Chapter 6

On k-type distributional chaos in a sequence for $\mathbb{Z}^d$-actions

In case of an abstract topological group S acting on a space, to generalize the idea of ω-limit sets, Gottschalk and Hedlund considered a subgroup P of S, and introduced P-limit set [23]. Following the idea, Oprocha considered the d-dimensional quarter planes of $\mathbb{Z}^d$ and introduced the concept of k-type limit sets, where k denotes the number of d-dimensional quarter plane considered [41]. Oprocha also proved the multidimensional version of Spectral Decomposition Theorem. Working in this direction, Kim and Lee have proved the Spectral Decomposition Theorem for the set of k-type nonwandering points of $\mathbb{Z}^2$-actions [33]. In [50, 51], the authors introduced and studied the notion of k-type Devaney chaos for $\mathbb{Z}^d$-actions. The notion of k-type collective sensitivity for $\mathbb{Z}^d$-actions is defined and studied in [52].

Continuing along this direction, in this chapter we generalize the notion of distributional chaos in terms of a k-type sequence of points in $\mathbb{Z}^d$, for continuous $\mathbb{Z}^d$-actions. In Section 6.1, we introduce and study the notion of k-type distributional chaos in a sequence for a continuous $\mathbb{Z}^d$-action. In Section 6.2, we give an equivalent definition of k-type Li-Yorke pair and study the connections between the existing notions of k-type weakly mixing and k-type Li-Yorke chaos. In section 6.3, we study the relation between the notions of k-type Li-Yorke chaos and k-type distributional chaos in a sequence.

Throughout this chapter, by a dynamical system we mean an ordered pair (X, T), where X is a compact metric space with metric ρ and T is a continuous $\mathbb{Z}^d$-action on X, $d > 1$.

6.1 k-type distributional chaos in a sequence

In this section, we define and study the notion of k-type distributional chaos in a sequence. In order to work with sequences in $\mathbb{Z}^d$, we consider the d-dimensional quarter planes of $\mathbb{Z}^d$. For $d \in \mathbb{N}$, let $k \in \{1, 2, \ldots, 2^d\}$ and let k^b represent $k-1$ in the d-positional binary system (i.e. $k^b \in \{0,1\}^d$, $k = 1 + \sum_{i=1}^{d} k_i^b 2^{i-1}$). Let $k \in \{1, 2, \ldots, 2^d\}$ and let $m = (m_1, m_2, \ldots, m_d)$, $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$. We write $m \overset{k}{\succeq} n$ if $(-1)^{k_i^b} m_i \geq (-1)^{k_i^b} n_i$, for $i = 1, 2, \ldots, d$. If all inequalities are strict, we write $m \overset{k}{\succ} n$. A sequence $\{p_i\}$ of points in $\mathbb{Z}^d$ is said to be a k-type increasing sequence if $p_{i+1} \overset{k}{\succ} p_i$, for all $i \in \mathbb{N}$. For $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$, we denote $\|n\| = \max\{|n_i| \mid i = 1, 2, \ldots, d\}$.

For a k-type increasing sequence $\{p_i\}$ of points in $\mathbb{Z}^d$, with $p_1 \overset{k}{\succ} \mathbf{0}$ and $p_0 = \mathbf{0}$, points x, $y \in X$, a real number $\epsilon > 0$ and any positive integer n, define

$$\Phi_{xy}^{(n)}(T, \epsilon, \{p_i\}) = \frac{1}{n} \#\{0 \leq i < n \mid \rho(T^{p_i}(x), T^{p_i}(y)) < \epsilon\},$$

where $\#A$ denotes the cardinality of the set A. Let

$$\Phi_{xy}^{k}(T, \epsilon, \{p_i\}) = \liminf_{n \to \infty} \Phi_{xy}^{(n)}(T, \epsilon, \{p_i\}),$$

$$\Phi_{xy}^{k,*}(T, \epsilon, \{p_i\}) = \limsup_{n \to \infty} \Phi_{xy}^{(n)}(T, \epsilon, \{p_i\}),$$

then $\Phi_{xy}^{k}(T, \epsilon, \{p_i\})$ and $\Phi_{xy}^{k,*}(T, \epsilon, \{p_i\})$ are called the lower and the upper distribution functions for T with respect to the k-type increasing sequence $\{p_i\}$, respectively. Clearly, for every $\epsilon > 0$, $\Phi_{xy}^{k}(T, \epsilon, \{p_i\}) \leq \Phi_{xy}^{k,*}(T, \epsilon, \{p_i\})$.

Definition 6.1.1. A subset D of X is said to be *k-type distributionally scrambled set for T in the k-type increasing sequence $\{p_i\}$*, if for any $x, y \in D$ with $x \neq y$ we have

(i) $\Phi^k_{xy}(\delta, \{p_i\}) = 0$, for some $\delta > 0$, and

(ii) $\Phi^{k,*}_{xy}(\epsilon, \{p_i\}) = 1$, for all $\epsilon > 0$.

