Chapter 5

Distributional chaos for $\mathbb{Z}^d$-actions

In the classical theory of discrete dynamical systems, a continuous action of $\mathbb{Z}$ on a metric space is considered. To capture more complex and rich dynamics, the natural generalization is to study the $\mathbb{Z}^d$-actions on a metric space, for $d > 1$. This allows for a wider range of possible behaviors in the system, making it more flexible for modelling real-world phenomena. Various authors have studied the dynamics for such systems (see [47, 44, 21, 49, 30]).

This chapter is devoted to the study of relationship between the notions of distributional chaos and specification property for a continuous $\mathbb{Z}^d$-action defined on a compact metric space. In Section 5.1, we study the notion of distributional chaos for a continuous $\mathbb{Z}^d$-action defined on a compact metric space. In Section 5.2, we relate the notions of distributional chaos and specification property defined for multidimensional discrete-time dynamical systems. Essentially, we prove that a $\mathbb{Z}^d$-action on a compact metric space with weak specification property and with a pair of distal points admits a dense distributionally scrambled set of type 1. This extends the result proved by Oprocha and Štefánková in [43], for a $\mathbb{Z}$-action defined on a compact metric space.

Throughout this chapter, by a dynamical system we mean a pair (X, T), where X is a compact metric space with metric ρ and T is a continuous $\mathbb{Z}^d$-action on X, $d \in \mathbb{N}$, i.e., $T : \mathbb{Z}^d \times X \to X$ satisfies the following three conditions:

(i) $T(n,\cdot)$ is a homeomorphism on X, for any $n \in \mathbb{Z}^d$,

(ii) $T(\mathbf{0},x) = x$, for any $x \in X$ ($\mathbf{0} = (0,0,\ldots,0)$ is the identity element in $\mathbb{Z}^d$),

(iii) $T(m,T(n,\cdot)) = T(m+n,\cdot)$, for any $m,n \in \mathbb{Z}^d$.

For $x \in X$, $n \in \mathbb{Z}^d$, we denote $T(n,x)$ by $T^n(x)$ and the homeomorphism $T(n,\cdot)$ is denoted by T^n.

5.1 Multidimensional distributional chaos

The notion of distributional chaos has a natural generalization to the case of multidimensional discrete-time dynamical systems. In [28], author has introduced the notion of distributional chaos for continuous $\mathbb{Z}^d$-actions and have studied it in the context of subshifts of $\Sigma^{\mathbb{Z}^d}$, where $\Sigma^{\mathbb{Z}^d}$ denotes the space of maps $x : \mathbb{Z}^d \to \Sigma$ with the product topology and Σ is a finite set. In this section, we recall and study the notion of distributional chaos for a continuous $\mathbb{Z}^d$-action on a compact metric space.

For $n \in \mathbb{N}$, points x, $y \in X$, and $t \in \mathbb{R}$, define

$$\Phi_{xy}^{(n)}(t) = \frac{1}{\#\Lambda(n)}\#\{m \in \Lambda(n) \mid \rho(T^m(x),T^m(y)) < t\},$$

where $\Lambda(n) = \{-(n-1),-(n-2),\ldots,0,\ldots,(n-2),(n-1)\}^d \subset \mathbb{Z}^d$, and $\#A$ denotes the cardinality of the set A. Let

$$\Phi_{xy}(t) = \liminf_{n\to\infty} \Phi_{xy}^{(n)}(t), \text{ and}$$
$$\Phi_{xy}^*(t) = \limsup_{n\to\infty} \Phi_{xy}^{(n)}(t).$$

These non-decreasing functions Φ_{xy} and Φ_{xy}^* are called the lower and the upper distribution functions for T, respectively.

Definition 5.1.1. [28] A pair of points x, $y \in X$ is called

(i) *distributionally chaotic of type 1* (DC1) if $\Phi_{xy}(t) = 0$, for some $t > 0$, and $\Phi_{xy}^*(t) = 1$, for all $t > 0$.

(ii) *distributionally chaotic of type 2* (DC2) if $\Phi_{xy}(t) < \Phi^*_{xy}(t)$, for some $t > 0$, and $\Phi^*_{xy}(t) = 1$, for all $t > 0$.

