Chapter 4

Various forms of specification on uniform spaces

Specification is one of the most important and extensively studied notion of chaos. In the past few decades, various types of specification properties have been introduced and their relation with the other well known notions of chaos has been studied. This chapter is devoted to the study of different forms of specification property for uniformly continuous surjective self-maps on uniform spaces.

Most definitions of chaos present in the literature are based on the global behavior of dynamical systems. However, in recent years many authors have shifted their attention to study the impact of local behavior of a system on the global behavior of a system (see [46, 73, 39, 16, 32, 7]). In a related direction, we introduce and study the notion of topological specification point for uniformly continuous surjective self-maps on uniform spaces in Section 4.1. In Section 4.2, we study the relation between the notion of pointwise topological specification and other notions of chaos. Section 4.3 deals with the limiting behavior of a topological specification point under orbital convergence of maps is discussed. In Section 4.4, we introduce and study some weaker forms of specification for uniformly continuous surjective self-maps on uniform spaces. This chapter is based on the work published in Dynamical Systems: An International Journal [69].

4.1 Pointwise topological specification property

In this section, we define and study the topological notion of periodic specification point for uniformly continuous surjective self-maps on uniform spaces.

Definition 4.1.1. Let f be a uniformly continuous surjective self-map on a uniform space $(X, \mathcal{U})$. A point $x \in X$ is said to be a *topological periodic specification point* of f if for every $E \in \mathcal{U}$ there exists a positive integer M such that for any finite sequence $x = x_1, x_2, \ldots, x_k$ of points in X, and any set of integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_j - b_{j-1} \geq M$, for all $2 \leq j \leq k$, and any positive integer $p > M + (b_k - a_1)$, there exists $y \in X$ such that $f^p(y) = y$ and $(f^i(y), f^i(x_j)) \in E$, for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. A map f is said to have *pointwise topological periodic specification property* if every point is a topological periodic specification point.

We denote the set of all topological periodic specification points of f by $\mathrm{TPSP}(f)$. Clearly, if f has topological periodic specification property, then every point of X is a topological periodic specification point of f. If we drop the periodicity condition in the definition of topological periodic specification point, then the point is said to be a *topological specification point.* We denote the set of all topological specification points of f by $\mathrm{TSP}(f)$.

Proposition 4.1.2. *If $f : X \to X$ is a uniformly continuous surjective map on a uniform space $(X,\mathcal{U})$, then $x \in \mathrm{TPSP}(f)$ implies $x \in \mathrm{TPSP}(f^m)$ for every $m \geq 2$.*

Proof. Let $U \in \mathcal{U}$ be any symmetric neighborhood of Δ_X. For $U \in \mathcal{U}$, choose a positive integer M as in the definition of topological periodic specification point corresponding to x. Define $M_1 = \left[\frac{M}{m}\right] + 1$, where $[\cdot]$ denotes the greatest integer function. Also, for any finite sequence $x = x_1, x_2, \ldots, x_k$ of points in X, let $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ be integers with $a_j - b_{j-1} \geq M_1$, for all $2 \leq j \leq k$ and let $p > M_1 + (b_k - a_1)$ be any positive integer. Consider the set of integers $0 \leq a'_1 \leq b'_1 < a'_2 \leq b'_2 < \cdots < a'_k \leq b'_k$, with $a'_j = ma_j$ and $b'_j = mb_j$, so that

$a'_j - b'_{j-1} = ma_j - mb_{j-1} \geq mM_1 > M$, for all $2 \leq j \leq k$ and $pm > M + (b'_k - a'_1)$. Since x is a topological periodic specification point of f, there exists $z \in X$ such that $f^{pm}(z) = (f^m)^p(z) = z$ and $(f^i(z), f^i(x_j)) \in U$, for all $a'_j \leq i \leq b'_j$ and for all $1 \leq j \leq k$. This implies that $((f^m)^i(z), (f^m)^i(x_j)) \in U$, for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. Thus, x is a topological periodic specification point of f^m, for every $m \geq 2$. □

Proposition 4.1.3. *If $f : X \to X$ is a uniformly continuous surjective map on a uniform space $(X,\mathcal{U})$, then* TSP*(f)=*TSP*(f^m) for every $m \geq 2$.*

Proof. From Proposition 4.1.2, it follows that if x is a topological specification point of f then x is a topological specification point of f^m for every $m \geq 2$.

Conversely, suppose that x is a topological specification point of f^m, for some $m \geq 2$. For $U \in \mathcal{U}$, choose a positive integer M as in the definition of topological specification point corresponding to x. Define $M_1 = m(M + 2)$. Now for any finite sequence $x = x_1, x_2, \ldots, x_k$ of points in X, let $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ be integers with $a_j - b_{j-1} \geq M_1$, for all $2 \leq j \leq k$. Consider the set of integers $0 \leq a'_1 \leq b'_1 < a'_2 \leq b'_2 < \cdots < a'_k \leq b'_k$, with $a'_j = \left[\frac{a_j}{m}\right]$ and $b'_j = \left[\frac{b_j}{m}\right] + 1$, so that $a'_j - b'_{j-1} = \left[\frac{a_j}{m}\right] - \left[\frac{b_{j-1}}{m}\right] > \left(\frac{a_j}{m} - 1\right) - \left(\frac{b_{j-1}}{m} + 1\right) \geq M$, for all $2 \leq j \leq k$. Since x is a topological specification point of f^m, there exists $z \in X$ such that $((f^m)^i(z), (f^m)^i(x_j)) \in U$, for all $a'_j \leq i \leq b'_j$ and for all $1 \leq j \leq k$. That is $(f^{mi}(z), f^{mi}(x_j)) \in U$, for all integers i such that $\frac{a_j}{m} \leq i \leq \frac{b_j}{m}$. Thus $(f^i(z), f^i(x_j)) \in U$, for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. Hence x is a topological specification point of f. □

Proposition 4.1.4. *Let f and g be uniformly continuous surjective self-maps on uniform spaces $(X,\mathcal{U})$, $(Y,\mathcal{V})$, respectively. Then x and y are topological periodic specification points of f and g, respectively if and only if (x, y) is a topological periodic specification point of $f \times g$.*

Proof. Suppose that x and y are topological periodic specification points of f and

g, respectively. Let W be a symmetric neighborhood of the diagonal $\Delta_{X\times Y}$. Define

$$U = \{(x, x') \in X \times X | (x, y, x', y') \in W, \text{ for some } y,\ y' \in Y\},$$
$$V = \{(y, y') \in Y \times Y | (x, y, x', y') \in W, \text{ for some } x,\ x' \in X\}.$$

Then U and V are symmetric neighborhoods of Δ_X and Δ_Y, respectively. Let $M = \max\{M_x, M_y\}$, where M_x and M_y are numbers as in the definition of topological periodic specification point corresponding to x and y, respectively.