Such a pair (x, y) is called *k-type distributionally chaotic pair for T in the sequence* $\{p_i\}$. We denote by $DCR^k(T, \{p_i\})$ the collection of all points $(x, y) \in X \times X$ such that (x, y) is k-type distributionally chaotic pair for T in the k-type increasing sequence $\{p_i\}$ and call it the k-type distributionally chaotic relation for T with respect to the k-type increasing sequence $\{p_i\}$.

Definition 6.1.2. A $\mathbb{Z}^d$-action T on X is said to be *k-type distributionally chaotic in a sequence* if there exists a k-type increasing sequence $\{p_i\}$ such that T has an uncountable k-type distributionally scrambled set in the sequence $\{p_i\}$.

Remark 6.1.3. For $d = 1$, the given definition of 1-type distributional chaos in a sequence coincides with the definition of distributional chaos in a sequence (See [65]).

Example 6.1.4. Consider $T : \mathbb{Z}^2 \times S^1 \to S^1$ defined by $T((n_1, n_2), \theta) = 2^{n_1+n_2}\theta$. Then T is k-type distributionally chaotic in a sequence for $k \in \{1, 2, 3\}$ but not 4-type distributionally chaotic in a sequence. Observe that, for any 4-type increasing sequence $\{p_i\}$ in $\mathbb{Z}^2$ and points $\theta_1, \theta_2 \in S_1$, $\rho(T^{p_i}(\theta_1), T^{p_i}(\theta_2)) \to 0$ as $i \to \infty$, where ρ is a usual metric on S_1. This implies that $\Phi^4_{\theta_1\theta_2}(T, \epsilon, \{p_i\}) = \Phi^{4,*}_{\theta_1\theta_2}(T, \epsilon, \{p_i\}) = 1$, for all $\epsilon > 0$.

Proposition 6.1.5. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be two $\mathbb{Z}^d$-actions on compact metric spaces (X, ρ_1) and (Y, ρ_2), respectively. If T_1 and T_2 are topologically conjugate then T_1 is k-type distributionally chaotic in a sequence implies T_2 is k-type distributionally chaotic in a sequence.*

Proof. Suppose that T_1 is k-type distributionally chaotic in a sequence $\{p_i\}$, where $\{p_i\}$ is a k-type increasing sequence of points in $\mathbb{Z}^d$, with $p_1 \overset{k}{\succ} \mathbf{0}$ and $p_0 = \mathbf{0}$. Since T_1 and T_2 are topologically conjugate, there exists a homeomorphism $h : X \to Y$

such that $h \circ T_1^n = T_2^n \circ h$, for all $n \in \mathbb{Z}^d$. By uniform continuity of h, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\rho_1(x_1, x_2) < \delta$ implies $\rho_2(h(x_1), h(x_2)) < \epsilon$. For $x_1, x_2 \in X$, choose $y_1 = h(x_1)$ and $y_2 = h(x_2)$. Then,

$$\begin{aligned}
\Phi^{(n)}_{x_1x_2}(T_1, \delta, \{p_i\}) &= \frac{1}{n}\#\{0 \le i < n \mid \rho_1(T_1^{p_i}(x_1), T_1^{p_i}(x_2)) < \delta\} \\
&\le \frac{1}{n}\#\{0 \le i < n \mid \rho_2(h(T_1^{p_i}(x_1)), h(T_1^{p_i}(x_2))) < \epsilon\} \\
&= \frac{1}{n}\#\{0 \le i < n \mid \rho_2(T_2^{p_i}(h(x_1)), T_2^{p_i}(h(x_2))) < \epsilon\} \\
&= \frac{1}{n}\#\{0 \le i < n \mid \rho_2(T_2^{p_i}(y_1), T_2^{p_i}(y_2)) < \epsilon\} \\
&= \Phi^{(n)}_{y_1y_2}(T_2, \epsilon, \{p_i\}).
\end{aligned}$$

Similarly by uniform continuity of h^{-1}, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\Phi^{(n)}_{y_1y_2}(T_2, \delta, \{p_i\}) \le \Phi^{(n)}_{x_1x_2}(T_1, \epsilon, \{p_i\})$.

Since T_1 is k-type distributionally chaotic in a sequence $\{p_i\}$, X has an uncountable k-type distributionally scrambled set D in sequence $\{p_i\}$ for T_1. Then $h(D)$ is an uncountable subset of Y. Moreover, for any $y_1 = h(x_1), y_2 = h(x_2) \in h(D)$ with $y_1 \neq y_2$, $\Phi^{k,*}_{y_1y_2}(T_2, t, \{p_i\}) = 1$, for all $t > 0$, and $\Phi^{k}_{y_1y_2}(T_2, \delta, \{p_i\}) = 0$, for some $\delta > 0$, as $\Phi^{k,*}_{x_1x_2}(T_1, t, \{p_i\}) = 1$, for all $t > 0$, and $\Phi^{k}_{x_1x_2}(T_1, \epsilon, \{p_i\}) = 0$, for some $\epsilon > 0$. Thus, $h(D)$ is uncountable k-type distributionally scrambled set in sequence $\{p_i\}$ for T_2. Hence T_2 is k-type distributionally chaotic in a sequence $\{p_i\}$. □