(iii) *distributionally chaotic of type 3* (DC3) if $\Phi_{xy}(t) < \Phi^*_{xy}(t)$, for some $t > 0$.

A set containing at least two points is called a *distributionally scrambled set of type* k (briefly DCk scrambled set) for T if any pair of its distinct points is distributionally chaotic of type k, where $k \in \{1, 2, 3\}$.

Definition 5.1.2. A $\mathbb{Z}^d$-action T is said to be *distributionally chaotic of type* k, where $k \in \{1, 2, 3\}$, if there exists an uncountable distributionally scrambled set of type k for T.

In order to emphasize the $\mathbb{Z}^d$-action T on X, we denote the distribution function by $\Phi^{(n)}_{xy}(T, t)$, in place of $\Phi^{(n)}_{xy}(t)$. Recall that, for dynamical systems (X, T_1) and (Y, T_2), the $\mathbb{Z}^d$-actions $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ are said to be *topologically conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ T_1^n = T_2^n \circ h$, for all $n \in \mathbb{Z}^d$. The map h is called *topological conjugacy* between T_1 and T_2.

Proposition 5.1.3. *Let (X, T_1) and (Y, T_2) be two dynamical systems, where X and Y are compact metric spaces with metrics ρ_1 and ρ_2, respectively. If T_1 and T_2 are topologically conjugate then T_1 is DCk implies T_2 is DCk, $k \in \{1, 2\}$.*

Proof. Since T_1 and T_2 are topologically conjugate, there exists a homeomorphism $h : X \to Y$ such that $h \circ T_1^n = T_2^n \circ h$, for all $n \in \mathbb{Z}^d$. By uniform continuity of h, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\rho_1(x_1, x_2) < \delta$ implies $\rho_2(h(x_1), h(x_2)) < \epsilon$.

For any $x_1, x_2 \in X$, let $y_1 = h(x_1)$ and $y_2 = h(x_2)$. Then,

$$\begin{aligned}
\Phi^{(n)}_{x_1x_2}(T_1, \delta) &= \frac{1}{\#\Lambda(n)}\#\{m \in \Lambda(n) | \rho_1(T_1^m(x_1), T_1^m(x_2)) < \delta\} \\
&\leq \frac{1}{\#\Lambda(n)}\#\{m \in \Lambda(n) | \rho_2(h(T_1^m(x_1)), h(T_1^m(x_2))) < \epsilon\} \\
&= \frac{1}{\#\Lambda(n)}\#\{m \in \Lambda(n) | \rho_2(T_2^m(h(x_1)), T_2^m(h(x_2))) < \epsilon\} \\
&= \frac{1}{\#\Lambda(n)}\#\{m \in \Lambda(n) | \rho_2(T_2^m(y_1), T_2^m(y_2)) < \epsilon\} \\
&= \Phi^{(n)}_{y_1y_2}(T_2, \epsilon)
\end{aligned}$$

Similarly by uniform continuity of h^{-1}, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\Phi^{(n)}_{y_1y_2}(T_2, \delta) \leq \Phi^{(n)}_{x_1x_2}(T_1, \epsilon)$.

Since T_1 is $DC1$, X has an uncountable subset D, which is distributionally scrambled set of type 1 for T_1. Then $h(D)$ is an uncountable subset of Y. Moreover, if $y_1, y_2 \in h(D)$, then $y_1 = h(x_1)$ and $y_2 = h(x_2)$, for some $x_1, x_2 \in D$. Since D is distributionally scrambled set of type 1, $\Phi^*_{x_1x_2}(T_1, t) = 1$, for all $t > 0$ and $\Phi_{x_1x_2}(T_1, \epsilon) = 0$, for some $\epsilon > 0$. This implies that $\Phi^*_{y_1y_2}(T_2, t) = 1$, for all $t > 0$, and $\Phi_{y_1y_2}(T_2, \delta) = 0$, for some $\delta > 0$. Thus, $h(D)$ is uncountable distributionally scrambled set of type 1 for T_2. Hence T_2 is $DC1$. On similar lines, we can prove that T_1 is $DC2$ implies that T_2 is $DC2$. □

Remark 5.1.4. In reference to the example given in [8], we can say that $DC3$ is not preserved under topological conjugacy.