Let (x_1, y_1), $(x_2, y_2), \ldots, (x_k, y_k)$ be any finite sequence of points in $X \times Y$, where $(x_1, y_1) = (x, y)$. Choose integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_j - b_{j-1} \geq M$, for all $2 \leq j \leq k$, and any positive integer $p > M + (b_k - a_1)$. Since x and y are topological periodic specification points of f and g, respectively, there exist $z_1 \in X$ and $z_2 \in Y$ such that $f^p(z_1) = z_1$, $g^p(z_2) = z_2$, $(f^i(z_1), f^i(x_j)) \in U$, $(g^i(z_2), g^i(y_j) \in V$, for all $a_j \leq i \leq b_j$ and $1 \leq j \leq k$. Then (z_1, z_2) is a point in $X \times Y$ such that $(f \times g)^p(z_1, z_2) = (f^p(z_1), g^p(z_2)) = (z_1, z_2)$ and $((f \times g)^i(z_1, z_2), (f \times g)^i(x_j, y_j)) \in W$, for all $a_j \leq i \leq b_j$ and $1 \leq j \leq k$. Hence, (x, y) is a topological periodic specification point of $f \times g$.

Conversely, suppose that (x, y) is a topological periodic specification point of $f \times g$. Let U and V be symmetric neighborhoods of Δ_X and Δ_Y, respectively. Consider

$$W = \{(x, y, x', y') \in (X \times Y) \times (X \times Y) | (x, x') \in U, (y, y') \in V\}.$$

Then W is a symmetric neighborhood of the diagonal $\Delta_{X\times Y}$. Let M be a positive integer as in the definition of topological periodic specification point corresponding to (x, y). Now for any finite sequence (x_1, y_1), $(x_2, y_2), \ldots, (x_k, y_k)$ of points in $X \times Y$, where $(x_1, y_1) = (x, y)$, let $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ be any set of integers with $a_j - b_{j-1} \geq M$, for all $2 \leq j \leq k$, and let $p > M + (b_k - a_1)$. Since (x, y) is a topological periodic specification point of $f \times g$, there exists $(z_1, z_2) \in X \times Y$ such that $(f \times g)^p(z_1, z_2) = (f^p(z_1), g^p(z_2)) = (z_1, z_2)$ and $((f \times g)^i(z_1, z_2), (f \times g)^i(x_j, y_j)) \in W$, for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. Thus, $(f^i(z_1), f^i(x_j)) \in U$, $(g^i(z_2), g^i(y_j)) \in V$, for all $a_j \leq i \leq b_j$ and for all

$1 \leq j \leq k$. Moreover, $f^p(z_1) = z_1$, $g^p(z_2) = z_2$. Hence x and y are topological periodic specification points of f and g, respectively. □

Proposition 4.1.5. *Let f and g be uniformly continuous surjective self-maps on uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$, respectively and let f be topologically conjugate to g by a uniform homeomorphism $\varphi : Y \to X$. If y is topological periodic specification point of g, then $\varphi(y)$ is a topological periodic specification point of f.*

Proof. Let $U \in \mathcal{U}$ be any symmetric neighborhood of Δ_X. Since φ is a uniform homeomorphism, there exists a symmetric neighborhood V of Δ_Y such that $\Phi(V) \subset U$, where $\Phi = \varphi \times \varphi$. For $V \in \mathcal{V}$, choose a positive integer M as in the definition of topological periodic specification point corresponding to y. For $x = \varphi(y)$, consider the sequence $x = x_1, x_2, \dots, x_k$ and integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_j - b_{j-1} \geq M$, for all $2 \leq j \leq k$ and let $p > M + (b_k - a_1)$ be any positive integer. Since φ is surjective, there exist $y = y_1, y_2, \dots, y_k$ in Y such that $\varphi(y_i) = x_i$. Now, y is a topological periodic specification point of g implies that there exists $z \in Y$ such that $g^p(z) = z$ and $(g^i(z), g^i(y_j)) \in V$, for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. Let $\varphi(z) = w \in X$. Then using the fact that f is topologically semi-conjugate to g, we get

$$\begin{aligned}\Phi(g^i(z), g^i(y_j)) &= (\varphi g^i(z), \varphi g^i(y_j))\\ &= (f^i(\varphi(z)), f^i(\varphi(y_j)))\\ &= (f^i(w), f^i(x_j)) \in U,\end{aligned}$$

for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. Further,

$$\begin{aligned}f^p(w) &= f^p(\varphi(z))\\ &= \varphi(g^p(z))\\ &= \varphi(z)\\ &= w.\end{aligned}$$

Thus, $\varphi(y)$ is a topological periodic specification point of f. □

In the following result, we prove the invariance of the set of all topological periodic specification points for self-homeomorphisms on uniform spaces.

Proposition 4.1.6. *If $f : X \to X$ is a homeomorphism on a uniform space $(X,\mathcal{U})$, then the set of all topological periodic specification points of f is an invariant set under the map f.*

Proof. Let x be a topological periodic specification point of f. We now prove $f(x)$ is a topological periodic specification point of f.

Let $U \in \mathcal{U}$ be any symmetric neighborhood of Δ_X. Since f is uniformly continuous, for $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $(f(y), f(z)) \in U$ whenever $(y, z) \in V$. For $V \in \mathcal{U}$, choose a positive integer M as in the definition of topological periodic specification point corresponding to x. Consider the sequence $f(x) = y_1, y_2, \ldots, y_k$ and integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_j - b_{j-1} \geq M$, for all $2 \leq j \leq k$ and let $p > M + (b_k - a_1)$ be any positive integer. Since f is surjective, there exist $x = x_1, x_2, \ldots, x_k$ in X such that $f(x_i) = y_i$, for all $1 \leq i \leq k$. As x is a topological periodic specification point of f, there exists $z \in X$ such that $f^p(z) = z$ and $(f^i(z), f^i(x_j)) \in V$, for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. This implies that $(f^i(f(z)), f^i(f(x_j))) = (f^i(f(z)), f^i(y_j)) \in U$, for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. Note that, $f(z)$ is also a periodic point of period p. Thus, $f(x)$ is a topological periodic specification point of f.