We recall here some definitions and notions defined for continuous $\mathbb{Z}^d$-actions, in this direction. For a point $x \in X$, the set $O_T^k(x) = \{T^n(x) \mid n \in \mathbb{Z}^d, n \overset{k}{\succeq} \mathbf{0}\}$ is called *k-type orbit* of x. A point $x \in X$ is said to be a *k-type periodic point* if there exists an $n \in \mathbb{Z}^d$ with $n \overset{k}{\succ} \mathbf{0}$ such that $T^n(x) = x$.

Definition 6.1.6 [41]**.** A $\mathbb{Z}^d$-action T on X is said to be *k-type transitive* if for any two nonempty open sets $U, V \subset X$ there exists an $n \in \mathbb{Z}^d$ with $n \overset{k}{\succ} \mathbf{0}$ such that $T^n(U) \cap V \neq \emptyset$.

Definition 6.1.7 [50]**.** A $\mathbb{Z}^d$-action T on X is said to be *k-type weakly mixing* if for every pair of nonempty open sets $(G_1, G_2), (H_1, H_2) \subset X \times X$ there exists an $n \in \mathbb{Z}^d$ with $n \overset{k}{\succ} \mathbf{0}$ such that $T^n(G_i) \cap H_i \neq \emptyset$ for $i = 1, 2$.

6.2 k-type Li-Yorke Chaos

This section is devoted to the study of k-type Li-Yorke chaos defined for a continuous $\mathbb{Z}^d$-action on metric space. In [51], the authors introduced the notion of k-type Li-Yorke pairs for $\mathbb{Z}^d$-actions on metric spaces. A pair $(x, y) \in X \times X$, $x \neq y$, is said to be a *k-type Li-Yorke pair* if there exists a k-type increasing sequence $\{p_i\}$, such that

$$\liminf_{i\to\infty} \rho(T^{p_i}(x), T^{p_i}(y)) = 0 \text{ and } \limsup_{i\to\infty} \rho(T^{p_i}(x), T^{p_i}(y)) > 0.$$

In the above definition, for $d = 1$, if we consider the sequence of positive integers, then it coincides with the definition of Li-Yorke pair as defined in [36].

Definition 6.2.1. For a k-type increasing sequence $\{p_i\}$ of points in $\mathbb{Z}^d$, with $p_1 \overset{k}{\succ} \mathbf{0}$, a pair of points x, y in X is said to be

(i) *k-type proximal with respect to sequence* $\{p_i\}$ if for any $\epsilon > 0$ there exists $i \in \mathbb{N}$ such that $\rho(T^{p_i}(x), T^{p_i}(y)) < \epsilon$.

(ii) *k-type asymptotic with respect to sequence* $\{p_i\}$ if for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\rho(T^{p_i}(x), T^{p_i}(y)) < \epsilon$, for all $i \geq n$.

We denote the set of all k-type proximal pairs with respect to sequence $\{p_i\}$ by $PR^k(T, \{p_i\})$ and the set of all k-type asymptotic pairs with respect to sequence $\{p_i\}$ by $AR^k(T, \{p_i\})$.

Definition 6.2.2. A pair of points x, y in X is said to be *k-type distal with respect to sequence* $\{p_i\}$ if they are not k-type proximal with respect to sequence $\{p_i\}$.

The set of all k-type distal pairs with respect to sequence $\{p_i\}$ is denoted by $DR^k(T, \{p_i\})$. In the above given terminologies, we reformulate the definition of Li-Yorke pair in the following manner:

Definition 6.2.3. A pair $(x, y) \in X \times X$ to be *k-type Li-Yorke pair* if there exists k-type increasing sequences $\{m_i\}$, $\{n_i\}$, where $m_1 \overset{k}{\succ} \mathbf{0}$, $n_1 \overset{k}{\succ} \mathbf{0}$, such that

$$(x, y) \in AR^k(T, \{m_i\}) \cap DR^k(T, \{n_i\}).$$

A subset $S \subset X$ such that any pair of distinct points x, y in S is a k-type Li-Yorke pair is called *k-type Li-Yorke scrambled set.*

Definition 6.2.4. A $\mathbb{Z}^d$-action T on a metric space (X, ρ) is said to be *k-type Li-Yorke chaotic* if there exists an uncountable k-type Li-Yorke scrambled set $S \subset X$.