5.2 Relation between the notions of specification and distributional chaos for $\mathbb{Z}^d$-actions

In [28], Hunter studied the relation between weak specification property and distributional chaos in case of multidimensional subshifts. In this section, we establish the relation between weak specification property and distributional chaos for a $\mathbb{Z}^d$-action defined on a compact metric space.

Recall that, a point $x \in X$ is said to be a *periodic point* if the set $\{T^n(x) : n \in \mathbb{Z}^d\}$ is finite. We call a pair of points x, y in X *proximal* if for any $\epsilon > 0$ there exists $n \in \mathbb{Z}^d$ such that $\rho(T^n(x), T^n(y)) < \epsilon$. A pair of points x, y in X is said to be *distal* if it is not proximal. Note that, the distance between two subsets A and B of $\mathbb{Z}^d$ is given by $\varrho(A, B) = \min_{a \in A, b \in B} \|a - b\|$, where $\|a - b\| = \max_{0 \leq i \leq d} |a_i - b_i|$.

Definition 5.2.1 [28]**.** A $\mathbb{Z}^d$-action T is said to have the *strong specification property* (briefly *SSP*) if for every $\epsilon > 0$ there exists a positive integer N_ϵ such that for every finite collection of sets $Q_1, Q_2, \ldots, Q_k$ in $\mathbb{Z}^d$ satisfying $\varrho(Q_i, Q_j) \geq N_\epsilon$, for $i \neq j$, $i, j \in \{1, 2, \ldots, k\}$, any finite collection of points $x_1, x_2, \ldots, x_k$ in X, and any subgroup $\Lambda \subset \mathbb{Z}^d$ with $\varrho(Q_i + q, Q_j) \geq N_\epsilon$, for $i, j \in \{1, 2, \ldots, k\}$ and $q \in \Lambda \setminus \{\mathbf{0}\}$, there exists $y \in X$ such that $\rho(T^t(y), T^t(x_j)) < \epsilon$, for all $t \in Q_j$, $j \in \{1, 2, \ldots, k\}$ and $T^q(y) = y$, for every $q \in \Lambda$.

If the periodicity condition given in terms of subgroup $\Lambda \subset \mathbb{Z}^d$ in above definition is omitted, T is said to have *specification property* (briefly SP). For the special case $k = 2$, the notion of specification property is termed as weak specification property, which can be reformulated in the following manner:

Definition 5.2.2. A $\mathbb{Z}^d$-action T is said to have the *weak specification property* (briefly WSP) if for every $\epsilon > 0$ and every positive integer k there exists a positive integer $N = N(\epsilon, k)$ such that for every finite collection of sets $Q_1, Q_2, \ldots, Q_k$ in $\mathbb{Z}^d$ satisfying $\varrho(Q_i, Q_j) \geq N$, for $i \neq j$, $i, j \in \{1, 2, \ldots, k\}$, any pair of points u, v in X, there exists $y \in X$ such that $\rho(T^t(y), T^t(u)) < \epsilon$, for all $t \in Q_j$, $j = 2i + 1 \leq k$, and $\rho(T^t(y), T^t(v)) < \epsilon$, for all $t \in Q_j$, $j = 2i \leq k$.

For any $\epsilon > 0$, if $\rho(T^t(x), T^t(y)) < \epsilon$, for all $t \in Q_i \subset \mathbb{Z}^d$, we say that x *ϵ-traces* y in Q_i. Note that, the definition of WSP can be easily reformulated using any finite sequence of points x_1, x_2, $\ldots, x_k$ instead of just a pair of two points u, v.

The following theorems extend the results proved by Oprocha and Štefánková in [43], for continuous self-maps defined on compact metric spaces to the case of continuous $\mathbb{Z}^d$-actions on compact metric spaces.

Theorem 5.2.3. *Let (X, ρ) be a compact metric space without isolated points and let T be a continuous $\mathbb{Z}^d$-action on X, $d \in \mathbb{N}$, $d > 1$, having a distal pair. If T has weak specification property, then T is distributionally chaotic of type 1.*

Proof. Let u, $v \in X$ be a distal pair and let $G \subset X$ be an open ball with center at g and of radius $\epsilon > 0$. Set $\epsilon_i = \frac{\epsilon}{2^i}$, $i \geq 1$. Let $M_i = M(\epsilon_i, 2^i)$, as in the definition of WSP, for all $i \geq 1$. Also, let $a_1 < a_2 < a_3 < \cdots$ be an increasing sequence of positive integers such that $a_{i+1} - a_i > M_{i+1}$, for any $i \geq 1$, and $\lim\limits_{i\to\infty} \frac{a_{i+1}-M_{i+1}}{a_i} = \infty$. Further, we consider a subsequence $\{m(i)\}_{i=1}^{\infty}$ of the sequence of positive integers, with $m(1) > 1$, such that $m(i+1) > m(i) + i$.