One can argue similarly to observe that $f^{-1}(x)$ is a topological periodic specification point of f if x is a topological periodic specification point of f. □

For uniformly continuous maps on uniform spaces, it is not known whether pointwise topological periodic specification property implies topological periodic specification property. However, for homeomorphisms on uniform space, existence of a topological periodic specification point alone guarantees that the map has topological periodic specification property. This is proved in the following theorem:

Theorem 4.1.7. *If $f : X \to X$ is a homeomorphism on a uniform space $(X,\mathcal{U})$ and f has a topological periodic specification point, then f has topological periodic* *specification property.*

Proof. Suppose that $x \in X$ is a topological periodic specification point of f. For $E \in \mathcal{U}$, choose a positive integer $M = M_x$ as in the definition of topological periodic specification point. Consider a sequence $x_1, x_2, \ldots, x_k$ of points in X, and any set of integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_j - b_{j-1} \geq M' = 2M$, for all $2 \leq j \leq k$, and let $p > M' + (b_k - a_1)$ be any positive integer. Since x is a topological periodic specification point of f, for the sequence $y_0 = x, y_1 = f^{-M}(x_1), y_2 = f^{-M}(x_2), \ldots, y_k = f^{-M}(x_k)$, and integers $a'_0 = b'_0 = a_1 < a'_1 = a_1 + M \leq b'_1 = b_1 + M < \cdots < a'_k = a_k + M \leq b'_k = b_k + M$ with $a'_j - b'_{j-1} = a_j - b_{j-1} \geq M$, for all $1 \leq j \leq k$, and positive integer $p > M + b'_k - a'_0 = 2M + b_k - a_1$, there exists $y \in X$ such that $f^p(y) = y$ and $(f^i(y), f^i(y_j)) \in E$, for all $a'_j \leq i \leq b'_j$ and for all $0 \leq j \leq k$. Then, for $z = f^M(y)$, $(f^{i-M}(z), f^{i-M}(x_j)) \in E$, for all $a'_j \leq i \leq b'_j$ and for all $0 \leq j \leq k$. But $a_j + M \leq i \leq b_j + M$ implies $a_j \leq i - M \leq b_j$. Thus, we have $(f^l(z), f^l(x_j)) \in E$, for all $a_j \leq l \leq b_j$ and for all $1 \leq j \leq k$. Also, $f^p(y) = y$ implies that $f^p(z) = z$.

Thus, for $E \in \mathcal{U}$ there exists a positive integer M' such that for any sequence of points $x_1, x_2, \ldots, x_k$ in X, and any set of integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_j - b_{j-1} \geq M'$, for all $2 \leq j \leq k$, there exists a periodic point $z \in X$ such that $(f^l(z), f^l(x_j)) \in E$, for all $a_j \leq l \leq b_j$ and for all $1 \leq j \leq k$. Hence f has topological periodic specification property. □

4.2 Relation between pointwise topological specification property and other notions of chaos

It is always useful to study the relationship between the newly defined notion and the existing notions present in the literature. In this section, we study the connection between the notion of pointwise topological specification property and other notions of chaos. To avoid confusion, henceforth in the chapter we denote the subsets of $X \times X$ by E, U, V and subsets of X by $\dot{E}$, $\dot{U}$, $\dot{V}$.

Definition 4.2.1. Let f be uniformly continuous self-map defined on a uniform space $(X,\mathcal{U})$ and let $x \in X$. Then,

(i) x is said to be a *topologically transitive point* of f, if for any open set $\dot{U}$ containing x and any nonempty open set $\dot{V}$ there exists a positive integer n such that $f^n(\dot{U}) \cap \dot{V} \neq \emptyset$.

(ii) x is said to be a *mixing point* of f, if for any open set $\dot{U}$ containing x and any nonempty open set $\dot{V}$ there exists a positive integer N such that $f^n(\dot{U}) \cap \dot{V} \neq \emptyset$, for all $n \geq N$.

(iii) f is said to have a *dense set of periodic points at* x if for any open set $\dot{U}$ containing x, the set $\dot{U} \setminus \{x\}$ contains a periodic point of f.

(iv) x is said to be a *sensitive point* of f, if there exists an entourage $D_x \in \mathcal{U}$ such that for any open set $\dot{U}$ containing x there exists $y \in \dot{U}$ such that $(f^n(x), f^n(y)) \notin D_x$, for some positive integer n. Such an entourage D_x is said to be a sensitivity entourage for f at x.

The set of all mixing points of f is denoted by $\mathrm{M}(f)$. Note that, a map f is mixing (respectively, topologically transitive) if and only if every point is mixing point (respectively, topologically transitive point) of f. Also, every mixing point of f is a topologically transitive point of f.

Lemma 4.2.2. *Let f be a uniformly continuous surjective self-map on a uniform space $(X,\mathcal{U})$. If $x \in X$ is a topological specification point of f, then x is a mixing point of f.*

Proof. Let $\dot{U}$ be any open subset of X containing x and $\dot{V}$ be any nonempty open set in X. Choose any point $y \in \dot{U}$. Then for $x \in \dot{U}$, $y \in \dot{V}$, there exist E_1, $E_2 \in \mathcal{U}$ such that $E_1[x] \subset \dot{U}$ and $E_2[y] \subset \dot{V}$. Let $E = E_1 \cap E_2 \in \mathcal{U}$. Then $E[x] \subset \dot{U}$ and $E[y] \subset \dot{V}$. Now for $E \in \mathcal{U}$, choose a positive integer M as in the definition of topological specification point. Since f is surjective, for any $n \geq M$, there exists

$z \in X$ such that $y = f^n(z)$. Then for $x_1 = x, x_2 = z$, and $a_1 = b_1 = 0, a_2 = b_2 = n$, as x is a topological specification point, there exists a point $w \in X$ such that $(w, x) \in E$ and $(f^n(w), f^n(z)) \in E$. But $(w, x) \in E$ implies $w \in E[x] \subset \dot{U}$ and $(f^n(w), f^n(z)) \in E$ implies $f^n(w) \in E[y] \subset \dot{V}$. Hence $f^n(\dot{U}) \cap \dot{V} \neq \emptyset$, for all $n \geq M$. Thus, x is a mixing point of f. □