Lemma 6.2.5. *Let (X, ρ) be a compact metric space and let T be a continuous $\mathbb{Z}^d$-action on X. If $(Y, T|_Y)$ is k-type transitive subsystem of (X, T) and $x \in X$ is such that $O_T^k(x)$ is dense in Y then for each open set U in Y, the set $\{t \overset{k}{\succ} \mathbf{0} \mid T^t(x) \in U\}$ is not k-type bounded above.*

Proof. Suppose U is open in Y and $m_0 \overset{k}{\succ} \mathbf{0}$. We will prove that there exists a $t \overset{k}{\succ} m_0$ such that $T^t(x) \in U$. For $x \in X$ and $m \overset{k}{\succ} \mathbf{0}$, we write

$$S(x, m) = \{T^t(x) \mid t \overset{k}{\succeq} \mathbf{0} \text{ and } \|t\| \leq \|m\|\}.$$

First, we assume that $S(x, m_0) \subset Y$. Then the set $V = Y - S(x, m_0)$ is a nonempty open set in Y. Since $T|_Y$ is k-type transitive, there exist $v \in V$ and $t_v \overset{k}{\succ} \mathbf{0}$ such that $T^{t_v}(v) \in U$. If $v \in O_T^k(x)$, then there exists $t_0 \overset{k}{\succ} m_0$ such that $v = T^{t_0}(x)$. Let $t = t_0 + t_v$, then $t \overset{k}{\succ} m_0$ and we have $T^t(x) = T^{t_v}(T^{t_0}(x)) = T^{t_v}(v) \in U$. Also if $v \notin O_T^k(x)$, then there exist a k-type increasing sequence $\{t_i\}$ such that $T^{t_i}(x) \to v$ for $i \to \infty$, which implies that $T^{t_v}(T^{t_i}(x)) \to T^{t_v}(v)$. Thus, in either case we have $T^t(x) \in U$, for some $t \overset{k}{\succ} m_0$.

Next, we assume that $Y \subset S(x, m_0)$. Since $O_T^k(x)$ is dense in Y, $U \cap O_t^k(x) \neq \emptyset$. Choose $y \in U \cap O_T^k(x)$, so $y = T^{t_y}(x)$, for some $t_y \overset{k}{\succ} \mathbf{0}$. As $T : Y \to Y$, $T^t(y) \in Y$, for each $t \overset{k}{\succ} m_0$. Also $Y \subset S(x, m_0)$ implies that there exist a periodic semi-orbit $P \subset S(x, m_0)$ such that $T^t(y) \in P$, for some $t \overset{k}{\succ} \mathbf{0}$. Let

$$\lambda = \min\{\|t\| \mid t \overset{k}{\succ} \mathbf{0} \text{ and } T^t(y) \in P\},$$

and define

$$B\left(y, \frac{\lambda}{2}\right) = \left\{T^t(y) \mid t \overset{k}{\succ} \mathbf{0} \text{ and } \|t\| \geq \frac{\lambda}{2}\right\}.$$

Further, let $V_1 = Y - \{T^t(x) \mid t \overset{k}{\succeq} \mathbf{0} \text{ and } \|t\| \leq \|m\| + \frac{\lambda}{2}\}$, $V_2 = Y - B(y, \frac{\lambda}{2})$. If $\lambda > 0$, then V_1, V_2 are both nonempty open subsets of Y. For any $q \in V_1$, $q = T^{t_1}(x)$, where $\|t_1\| > \|t_y\| + \frac{\lambda}{2}$. Therefore $q = T^{t_q}(y)$, where $t_q = t_1 - t_y$ and $\|t_q\| > \frac{\lambda}{2}$. Thus for any $t \overset{k}{\succ} \mathbf{0}$, $T^t(q) = T^{t+t_q}(y) \in B(y, \frac{\lambda}{2})$, since $\|t + t_q\| > \frac{\lambda}{2}$. By definition of V_2, $T^t(q) \notin V_2$, for all $t \overset{k}{\succ} \mathbf{0}$. Hence for each $t \overset{k}{\succ} \mathbf{0}$, $T^t(V_1) \cap V_2 = \emptyset$, which contradicts the transitivity of $T|_Y$. Thus $\lambda = 0$ and hence $T^{\mathbf{0}}(y) = y \in P$. Since P is periodic, there exists a $t \overset{k}{\succ} t_0$ such that $T^t(x) = y \in U$. □

Theorem 6.2.6. *Let (X, ρ) be a compact metric space without isolated points and let T be a continuous $\mathbb{Z}^d$-action on X. If T is k-type weakly mixing then T is k-type Li-Yorke chaotic.*

Proof. Since X is a second countable space, $X \times X$ has a countable open base, say $G_1, G_2, \ldots$. For each $n \in \mathbb{N}$, let

$$D_n = \bigcup \{(T^{-t} \times T^{-t})(G_n) \mid t \in \mathbb{Z}^d, t \overset{k}{\succ} \mathbf{0}\}.$$

Clearly for each $n \in \mathbb{N}$, D_n is a nonempty open set in $X \times X$. Further, since T is k-type weakly mixing, $T \times T$ is k-type transitive and hence each D_n is dense in $X \times X$. Consider the set $D = \bigcap_{n=1}^{\infty} D_n$. Then D is a countable intersection of dense sets in $X \times X$. Choose any x_0, $y_0 \in X$, with $x_0 \neq y_0$.