Step 1: Consider

$$Q_0^1 = \{t \in \mathbb{Z}^d \mid 0 \leq \|t\| \leq a_1\},$$
$$Q_1^1 = \{t \in \mathbb{Z}^d \mid a_{m(1)} \leq \|t\| \leq a_{m(1)+1} - M_2\},$$
$$Q_2^1 = \{t \in \mathbb{Z}^d \mid a_{m(2)} \leq \|t\| \leq a_{m(2)+1} - M_2\},$$
$$Q_3^1 = \{t \in \mathbb{Z}^d \mid a_{m(2)+1} \leq \|t\| \leq a_{m(2)+2} - M_2\}.$$

Since T has WSP, for $\epsilon_2 > 0$ and the sequences of points g, u, v, u and g, u, v, v, we get points $x_u, x_v \in G$, respectively, so that the points x_u, x_v: ϵ_2-trace g in Q_0^1, ϵ_2-trace u in Q_1^1, ϵ_2-trace v in Q_2^1. Moreover, x_u ϵ_2-trace u in Q_3^1 and x_v ϵ_2-trace v in Q_3^1.

Further, since T^k, $k \in \mathbb{Z}^d$, are all uniformly continuous, there are disjoint compact balls B_u and B_v in G centered at x_u and x_v, respectively, such that each $x \in B_\alpha$ ϵ_3-traces x_α in $Q^{(1)} = \{t \in \mathbb{Z}^d | 0 \leq \|t\| \leq a_{m(2)+2} - M_2\}$, where $\alpha \in \{u, v\}$. Without loss of generality we can assume that the diameter of either B_u or B_v is $\epsilon_{j(1)}$, where $j(1) \geq 3$.

Step 2: Again by WSP, for $\epsilon_{j(1)} > 0$ and for

$$Q_1^2 = \{t \in \mathbb{Z}^d \mid a_{m(3)} \leq \|t\| \leq a_{m(3)+1} - M_{j(1)}\},$$
$$Q_2^2 = \{t \in \mathbb{Z}^d \mid a_{m(4)} \leq \|t\| \leq a_{m(4)+1} - M_{j(1)}\},$$
$$Q_3^2 = \{t \in \mathbb{Z}^d \mid a_{m(4)+1} \leq \|t\| \leq a_{m(4)+2} - M_{j(1)}\},$$
$$Q_4^2 = \{t \in \mathbb{Z}^d \mid a_{m(4)+2} \leq \|t\| \leq a_{m(4)+3} - M_{j(1)}\}.$$

there are points $x_{uu}, x_{uv} \in B_u$ and $x_{vu}, x_{vv} \in B_v$, which $\epsilon_{j(1)}$-trace u in Q_1^2 and $\epsilon_{j(1)}$-trace v in Q_2^2. Additionally,

$$x_{uu}, x_{uv}\, \epsilon_{j(1)}\text{-trace } u, \text{ and } x_{vu}, x_{vv}\, \epsilon_{j(1)}\text{-trace } v \text{ in } Q_3^2,$$
$$x_{uu}, x_{vu}\, \epsilon_{j(1)}\text{-trace } u, \text{ and } x_{uv}, x_{vv}\, \epsilon_{j(1)}\text{-trace } v \text{ in } Q_4^2.$$

Again by uniform continuity of T^k's, $k \in \mathbb{Z}^d$, there are disjoint compact balls B_{uu}, B_{uv} in interior of B_u centered at x_{uu} and x_{uv}, respectively, and B_{vu}, B_{vv} in interior of B_v centered at x_{vu} and x_{vv}, respectively, such that each $x \in B_\alpha$ $\epsilon_{j(1)+1}$-traces x_α in $Q^{(2)} = \{t \in \mathbb{Z}^d \mid 0 \le \|t\| \le a_{m(4)+3} - M_{j(1)}\}$. Without loss of generality we can assume that the diameter of either B_α is $\epsilon_{j(2)}$, where $j(2) > j(1) + 1$.