Lemma 4.2.3. *Let f be a uniformly continuous surjective self-map on a uniform space $(X, \mathcal{U})$. If $x \in X$ is a topological periodic specification point of f, then f has a dense set of periodic points at x.*

Proof. Let $\dot{U}$ be any open subset of X containing x. Choose an entourage $E \in \mathcal{U}$ such that $E[x] \subset \dot{U}$. For $E \in \mathcal{U}$, let M be a positive integer as in the definition of topological periodic specification point. Consider the sequence $x = x_1, x_2, \ldots x_k$ of points in X, any set of integers $0 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_j - b_{j-1} \geq M$, for all $2 \leq j \leq k$ and let $p > M + (b_k - a_1)$ be any positive integer. Since x is a topological specification point, there exists a point $z \in X$ such that $f^p(z) = z$ and $(f^i(z), f^i(x_j)) \in E$, for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. But $(f^0(z), f^0(x)) = (z, x) \in E$, which implies that $z \in E[x] \subset \dot{U}$. Thus, every open set $\dot{U}$ of X containing x contains a periodic point. Hence, f has a dense set of periodic points at x. □

Definition 4.2.4. Let f be a uniformly continuous self-map defined on a uniform space $(X, \mathcal{U})$. Then a point $x \in X$ is said to be a *Devaney chaotic point* of f, if x is topologically transitive point of f, f has a dense set of periodic points at x and x is a sensitive point of f.

Definition 4.2.5. A map f is said to be *pointwise Devaney chaotic* if every point in X is Devaney chaotic.

Remark 4.2.6. A map f is Devaney chaotic if and only if every point in X is Devaney chaotic.

Lemma 4.2.7. *Let f be a uniformly continuous self-map on a Hausdorff uniform space $(X,\mathcal{U})$. If f has dense set of periodic points at $x \in X$, then there exists an entourage $U \in \mathcal{U}$ and a periodic point q such that $U[x] \cap O(q) = \emptyset$.*

Proof. Choose two arbitrary periodic points q_1 and q_2 with disjoint orbits $O(q_1)$ and $O(q_2)$. Since X is Hausdorff, there exists an entourage $W \in \mathcal{U}$ such that $O(q_1) \times O(q_2)$ does not meet W. If we take a symmetric entourage $U \in \mathcal{U}$ such that $U \circ U \subset W$, then either $U[x] \cap O(q_1) = \emptyset$ or $U[x] \cap O(q_2) = \emptyset$. For, if $a \in U[x]$ and $b \in U[x]$ are such that $(a,b) \in O(q_1) \times O(q_2)$, then $(x,a) \in U$ and $(x,b) \in U$, which implies that $(a,b) \in U \circ U \subset W$, which is a contradiction. Thus, in either case we get a periodic point q such that $U[x] \cap O(q) = \emptyset$. □

Theorem 4.2.8. *Let f be a uniformly continuous self-map on a Hausdorff uniform space $(X,\mathcal{U})$. If $x \in X$ is topologically transitive point of f and f has dense set of periodic points at x, then x is a sensitive point of f.*

Proof. Since f has dense set of periodic points at x, using Lemma 4.2.7 there exist an entourage $U \in \mathcal{U}$ and a periodic point q such that $U[x] \cap O(q) = \emptyset$. For $U \in \mathcal{U}$, let $V \in \mathcal{U}$ be an entourage such that $V \circ V \circ V \circ V \subset U$. We claim that V is a sensitivity entourage for f at x. Since f has a dense set of periodic points at x, for the neighborhood $\dot{U} = V[x]$ of x, the set $\dot{U} \setminus \{x\}$ contains a periodic point p. Note that $(x,p) \in V$. Let n denote the period of p and let $\dot{V} = \bigcap_{i=0}^{n} f^{-i}(V[f^i(q)])$. Clearly, $\dot{V}$ is a nonempty open subset of X. Since x is a topologically transitive point, there exist y in $\dot{U}$ and a positive integer k such that $f^k(y) \in \dot{V}$. Let j be the integer part of $(\frac{k}{n}+1)$. Then $1 \leq nj-k \leq n$ and

$$f^{nj}(y) = f^{nj-k}(f^k(y)) \in f^{nj-k}(\dot{V}) \subseteq V[f^{nj-k}(q)].$$

This implies that $(f^{nj}(y), f^{nj-k}(q)) \in V$. Then, either $(f^{nj}(x), f^{nj}(p)) = (f^{nj}(x), p) \notin V$ or $(f^{nj}(x), f^{nj}(y)) \notin V$. For, if both $(f^{nj}(x), p)$ and $(f^{nj}(x), f^{nj}(y))$ are in V, then $(p, f^{nj}(y)) \in V \circ V$. Consequently, $(p, f^{nj-k}(q)) \in V \circ V \circ V$ implying that $(x, f^{nj-k}(q)) \in U$. Thus $f^{nj-k}(q) \in U[x]$, which contradicts the fact that

$U[x] \cap O(q) = \emptyset$. In either case, we find a point $z \in \dot{U}$ such that $(f^{nj}(x), f^{nj}(z)) \notin V$ ($z = p$ or y). Hence x is a sensitive point of f. □

Theorem 4.2.9. *Let f be a uniformly continuous surjective self-map on a Hausdorff uniform space $(X,\mathcal{U})$. If f has pointwise topological periodic specification property, then f is Devaney chaotic.*

Proof. Since f has pointwise topological periodic specification property, each point x in X is a topological periodic specification point. From Lemma 4.2.2, it follows that each point x in X is a mixing point, and hence a transitive point. Moreover from Lemma 4.2.2, it follows that f has dense set of periodic points at each point x in X. From Theorem 4.2.8 we get that each point x in X is a sensitive point and thus each point x in X is a Devaney chaotic point of f. Hence f is Devaney chaotic. □

We now give the definitions of topological expansive point and topological shadowable point for a uniformly continuous surjective self-map f defined on a uniform space $(X,\mathcal{U})$.

Definition 4.2.10. A point $x \in X$ is said to be a *topological expansive point* if there exists an entourage $U_x \in \mathcal{U}$ such that for each $y \neq x$, we have $(f^n(x), f^n(y)) \notin U_x$, for some positive integer n. The entourage U_x is called an expansivity entourage corresponding to x.