By choice of D, orbit of any pair in D is dense in $X \times X$. Thus, for any $(x, y) \in D$ with $x \neq y$, from Lemma 6.2.5, it follows that any open set containing (x_0, x_0) contains infinite number of points of type $(T^t(x), T^t(y))$, $t \overset{k}{\succ} \mathbf{0}$. Therefore, there exist a k-type increasing sequence $\{m_i\}$, where $m_1 \overset{k}{\succ} \mathbf{0}$ such that

$$\lim_{i \to \infty} \rho(T^{m_i}(x), T^{m_i}(y)) = \rho(x_0, x_0) = 0.$$

Thus, $(x, y) \in AR^k(T, \{m_i\})$.

Similarly, there exist a k-type increasing sequence $\{n_i\}$ with $n_1 \overset{k}{\succ} \mathbf{0}$ such that

$$\lim_{i \to \infty} \rho(T^{n_i}(x), T^{n_i}(y)) = \rho(x_0, y_0) > 0.$$

Thus, $(x, y) \in DR^k(T, \{n_i\})$. This proves that

$$(x, y) \in AR^k(T, \{m_i\}) \cap DR^k(T, \{n_i\}).$$

Hence (x, y) is a Li-Yorke scrambled pair. Since $(x, y) \in D$ with $x \neq y$ is arbitrary, it follows that T is k-type Li-Yorke chaotic. $\square$

6.3 Relation between k-type Li-Yorke chaos and k-type distributional chaos in sequence

In this section, we study the connection between the notions of k-type Li-Yorke chaos and k-type distributional chaos in sequence. Essentially, we prove that the notion of k-type Li-Yorke chaos is equivalent to k-type distributional chaos in sequence. From the definitions, it follows that if T is k-type distributionally chaotic in a sequence, then T is k-type Li-Yorke chaotic.

In order to prove that k-type Li-Yorke chaos implies k-type distributional chaos in sequence, we consider an equivalent definition of k-type distributional chaotic pair in terms of relative density of sequences. For a k-type increasing sequence $Q = \{q_i\}$ of points in $\mathbb{Z}^d$ with $q_1 \overset{k}{\succ} \mathbf{0}$ and any positive integer t, let

$$C_Q(t) = \#\{i \geq 1 \mid \|q_i\| \leq t\},$$

If $\lim\limits_{t\to\infty} \frac{C_Q(t)}{t^d}$ exists, then this limit is called the k density of sequence Q and is denoted by $\tau_k(Q)$.

Definition 6.3.1. Suppose $Q = \{q_i\}$ is a k-type increasing sequence of points in $\mathbb{Z}^d$ with $q_1 \overset{k}{\succ} \mathbf{0}$ and $P = \{p_j\}$ is a subsequence of Q. Then the limit

$$\limsup_{i\to\infty} \frac{\#\{j \geq 1 \mid q_i \overset{k}{\succ} p_j\}}{i},$$

is called the upper density of sequence P relative to sequence Q and is denoted by $\tau^*(P|Q)$.

The notion of k-type distributionally chaotic pair for T in a sequence Q can be reformulated in the following manner:

Let $Q = \{q_i\}$ be a k-type increasing sequence of points in $\mathbb{Z}^d$ with $q_1 \overset{k}{\succ} \mathbf{0}$. For every $a \in [0, 1]$, let

$$D_Q(a) = \{P \subseteq \mathbb{Z}^d \mid P \cap Q \text{ is infinite and } \tau^*(P \mid Q) \geq a\}.$$

For $x \in X$ and $A \subseteq X$, define

$$N^k(x, A) = \{n \in \mathbb{Z}^d \mid n \overset{k}{\succ} \mathbf{0} \text{ and } T^n(x) \in A\} \text{ and}$$

$$\mathcal{F}^k(A, Q, a) = \{x \in X \mid N^k(x, A) \in D_Q(a)\}.$$

Recall that, for a subset A of X, the set $A_\delta = \{x \in X \mid \inf_{y \in A} \rho(x, y) < \delta\}$ is called the δ neighborhood of A.

In the above given terminologies, we get a pair $(x,\, y) \in X \times X$ is k-type distributionally chaotic pair for T in a sequence Q if and only if

(i) for some $\delta > 0$, $(x,\, y) \in \mathcal{F}^k(X \times X \setminus \overline{[\Delta]}_\delta, Q, 1)$, and

(ii) for any $\epsilon > 0$, $(x,\, y) \in \mathcal{F}^k([\Delta]_\epsilon, Q, 1)$,

where $\Delta = \{(x, x) \mid x \in X\}$ denotes the diagonal of the product space $X \times X$, and $[\Delta]_\epsilon$ denotes the ϵ neighborhood of Δ in the product space $X \times X$. The metric on the product space $X \times X$ is given by $\varrho((x_1, y_1), (x_2, y_2)) = \max\{\rho(x_1, x_2), \rho(y_1, y_2)\}$.

