification for uniformly continuous surjective self-maps on uniform spaces, namely topological quasi-weak specification property (TQWSP), topological semi-weak specification property (TSWSP) and topological periodic quasi-weak specification property (TPQWSP). We have obtained the following result:

Theorem 8. *Let f be a uniformly continuous surjective self-map on a totally bounded uniform space $(X, \mathcal{U})$. Then the following statements are equivalent:*

1. *f is mixing;*

2. *f has TQWSP;*

3. *f has TSWSP.*

Ruelle introduced and studied the notion of specification property for continuous $\mathbb{Z}^d$-actions defined on a compact metric space [16]. In [8], Hunter introduced the notion of distributional chaos for a continuous $\mathbb{Z}^d$-action on a compact metric space and explored the relation between weak specification property and distributional chaos for the case of multidimensional subshifts. Chapter 5 is devoted to the study of relationship between the notions of distributional chaos and specification property for a continuous $\mathbb{Z}^d$-action defined on a compact metric space. We extend the results proved by Oprocha and Štefánková in [14], to the case of continuous $\mathbb{Z}^d$-actions on compact metric spaces. Essentially, we prove the following results:

Theorem 9. *Let (X, ρ) be a compact metric space without isolated points and T be a continuous $\mathbb{Z}^d$-action on X, $d \in \mathbb{N}$, $d > 1$, having a distal pair. If T has weak specification property, then T is distributionally chaotic of type 1.*

Theorem 10. *Let (X, ρ) be a compact metric space without isolated points and T be a continuous $\mathbb{Z}^d$-action on X, $d \in \mathbb{N}$, $d > 1$, having a distal pair. If T has weak specification property, then X has a dense $DC1$ scrambled set.*

In Chapter 6, we introduce and study the notion of k-type distributional chaos in a sequence for a continuous $\mathbb{Z}^d$-action on a compact metric space. We further reformulate the notion of k-type Li-Yorke chaos in terms of k-type sequences in $\mathbb{Z}^d$. For a continuous $\mathbb{Z}^d$-action on a compact metric space, the connection between the notions of k-type weakly mixing and k-type Li-Yorke chaos in examined and the following result is obtained.

Theorem 11. *Let (X, ρ) be a compact metric space and let T be a continuous $\mathbb{Z}^d$-action on X. If T is k-type weakly mixing then T is k-type Li-Yorke chaotic.*

Moreover, we study the relation between the notions of k-type Li-Yorke chaos and k-type distributional chaos in a sequence for a continuous $\mathbb{Z}^d$-action on a compact metric space. We have obtained the following results:

Theorem 12. *Let (X, ρ) be a compact metric space and let T be a continuous $\mathbb{Z}^d$-action on X. Then T is k-type Li-Yoke chaotic if and only if T is k-type distributionally chaotic in a sequence.*

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2. N. Yadav and S. Shah, "Li-Yorke chaos and topological distributional chaos in a sequence", Turkish Journal of Mathematics, 46(4), 1360–1368 (2022). (**SCOPUS** Indexed, **SCIE Journal**, 2022 Impact Factor: **1.0**)

3. N. Yadav, "Weaker forms of specification for maps on uniform spaces", Dynamical Systems, 39(1), 150–165 (2024). (**SCOPUS** Indexed, **SCIE Journal**, 2022 Impact Factor: **0.5**)

4. N. Yadav and S. Shah, "A note on distributionally scrambled sets", Communicated.

5. N. Yadav, "Distributional chaos via specification for $\mathbb{Z}^d$-actions", Communicated.

6. N. Yadav, "A note on k-type distributional chaos in a sequence for $\mathbb{Z}^d$-actions", Communicated.