Recall that, for an entourage $D \in \mathcal{U}$, a sequence $\{x_i\}_{i=0}^{\infty}$ in X is said to be a D-pseudo orbit for f if $(f(x_i), x_{i+1}) \in D$, for all $i \in \mathbb{N}_0$. Further if $x_0 = x$ then $\{x_i\}_{i=0}^{\infty}$ is said to be a *D-pseudo orbit through x*. For $y \in X$ and an entourage $E \in \mathcal{U}$, the sequence $\{x_i\}_{i=0}^{\infty}$ in X is said to be E-shadowed by y if $(f^i(y), x_i) \in E$, for all $i \in \mathbb{N}_0$.

Definition 4.2.11. A point $x \in X$ is said to be a *topological shadowable point* if for every entourage $E \in \mathcal{U}$ there is an entourage $D \in \mathcal{U}$ such that every D-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ through x is E-shadowed by a point of X.

Theorem 4.2.12. *Let f be a uniformly continuous surjective self-map on a totally bounded uniform space $(X,\mathcal{U})$. If f is mixing and a point $x \in X$ is a topological expansive point and a topological shadowable point for f, then x is a topological periodic specification point of f.*

Proof. Let U_x be an expansivity entourage corresponding to x. Choose an entourage $E \in \mathcal{U}$ such that $E \circ E \subset U_x$. Since x is a topological shadowable point, for $E \in \mathcal{U}$ there exists an entourage $D \in \mathcal{U}$ such that every D-pseudo orbit through x is E-traced by some point of X. Since X is totally bounded, for $D \in \mathcal{U}$ there exists a finite subset $\dot{F} = \{v_1, v_2, \dots, v_m\}$ of X such that $X = \bigcup_{i=1}^{m} D[v_i]$. Since f is mixing, there exists a positive integer M such that $f^n(D[v_i]) \cap D[v_j] \neq \emptyset$, for all $n \geq M$ and for any i, $j \in \{1, 2, \dots, m\}$. Choose a finite sequence $x = x_1, x_2, \dots, x_k$ in X, integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_{j+1} - b_j \geq M$, for all $1 \leq j \leq k$, and a positive integer $p > M + (b_k - a_1)$. Set $a_{k+1} = b_{k+1} = p$, $x_{k+1} = f^{-p}(x)$. For any $z \in X$, we denote by $D(z)$ a set in $\{D[v_i] | i = 1, 2, \dots, n\}$ containing z. Since $a_{j+1} - b_j \geq M$ and f is mixing, we have $f^{a_{j+1}-b_j}(D(f^{b_j}(x_j))) \cap D(f^{a_{j+1}}(x_{j+1})) \neq \emptyset$. Choose $y_j \in D(f^{b_j}(x_j))$ such that $f^{a_{j+1}-b_j}(y_j) \in D(f^{a_{j+1}}(x_{j+1}))$. We define a sequence $\{z_i\}$ in X as follows:

$$z_i = \begin{cases} f^i(x), & 0 \leq i \leq a_1; \\ f^i(x_j), & a_j \leq i \leq b_j; \\ f^{i-b_j}(y_j), & b_j \leq i < a_{j+1} \end{cases}$$

and $z_{i+p} = z_i$ for all $i \in \mathbb{N}_0$. It is easy to observe that $\{z_i\}$ is a D-pseudo orbit of f through x. Since x is a topological shadowable point, there exist a $w \in X$ such that $(f^i(w), z_i) \in E$, for all $i \in \mathbb{N}_0$. Also $z_{i+p} = z_i$, $i \in \mathbb{N}_0$ implies that $(f^i(f^p(w)), z_i) \in E$ and hence $(f^i(f^p(w)), f^i(w)) \in U_x$, for all $i \in \mathbb{N}_0$. Thus, we have $f^p(w) = w$. Hence x is a topological periodic specification point. □

Remark 4.2.13. In proof of Theorem 4.2.12, expansiveness of point is used to guarantee that the obtained tracing point is periodic.

Corollary 4.2.14. *Let f be a uniformly continuous surjective self-map on a totally bounded uniform space $(X,\mathcal{U})$. If f is mixing and a point $x \in X$ is topological shadowable point for f, then x is a topological specification point of f.*

The following result follows from Theorem 4.1.7 and Theorem 4.2.12.

Corollary 4.2.15. *Let f be a self-homeomorphism on a totally bounded uniform space $(X,\mathcal{U})$. Suppose f is mixing and there is a point $x \in X$ which is both topological expansive point and topological shadowable point for f, then f has topological periodic specification property.*

Next, we relate the notion of topological specification point with the notion of uniform entropy. First we recall here the definition of uniform entropy. Let f be a uniformly continuous self-map on a uniform space $(X,\mathcal{U})$, and let $\mathcal{U}^s$ denote the set of symmetric entourages. For a positive integer n and any symmetric entourage $U \in \mathcal{U}^s$, a subset $\dot{E}$ of X is said to be (n,U)-separated with respect to f if for each pair of distinct points x, y in $\dot{E}$ there exists j such that $0 \leq j < n$ and $(f^j(x), f^j(y)) \notin U$. Let $\mathcal{K}(X)$ denote the set of all compact subsets of X. For $\dot{K} \in \mathcal{K}(X)$, let $s_n(U, \dot{K}, f)$ denote the maximal cardinality of (n,U)-separated sets contained in $\dot{K}$. Define

$$\overline{s}_f(U, \dot{K}) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(U, \dot{K}, f)$$

Note that, the collection $\mathcal{U}^s$ forms a directed set under set inclusion, and hence $\overline{s}_f(U, \dot{K})$, $U \in \mathcal{U}^s$ forms a net in $\mathbb{R}^+$.

Define $h(f, \dot{K}, \mathcal{U}) = \lim\limits_{U\in\mathcal{U}^s} \overline{s}_f(U, \dot{K})$. The *uniform entropy* of f with respect to the uniformity $\mathcal{U}$ is given by

$$h(f,\mathcal{U}) = \sup_{\dot{K}\in\mathcal{K}(X)} h(f, \dot{K}, \mathcal{U}).$$

For more details one can refer [26].