To prove the equivalence, we need the following key lemmas. Lemma 6.3.2 and Lemma 6.3.3 are proved along the lines of analogous result proved for sequence of positive integers in [24]. We present the proofs here for completion.

Lemma 6.3.2. *Let $Q_j = \{q_i^j\}$ be k-type increasing sequences of points in $\mathbb{Z}^d$ with $q_1^j \overset{k}{\succ} \mathbf{0}$, for each $j \in \{1, 2, \ldots l\}$. Then there exists a k-type increasing sequence $P = \{p_i\}$ of points in $\mathbb{Z}^d$ such that*

$$\tau^*(P \cap Q_j \mid P) = 1,$$

for each $j = 1, 2, \ldots, l$.

Proof. By Definition 6.3.1, we only need to prove that for each $j = 1, 2 \ldots l$, there exists a sequence $\{i_m\}$ of positive integer such that

$$\lim_{m\to\infty} \frac{\#\{i \geq 1 \mid p_{i_m} \overset{k}{\succeq} q_i^j,\ q_i^j \in P\}}{i_m} = 1.$$

Choose $i_m = m!$, and let $\{p_i\}$ be a k-type increasing sequence of points in $\mathbb{Z}^d$ such that,

$$\{p_s \mid i_m \leq s < i_{m+1}\} \subset Q_j,$$

for each $j = 1, 2, \ldots, l$ whenever $(m-1) \equiv j \bmod (l+1)$. For each fixed j, it is easy to see that $\#\{s \geq 1 \mid p_s \in P \cap Q_j\} \geq i_{h.l+j} - i_{h.l+(j-1)}$. Thus,

$$\begin{aligned}
\lim_{h\to\infty} \frac{C_{P\cap Q_j}(i_{h.l+j})}{i_{h.l+j}} &\geq \lim_{h\to\infty} \frac{i_{h.l+j} - i_{h.l+(j-1)}}{i_{h.l+j}} \\
&= \lim_{h\to\infty} \left(1 - \frac{i_{h.l+(j-1)}}{i_{h.l+j}}\right) \\
&= \lim_{h\to\infty} \left(1 - \frac{1}{h.l+j}\right) \\
&= 1.
\end{aligned}$$

□

Lemma 6.3.3. *Let $Q_j = \{q_i^j\}$ be k-type increasing sequences of points in $\mathbb{Z}^d$ with $q_1^j \overset{k}{\succ} \mathbf{0}$, for each $j \in \{1, 2, \ldots\}$. Then there exists a k-type increasing sequence $P = \{p_i\}$ of points in $\mathbb{Z}^d$ such that*

$$\tau^*(P \cap Q_j \mid P) = 1,$$

for each $j = 1, 2, \ldots$.

Proof. Let $i_m = m!$, and choose a k-type increasing sequence $\{p_i\}$ of points in $\mathbb{Z}^d$ as follows:

For each $j = 1, 2, \ldots$, when $\frac{(l-1)l}{2} = \sum_{j=1}^{l-1} j \leq s < \sum_{j=1}^{l} j = \frac{l(l+1)}{2}$,

$$\{p_s \mid i_{\frac{(l-1)l}{2}} \leq s < i_{\frac{(l-1)l}{2}+1}\} \subset Q_1,$$
$$\{p_s \mid i_{\frac{(l-1)l}{2}+1} \leq s < i_{\frac{(l-1)l}{2}+2}\} \subset Q_2,$$
$$\vdots$$
$$\{p_s \mid i_{\frac{l(l+1)}{2}-1} \leq s < i_{\frac{l(l+1)}{2}+1}\} \subset Q_l,$$

Then for each fixed j, we get

$$\begin{aligned}
\lim_{l\to\infty} \frac{C_{P\cap Q_j}(i_{\frac{(l-1)l}{2}+j})}{i_{\frac{(l-1)l}{2}+j}} &\geq \lim_{l\to\infty} \frac{i_{\frac{(l-1)l}{2}+j} - i_{\frac{(l-1)l}{2}+(j-1)}}{i_{\frac{(l-1)l}{2}+j}} \\
&= \lim_{l\to\infty} \left(1 - \frac{i_{\frac{(l-1)l}{2}+(j-1)}}{i_{\frac{(l-1)l}{2}+j}}\right) \\
&= \lim_{l\to\infty} \left(1 - \frac{1}{\frac{(l-1)l}{2} + j}\right) \\
&= 1.
\end{aligned}$$