Continuing like this, for each $k \in \mathbb{N}$ and any $\alpha \in \{u, v\}^k$, there is a compact ball B_α centered at x_α, and positive integers $j(1) < j(2) < \cdots < j(k)$ such that if x, y are distinct points in B_α, then x $\epsilon_{j(k)-1}$-traces y in

$$Q^{(k)} = \{t \in \mathbb{Z}^d \mid 0 \le \|t\| \le a_{m(2k)+k+1} - M_{j(k-1)}\}.$$

Also, if $\alpha, \beta \in \{u, v\}^k$ and $\alpha \ne \beta$ then $B_\alpha \cap B_\beta = \emptyset$ and $B_{\alpha u} \cup B_{\alpha v} \subset B_\alpha$.

Moreover, for any $x \in B_\alpha$, where $\alpha = \alpha_1 \alpha_2 \cdots \alpha_k$:

(i) x ϵ_i-traces u in Q_1^i, and ϵ_i-traces v in Q_2^i, $1 \le i < k$.

(ii) x ϵ_i-traces α_1 in Q_3^i, for $1 \le i \le k$, and ϵ_i-traces α_2 in Q_4^i for $2 \le i \le k$. In general, for any j, $1 \le j \le k$, x_α ϵ_i-traces α_j in Q_{j+2}^i, for $j \le i \le k$.

Take $S = \bigcap_{n=1}^\infty \bigcup_{\alpha \in \{u,v\}^n} B_\alpha$. Since $\operatorname{diam}(B_\alpha) \to 0$ as $n \to \infty$, $S \subset G$ is a Cantor set. Further, since each sequence in $\{u, v\}^{\mathbb{N}}$ corresponds to a distinct element in S, and since the set of all binary sequences is uncountable, it follows that S is uncountable. Observe that, any $s \in S$ can be uniquely determined as $s_\alpha = \bigcap_{n=1}^\infty B_{\alpha_1 \alpha_2 \cdots \alpha_n}$, where $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n \cdots \in \{u, v\}^{\mathbb{N}}$.

We now show that S is distributionally scrambled set of type 1. Let x and y be distinct points in S. Then for any $\delta > 0$, there will be a positive integer m such that $\epsilon_i < \delta$, for all $i > m$. Using condition (i), we get that both x, y, ϵ_i-traces u in Q_1^i, and ϵ_i-traces v in Q_2^i, for all $i \ge 1$. Thus, $\rho(T^p(x), T^p(y)) < \frac{\epsilon_i}{2} < \delta$, for all $p \in Q_1^i \cup Q_2^i$, and for all $i > m$. Hence, $\Phi^*_{xy}(\delta) = 1$, for any $\delta > 0$.

Further $x, y \in S$ implies $x = x_\alpha, y = y_\beta$ for some $\alpha = \alpha_1\alpha_2\cdots\alpha_n\cdots$ and $\beta = \beta_1\beta_2\cdots\beta_n\cdots$ in $\{u, v\}^{\mathbb{N}}$. Since x and y are distinct, there exist a positive integer j such that $\alpha_j \neq \beta_j$. Without loss of generality we assume that $\alpha_j = u$ and $\beta_j = v$. Using condition (ii), we have that x ϵ_i-traces u in Q^i_{j+2} and y ϵ_i-traces v in Q^i_{j+2}, for $j \leq i$. Since u, v is a distal pair, there exist a $\delta_0 > 0$ such that $\rho(T^n(u), T^n(v)) > \delta_0$, for all $n \in \mathbb{Z}^d$. Choose a positive integer m such that $\epsilon_i < \frac{\delta_0}{3}$, for all $i > m$. Then $\rho(T^p(x), T^p(y)) > \frac{\delta_0}{3}$, for any $p \in Q^i_{j+2}$, and for all $j \leq i$ and $i > m$. For if $\rho(T^p(x), T^p(y)) \leq \frac{\delta_0}{3}$, for any $p \in Q^i_{j+2}$, for $j \leq i$ and $i > m$, then using $\rho(T^p(x), T^p(u)) < \epsilon_i$ and $\rho(T^p(y), T^p(v)) < \epsilon_i$ we get $\rho(T^p(u), T^p(v)) < \delta_0$, a contradiction. Hence $\Phi_{xy}(\delta_0) = 0$.