Theorem 4.2.16. *Let $f : X \to X$ be a uniformly continuous surjective self-map on a uniform space $(X,\mathcal{U})$. If f has two distinct topological specification points, then f has positive uniform entropy.*

Proof. Let $x, y \in X$ be two distinct topological specification points of f and let U be a symmetric neighborhood of Δ_X such that $(x, y) \notin U \circ U \circ U$. Let $M = \max\{M_x, M_y\}$, where M_x and M_y are numbers as in the definition of topological specification point corresponding to x and y, respectively. Consider any n-tuple, $(z_1, z_2, \ldots, z_n)$ with $z_i \in \{x, y\}$, $1 \leq i \leq n$. Then for $a_1 = b_1 = M$, $a_2 = b_2 = 2M$, $\ldots$, $a_n = b_n = nM$, as x, y are both topological specification points, there exists a point $z \in X$ such that $(f^{iM}(z), z_i) \in U$, for all $1 \leq i \leq n$.

We claim that for distinct n-tuples there corresponds different z. Consider any two distinct n-tuples, $(z_1, z_2, \ldots, z_n)$ and $(z'_1, z'_2, \ldots, z'_n)$ with $z_i, z'_i \in \{x, y\}$, $1 \leq i \leq n$. Then $z_k \neq z'_k$ for some $k \in \{1, 2, \ldots, n\}$. Suppose $z_k = x$ and $z'_k = y$. Let $z, z' \in X$ be the tracing points corresponding to the above given n-tuples. Then $(f^{iM}(z), f^{iM}(z_i)) \in U$ and $(f^{iM}(z'), f^{iM}(z'_i)) \in U$, for all $1 \leq i \leq n$. If $z = z'$, then for $i = k$ we get $(f^{kM}(z), x) \in U$ and $(f^{kM}(z), y) \in U$ implying that $(x, y) \in U \circ U \subset U \circ U \circ U$, which is a contradiction. Thus $z \neq z'$. Moreover, z and z' are (nM, U)-separated. For, if $(f^{kM}(z), f^{kM}(z')) \in U$, then we obtain that $(x, y) \in U \circ U \circ U$, which is a contradiction. Thus there are at least 2^n points which are (nM, U)-separated. Hence

$$\begin{aligned}
h(f, \mathcal{U}) &= \sup\{h(f, \dot{K}, \mathcal{U}) | \dot{K} \in \mathcal{K}(X)\} \\
&\geq \lim_{U \in \mathcal{U}} \overline{s}_f(U, \dot{K}) \\
&= \lim_{n \to \infty} \sup \frac{1}{n} \log s_n(U, \dot{K}, f) \\
&= \lim_{n \to \infty} \sup \frac{1}{nM} \log 2^n \\
&= \frac{\log 2}{M} > 0.
\end{aligned}$$

□

Corollary 4.2.17. *Let f be a uniformly continuous surjective self-map on a totally bounded uniform space $(X, \mathcal{U})$. If f is mixing having two distinct topological shadowable points for f, then f has positive uniform entropy.*

Proof. By Corollary 4.2.14, f has two distinct topological specification points hence by Theorem 4.2.16, f has positive uniform entropy. $\square$

4.3 Pointwise specification under orbital convergence

In this section, for orbitally convergent sequence of uniformly continuous self-maps defined on uniform spaces, the necessary and sufficient condition for a point to exhibit topological specification is discussed.

Let $(X, \mathcal{U})$ be a uniform space and let $UC(X)$ denote the set of all uniformly continuous self-maps on X. Let $f, f_n \in UC(X)$, for each $n \in \mathbb{N}$. Recall that, f_n is uniformly convergent to f or $f_n \xrightarrow{uc} f$, if for every $E \in \mathcal{U}$ there exists an $N \in \mathbb{N}$ such that $(f_n(x), f(x)) \in E$, for all $n \geq N$ and for each $x \in X$.

We now define the notions of convergence for sequences of functions defined on uniform spaces. Let $f, f_n \in UC(X)$, for each $n \in \mathbb{N}$. Then:

(i) f_n is *orbitally convergent* to f or $f_n \xrightarrow{oc} f$, if for every $E \in \mathcal{U}$ there exists an $N \in \mathbb{N}$ such that $(f_n^k(x), f^k(x)) \in E$, for all $n \geq N$, for each $x \in X$ and for each $k \in \mathbb{N}$.

(ii) f_n is *weak orbitally convergent* to f or $f_n \xrightarrow{woc} f$, if for every $k \in \mathbb{N}$ and $E \in \mathcal{U}$, there exists an $N(E, k) = N \in \mathbb{N}$ such that $(f_n^k(x), f^k(x)) \in E$, for all $n \geq N$ and for each $x \in X$.

(iii) f_n is *pointwise weak orbitally convergent* to f or $f_n \xrightarrow{pwoc} f$, if for every $x \in X$, $k \in \mathbb{N}$ and $E \in \mathcal{U}$, there exists an $N(x, E, k) = N \in \mathbb{N}$ such that $(f_n^k(x), f^k(x)) \in E$, for all $n \geq N$.

We set $\mathrm{TSP}(f, x, E) = \{M \in \mathbb{N} : M$ corresponds to E in the definition of topological specification point $x\}$. Recall that, $\mathrm{TSP}(f)$ denotes the set of all topological specification points of f.

Theorem 4.3.1. *Let $\{f_n\}$ be a sequence of uniformly continuous surjective self-maps on a uniform space $(X,\mathcal{U})$. If sequence $\{f_n\}$ is orbitally convergent to $f \in UC(X)$, then $x \in \mathrm{TSP}(f)$ if and only if for every $E \in \mathcal{U}$, $\cup_{m\geq 1}\cap_{n\geq m}\mathrm{TSP}(f_n, x, E) \neq \emptyset$.*

Proof. Suppose that $x \in\mathrm{TSP}(f)$. For $E \in \mathcal{U}$, choose an entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subset E$. Since $f_n \xrightarrow{oc} f$, for $V \in \mathcal{U}$ there exists $N \in \mathbb{N}$ such that $(f_n^m(y), f^m(y)) \in V$, for all $n \geq N$, for each $y \in X$ and for each $m \in \mathbb{N}$. Now for $V \in \mathcal{U}$, choose $M \in \mathbb{N}$ as in the definition of topological specification point corresponding to x. Then for any finite sequence $x = x_1, x_2, \ldots, x_k$ in X, and any set of integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_j - b_{j-1} \geq M$, for all $2 \leq j \leq k$, there exists $y \in X$ such that $(f^i(y), f^i(x_j)) \in V$, for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. Clearly then, $(f_n^i(y), f_n^i(x_j)) \in E$, for all $a_j \leq i \leq b_j$, for all $1 \leq j \leq k$ and for all $n \geq N$. Hence $M \in \cap_{n\geq N}\mathrm{TSP}(f_n, x, E)$, which implies that $\cup_{m\geq 1}\cap_{n\geq m}\mathrm{TSP}(f_n, x, E) \neq \emptyset$.