□

Lemma 6.3.4. *Let (X, ρ) be a compact metric space and let T be a continuous $\mathbb{Z}^d$-action on X. If $\{m_i\}$ and $\{n_i\}$ are k-type increasing sequences of points in $\mathbb{Z}^d$, where $m_1 \overset{k}{\succ} \mathbf{0}$, $n_1 \overset{k}{\succ} \mathbf{0}$, then there exists a k-type increasing sequence $\{p_i\}$ of points in $\mathbb{Z}^d$ with $p_1 \overset{k}{\succ} \mathbf{0}$, such that*

$$DR^k(T, \{m_i\}) \cap AR^k(T, \{n_i\}) \subset DCR^k(T, \{p_i\}).$$

Proof. Let $b_1 = 2$, $b_i = 2^{b_1+b_2+\cdots+b_{i-1}}$ for $i > 1$. Then $\{b_i\}$ is an increasing sequence of positive integers. Let

$$p_i = \begin{cases} m_i', & \text{if } i \leq b_1 \text{ or } \sum_{j=1}^{2k} b_j < i \leq \sum_{j=1}^{2k+1} b_j,\ k \in \mathbb{N} \\ n_i', & \text{otherwise} \end{cases}$$

where $\{m_i'\}$ and $\{n_i'\}$ are sub-sequences of $\{m_i\}$ and $\{n_i\}$ respectively, with $m_l' \overset{k}{\succ} n_j'$, $n_l' \overset{k}{\succ} m_j'$ for any $l > j$. Then $\{p_i\}$ is a k-type increasing sequence of points in $\mathbb{Z}^d$.

Let $(x,y) \in DR^k(T,\{m_i\}) \cap AR^k(T,\{n_i\})$. Then $(x,y) \in DR^k(T,\{m_i\})$ implies that there exists a $\delta > 0$ such that $\rho(T^{m_i}(x), T^{m_i}(y)) > \delta$, for all $i \in \mathbb{N}$. Thus,

$$\begin{aligned}
\Phi_{xy}^k(T,\delta,\{p_i\}) &= \liminf_{n\to\infty} \frac{1}{n}\#\{0 \le i < n \mid \rho(T^{p_i}(x), T^{p_i}(y)) < \delta\} \\
&\le \lim_{i\to\infty} \frac{1}{j_i}\#\{0 \le k < j_i \mid \rho(T^{p_k}(x), T^{p_k}(y)) < \delta\} \text{ (where } j_i = \sum_{h=1}^{2i+1} b_h) \\
&\le \lim_{i\to\infty} \frac{b_1 + b_2 + \cdots + b_{2i}}{j_i} \\
&= \lim_{i\to\infty} \frac{b_1 + b_2 + \cdots + b_{2i}}{b_1 + b_2 + \cdots + b_{2i} + 2^{b_1+b_2+\cdots+b_{2i}}} \\
&= 0.
\end{aligned}$$

Further, $(x,y) \in AR^k(T,\{n_i\})$ implies that for any $\epsilon > 0$ there exist a positive integer $N > 0$ such that $\rho(T^{n_i}(x), T^{n_i}(y)) < \epsilon$, for all $i > N$. Thus,

$$\begin{aligned}
\Phi_{xy}^k(T,\epsilon,\{p_i\}) &= \limsup_{n\to\infty} \frac{1}{n}\#\{0 \le i < n \mid \rho(T^{p_i}(x), T^{p_i}(y)) < \epsilon\} \\
&\ge \lim_{i\to\infty} \frac{1}{l_i}\#\{0 \le k < l_i \mid \rho(T^{p_k}(x), T^{p_k}(y)) < \delta\} \text{ (where } l_i = \sum_{h=1}^{2i} b_h) \\
&\ge \lim_{i\to\infty} \frac{b_{2i}}{l_i} \\
&= \lim_{i\to\infty} \frac{2^{b_1+b_2+\cdots+b_{2i}}}{b_1 + b_2 + \cdots + b_{2i-1} + 2^{b_1+b_2+\cdots+b_{2i-1}}} \\
&= 1.
\end{aligned}$$

Hence $(x,y) \in DCR^k(T,\{p_i\})$. □

Lemma 6.3.5. *Let (X,ρ) be a compact metric space and let T be a continuous $\mathbb{Z}^d$-action on X. If $S \subset X$ is a countable k-type Li-Yorke scrambled set, then there exists a k-type increasing sequence $\{p_i\}$ of points in $\mathbb{Z}^d$ with $p_1 \overset{k}{\succ} \mathbf{0}$, such that S is k-type distributionally scrambled set in the sequence $\{p_i\}$.*

Proof. For any pair of distinct points $x,y \in S$, by definition of a Li-Yorke scrambled set, there exist k-type increasing sequences $\{m_i\}$ and $\{n_i\}$ of points in $\mathbb{Z}^d$ with $m_1 \overset{k}{\succ} \mathbf{0}$, $n_1 \overset{k}{\succ} \mathbf{0}$, such that $(x,y) \in AR^k(T,\{m_i\}) \cap DR^k(T,\{n_i\})$. Using Lemma 6.3.4, there exists a k-type increasing sequence $Q_{xy} = \{q_i^{(x,y)}\}$ of points in $\mathbb{Z}^d$ such that