Thus, S is an uncountable distributional scrambled set of type 1 for T. Hence T is distributionally chaotic of type 1. □

Theorem 5.2.4. *Let (X, ρ) be a compact metric space without isolated points and let T be a continuous $\mathbb{Z}^d$-action on X, $d \in \mathbb{N}$, $d > 1$, having a distal pair. If T has weak specification property, then X has a dense distributionally scrambled set of type 1.*

Proof. Since (X, ρ) is a compact metric space, it is second countable. Let $\{G_i\}_{i=1}^{\infty}$ denote a countable base for topology of X consisting of open balls.

For $G_1 = B(g_1, \epsilon^{(1)})$, using the arguments given in Theorem 5.2.3, we get a Cantor DC1 scrambled set S_1 and a sequence $\mu_1 = \{m(1, i)\}_{i=1}^{\infty}$ such that for any $x_1 \in S_1$:

x_1 $\epsilon_i^{(1)}$-traces u in $Q_1^{(1,i)} = \{t \in \mathbb{Z}^d \mid a_{m(1,2i-1)} \leq \|t\| \leq a_{m(1,2i-1)+1} - M_{j(1,i-1)}\}$ and
x_1 $\epsilon_i^{(1)}$-traces v in $Q_2^{(1,i)} = \{t \in \mathbb{Z}^d \mid a_{m(1,2i)} \leq \|t\| \leq a_{m(1,2i)+1} - M_{j(1,i-1)}\}$,

for all $i \geq 1$, where $j(1, 0) = 2$.

Again following the steps discussed in Theorem 5.2.3, for $G_2 = B(g_2, \epsilon^{(2)})$, and for the sequence $a_{m(1,1)} < a_{m(1,3)} < a_{m(1,5)} < \cdots$ of positive integers, we obtain a subsequence μ_2 of $\{m(1, 1), m(1, 3), m(1, 5), \ldots\}$ denoted by $\{m(2, i)\}_{i=1}^{\infty}$ and a

Cantor DC1 scrambled set S_2 such that for any $x_2 \in S_2$:

$x_2\, \epsilon_i^{(2)}$-traces u in $Q_1^{(2,i)} = \{t \in \mathbb{Z}^d \mid a_{m(2,2i-1)} \leq \|t\| \leq a_{m(2,2i-1)+1} - M_{j(2,i-1)}\}$ and

$x_2\, \epsilon_i^{(2)}$-traces v in $Q_2^{(2,i)} = \{t \in \mathbb{Z}^d \mid a_{m(2,2i)} \leq \|t\| \leq a_{m(2,2i)+1} - M_{j(2,i-1)}\}$,

for all $i \geq 1$. Continuing this way, we can find a sequence $S_1, S_2, \ldots$ of Cantor sets such that $S = \bigcup_{n=1}^{\infty} S_n$ is dense in X.

We now prove that S is a DC1 scrambled set. Clearly S_n is a DC1 scrambled set, for each $n \in \mathbb{N}$. Consider $y, z \in S$ such that $y \in S_p$ and $z \in S_q$, $p \neq q$. Assume that $p < q$. Then it follows that y ϵ_i-traces z in

$$Q_1^{(q,i)} = \{t \in \mathbb{Z}^d \mid a_{m(q,2i-1)} \leq \|t\| \leq a_{m(q,2i-1)+1} - M_{j(q,i-1)}\},$$

for any $i \geq 1$, where $\epsilon_i = \max\{\epsilon_i^{(p)}, \epsilon_i^{(q)}\}$. Moreover, y $\epsilon_i^{(p)}$-traces u and z $\epsilon_i^{(q)}$-traces v in

$$Q_2^{(q,i)} = \{t \in \mathbb{Z}^d\ a_{m(q,2i)} \leq \|t\| \leq a_{m(q,2i)+1} - M_{j(q,i-1)}\},$$

for any $i \geq 1$. Thus, we get that y and z are distributionally chaotic of type 1. Hence S is a DC1 scrambled set. □