Conversely, suppose that $\cup_{m\geq 1}\cap_{n\geq m}\mathrm{TSP}(f_n, x, E) \neq \emptyset$, for all $E \in \mathcal{U}$. Given $E \in \mathcal{U}$, choose an entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subset E$. Since $f_n \xrightarrow{oc} f$, for $V \in \mathcal{U}$ there exists $N \in \mathbb{N}$ such that $(f_n^m(y), f^m(y)) \in V$, for all $n \geq N$, for each $y \in X$ and for each $m \in \mathbb{N}$. Now, since $\cup_{m\geq 1}\cap_{n\geq m}\mathrm{TSP}(f_n, x, V) \neq \emptyset$, we can choose $K, M \in \mathbb{N}$ such that for all $n \geq K$, any finite sequence $x = x_1, x_2, \ldots, x_k$ in X, and any set of integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_j - b_{j-1} \geq M$, for all $2 \leq j \leq k$, there exists $y_n \in X$ such that $(f_n^i(y_n), f_n^i(x_j)) \in V$, for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. Set $q = \max\{N, K\}$. Then

$$(f^i(y_q), f_q^i(y_q)) \circ (f_q^i(y_q), f_q^i(x_j)) \circ (f_q^i(x_j), f^i(x_j)) = (f^i(y_q), f^i(x_j)) \in V \circ V \circ V,$$

for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. That is $(f^i(y_q), f^i(x_j)) \in E$, for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq k$. Hence $x \in\mathrm{TSP}(f)$. □

The following result follows from Theorem 4.1.7.

Corollary 4.3.2. *Let $\{f_n\}$ be a sequence of self-homeomorphisms on a uniform space $(X,\mathcal{U})$. If sequence $\{f_n\}$ is orbitally convergent to f, then f has topological* *specification property if and only if for every* $E \in \mathcal{U}$, $\cup_{m\geq 1} \cap_{n\geq m} \mathrm{TSP}(f_n, x, E) \neq \emptyset$, *for some* $x \in X$.

The above theorem suggests that the condition $\cup_{m\geq 1}\cap_{n\geq m}\mathrm{TSP}(f_n, x, E) \neq \emptyset$ is necessary and sufficient condition for the point $x \in X$ to be a topological specification point of f, whenever f_n is orbitally convergent to f. Note that the condition $\cup_{m\geq 1}\cap_{n\geq m}\mathrm{TSP}(f_n, x, E) \neq \emptyset$ is much stronger and thus serves as a sufficient condition for the point $x \in X$ to be a topological specification point of f, even when f_n is weak orbitally convergent to f or when f_n is pointwise weak orbitally convergent to f.

Theorem 4.3.3. *Let* $\{f_n\}$ *be a sequence of uniformly continuous surjective self-maps on a uniform space* $(X,\mathcal{U})$. *If sequence* $\{f_n\}$ *is weak orbitally convergent to* $f \in UC(X)$ *and for every* $E \in \mathcal{U}$, $\cup_{m\geq 1} \cap_{n\geq m} \mathrm{TSP}(f_n, x, E) \neq \emptyset$, *then* $x \in \mathrm{TSP}(f)$.

Proof. For $E \in \mathcal{U}$, choose an entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subset E$. Now, since $\cup_{m\geq 1}\cap_{n\geq m}\mathrm{TSP}(f_n, x, V) \neq \emptyset$, we can choose $K, M \in \mathbb{N}$ such that for all $n \geq K$, any finite sequence $x = x_1, x_2, \ldots, x_r$ in X, and any set of integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_r \leq b_r$ with $a_j - b_{j-1} \geq M$, for all $2 \leq j \leq r$, there exists $y_n \in X$ such that $(f_n^i(y_n), f_n^i(x_j)) \in V$, for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq r$. Now $f_n \xrightarrow{woc} f$, thus for $V \in \mathcal{U}$ and for any $a_1 \leq k \leq b_r$ there exists $N_k \in \mathbb{N}$ such that $(f_n^k(y), f^k(y)) \in V$, for all $n \geq N_k$ and for each $y \in X$. Choose $N = \max\{N_k \mid a_1 \leq k \leq b_r\}$. Thus $(f_n^k(y), f^k(y)) \in V$, for all $a_1 \leq k \leq b_r$, for all $n \geq N$ and for each $y \in X$. Set $q = \max\{N, K\}$. Then

$$(f^i(y_q), f^i(x_j)) = (f^i(y_q), f_q^i(y_q)) \circ (f_q^i(y_q), f_q^i(x_j) \circ (f_q^i(x_j), f^i(x_j)) \in V \circ V \circ V \subset E,$$

for all $a_j \leq i \leq b_j$ and for all $1 \leq j \leq r$. Hence $x \in$TSP(f). □

The following result follows from Theorem 4.1.7.

Corollary 4.3.4. *Let* $\{f_n\}$ *be a sequence of self-homeomorphisms on a uniform space* $(X,\mathcal{U})$. *If sequence* $\{f_n\}$ *is weak orbitally convergent to* f, *and for some* $x \in X$, $\cup_{m\geq 1} \cap_{n\geq m} \mathrm{TSP}(f_n, x, E) \neq \emptyset$, *for every* $E \in \mathcal{U}$, *then* f *has topological specification property.*

4.4 Weaker forms of topological specification property

In this section, we define and study some weaker notions of topological specification property for uniformly continuous surjective self-maps on uniform spaces, namely topological quasi-weak specification property, topological semi-weak specification property and topological periodic quasi-weak specification property.

Let $(X, \mathcal{U})$ be a uniform space and f a uniformly continuous self-map on X.

Definition 4.4.1. A map f is said to have *topological quasi-weak specification property* (briefly TQWSP) if for every entourage $E \in \mathcal{U}$, there exists a positive integer M such that for any two points x_1, x_2 and $n \geq M$, there is a point $x \in X$ such that $(x, x_1) \in E$ and $(f^n(x), f^n(x_2)) \in E$.