$(x,y) \in DCR^k(T,\{q_i^{(x,y)}\})$. Then for k-type increasing sequences $Q_{xy} = \{q_i^{(x,y)}\}$, $x \neq y$ in S, by Lemma 6.3.3 there exists a k-type increasing sequence $P = \{p_i\}$ of points in $\mathbb{Z}^d$ with $p_1 \overset{k}{\succ} \mathbf{0}$, such that

$$\tau^*(P \cap Q_{xy} \mid P) = 1,$$

for all $x \neq y$ in S. Thus $(x,y) \in DCR^k(T,\{p_i\})$, for all $x \neq y$ in S. Hence, S is k-type distributionally scrambled set in the sequence $\{p_i\}$. □

Lemma 6.3.6. *Let (X,ρ) be a compact metric space and let T be a continuous $\mathbb{Z}^d$-action on X. If $Q = \{q_i\}$ is a k-type increasing sequence of points in $\mathbb{Z}^d$ with $q_1 \overset{k}{\succ} \mathbf{0}$ and $a \in [0,1]$, then for any nonempty open subset A of X, $\mathcal{F}^k(A,Q,a)$ is a G_δ set.*

Proof. For $m \in \mathbb{N}$, let $g_m : X \to [0,1]$ be given by

$$x \mapsto \frac{1}{n}\#\{1 \leq i \leq m \mid T^{q_i}x \in A\}$$

and

$$g : X \to [0,1], \quad x \mapsto \tau^*(N^k(x,A) \mid Q).$$

Clearly, $g(x) = \limsup_{m\to\infty} g_m(x)$. Since A is open, we have that each g_m is continuous, and this implies that g is lower semi-continuous. From the definition of semi-continuity, it follows that $\{x \in X \mid g(x) > a - 1/r\}$ is open for every $r \in \mathbb{N}$. Therefore,

$$\mathcal{F}^k(A,Q,a) = \{x \in X \mid g(x) \geq a\} = \bigcap_{r=1}^{\infty} \left\{x \in X \mid g(x) > a - \frac{1}{r}\right\}$$

is a G_δ subset of X. □

Lemma 6.3.7 [27]**.** *Let (X,ρ) be a complete separable metric space without isolated points. If R is a symmetric relation on X which contains a dense G_δ subset A of X such that for each $x \in A$, $R(x)$ contains a dense G_δ set, then there is an uncountable dense subset B of X such that $B \times B \setminus \Delta \subset R$.*

Lemma 6.3.8 [27]**.** *Let (X, ρ) be a complete separable metric space without isolated points. If R is a symmetric relation on X which contains a dense G_δ subset of $X \times X$, then there is a dense G_δ subset A of X such that for each $x \in A$, there exists a dense G_δ subset A_x of X with $\{(x, y) \mid x \in A, y \in A_x\} \subset R$.*

Combining Lemma 6.3.7 and Lemma 6.3.8, we get the following.

Lemma 6.3.9. *Let (X, ρ) be a complete separable metric space without isolated points. If R is a symmetric relation on X which contains a dense G_δ subset of $X \times X$, then there is an uncountable dense subset B of X such that $B \times B \setminus \Delta \subset R$*

Theorem 6.3.10. *Let (X, ρ) be a compact metric space and let T be a continuous $\mathbb{Z}^d$-action on X. Then T is k-type Li-Yoke chaotic if and only if T is k-type distributionally chaotic in a sequence.*

Proof. Suppose that T is k-type Li-Yorke chaotic, then T has an uncountable k-type Li-Yorke scrambled set $D \subset X$. Since X is second countable, so is D, hence we can choose a countable dense subset S of D. By Lemma 6.3.5, there exist a k-type increasing sequence $Q = \{q_i\}$ of points in $\mathbb{Z}^d$ with $q_1 \overset{k}{\succ} \mathbf{0}$, such that S is k-type distributionally scrambled set in the sequence Q. Also from Lemma 6.3.6, it follows that $S \times S$ is a G_δ set in $X \times X$. Let E be the set of all k-type distributionally scrambled pairs in the sequence Q. Now $S \times S \setminus \Delta \subset E$ and S is dense in $\overline{D}$. By Lemma 6.3.9, there exists an uncountable dense set $K \subset \overline{D}$ such that $K \times K \setminus \Delta \subset E$. Thus T is k-type distributionally chaotic in a sequence. □

Corollary 6.3.11. *Let (X, ρ) be a compact metric space and let T be a continuous $\mathbb{Z}^d$-action on X. If T is k-type weakly mixing then T is k-type distributionally chaotic in a sequence.*

Proof. Since T is k-type weakly mixing, from Theorem 6.2.6 it follows that T is k-type Li-Yorke chaotic. Hence by Theorem 6.3.10, we have that T is k-type distributionally chaotic in a sequence. □

Following diagram summarizes the results obtained in the chapter.

$$\begin{array}{ccc} k\text{-type weakly mixing} & \Rightarrow & k\text{-type Li-Yorke chaos} \\ & & \Updownarrow \\ & & k\text{-type distributional chaos in a sequence} \end{array}$$