Definition 4.4.2. A map f is said to have *topological semi-weak specification property* (briefly TSWSP) if for every entourage $E \in \mathcal{U}$, there is a positive integer M such that for any two points x_1, x_2 and any sequence of integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2$ with $a_2 - b_1 \geq M$ there is a point $x \in X$ such that $(f^i(x), f^i(x_j)) \in E$, for some integer i with $a_j \leq i \leq b_j$, for each positive integer $j = 1, 2$.

Remark 4.4.3. Note that TSWSP implies TQWSP.

Proposition 4.4.4. *Let f and g be uniformly continuous surjective self-maps on uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$, respectively. Suppose that f and g are topologically conjugate. Then f has TQWSP implies g has TQWSP.*

Proof. Since f and g are topologically conjugate, there exists a uniform homeomorphism $\varphi : X \to Y$ such that $g \circ \varphi = \varphi \circ f$. Now for any entourage $E \in \mathcal{V}$, $\varphi : Y \to X$ is a uniform homeomorphism implies $U = \Phi^{-1}(E) \in \mathcal{U}$, where $\Phi = \varphi \times \varphi$. For $U \in \mathcal{U}$, choose a positive integer M as in the definition of TQWSP. Then for any two points $y_1, y_2 \in Y$ and $n \geq M$, consider the points $x_1 = \varphi^{-1}(y_1), x_2 = \varphi^{-1}(y_2)$ in X. Now f has TQWSP implies that there exists $x \in X$ such that $(x, x_1) \in U$ and

$(f^n(x), f^n(x_2)) \in U$. Then for $y = h(x)$ in Y, $(y, y_1) = (\varphi(x), \varphi(x_1)) \in \Phi(U) = E$ and $(g^n(y), g^n(y_2)) = (g^n(\varphi(x)), g^n(\varphi(x_2))) = (\varphi(f^n(x)), \varphi(f^n(x_2))) \in \Phi(U) = E$. Hence g has TQWSP. $\square$

Proposition 4.4.5. *Let f and g be uniformly continuous surjective self-maps on uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$, respectively. If f and g has TQWSP, then $f \times g : X \times Y \to X \times Y$ has TQWSP.*

Proof. Let W be any symmetric neighborhood of the diagonal $\Delta_{X\times Y}$. Define

$$U = \{(x, x') \in X \times X | (x, y, x', y') \in W, \text{ for some } y,\ y' \in Y\},$$
$$V = \{(y, y') \in Y \times Y | (x, y, x', y') \in W, \text{ for some } x,\ x' \in X\}.$$

Then U and V are symmetric neighborhoods of Δ_X and Δ_Y, respectively. Take $M = \max\{M_1, M_2\}$, where M_1 and M_2 are numbers as in the definition of TQWSP, respectively.

For any two points (x_1, y_1), (x_2, y_2) in $X \times Y$ and $n \geq M$. Since f and g has TQWSP, there exist $x \in X$ and $y \in Y$ such that $(x, x_1) \in U$, $(f^n(x), f^n(x_2)) \in U$ and $(y, y_1) \in V$, $(g^n(y), g^n(y_2)) \in V$. Then (x, y) is a point in $X \times Y$ such that $((x, y), (x_1, y_1)) \in W$ and $((f \times g)^n(x, y), (f \times g)^n(x_2, y_2)) \in W$. Hence $f \times g$ has TQWSP. $\square$

Proposition 4.4.6. *Let f be uniformly continuous surjective self-map on uniform spaces $(X,\mathcal{U})$. If f has TQWSP then f^m has TQWSP, for any $m \geq 2$.*

Proof. Let U be any symmetric entourage in $\mathcal{U}$. Let M' be the numbers as in the definition of TQWSP. For any $m \geq 2$, take $M = \left[\frac{M'}{m}\right]$, where $[\cdot]$ denotes the greatest integer function. Then for any $x_1, x_2 \in X$ and any $n > M$, we have $mn > M'$. Since f has TQWSP, there exists $x \in X$ such that $(x, x_1) \in U$ and $(f^n m(x), f^n m(x_2)) = ((f^m)^n(x), (f^m)^n(x_2)) \in U$. Hence f^m has TQWSP, for any $m \geq 2$. $\square$

Remark 4.4.7. Proposition 4.4.4-4.4.6 also holds for maps having TSWSP.

Lemma 4.4.8. *Let f be a uniformly continuous surjective self-map on a totally bounded uniform space $(X, \mathcal{U})$. Then f has* TQWSP *if and only if f is mixing.*

Proof. Let $\dot{U}$, $\dot{V} \subset X$ be any two nonempty open sets in X. Then for $x \in \dot{U}$, $y \in \dot{V}$, there exist E_1, $E_2 \in \mathcal{U}$ such that $E_1[x] \subset \dot{U}$ and $E_2[y] \subset \dot{V}$. Let $E = E_1 \cap E_2 \in \mathcal{U}$. Then $E[x] \subset \dot{U}$ and $E[y] \subset \dot{V}$. Now for $E \in \mathcal{U}$, choose a positive integer M as in the definition of TQWSP. Since f is surjective, for any $n \geq M$, there exists $z \in X$ such that $y = f^n(z)$. For points x, z, by definition of TQWSP, there exists a point $w \in X$ such that $(w, x) \in E$ and $(f^n(w), f^n(z)) \in E$. But $(w, x) \in E$ implies $w \in E[x] \subset \dot{U}$ and $(f^n(w), f^n(z)) \in E$ implies $f^n(w) \in E[y] \subset \dot{V}$, and hence $f^n(\dot{U}) \cap \dot{V} \neq \emptyset$, for all $n \geq M$. Thus, f is mixing.

Conversely, for given $E \in \mathcal{U}$, choose $V \in \mathcal{U}$ such that $V \circ V \subset E$. Since X is totally bounded, for $V \in \mathcal{U}$ there exists a finite subset $\dot{F} = \{x_1, x_2, \dots, x_m\}$ of X such that $X = \bigcup_{i=1}^{m} V[x_i]$. Also, f is mixing implies that there exists a positive integer M such that

$$f^n(V[x_i]) \cap V[x_j] \neq \emptyset, \tag{4.1}$$

for all $n \geq M$ and for any i, $j \in \{1, 2, \dots, m\}$. Choose $y_1, y_2 \in X$, and for any $n \geq M$, let $y = f^n(y_2)$. Since $X = \bigcup_{i=1}^{m} V[x_i]$, $y_1 \in V[x_k]$ and $y \in V[x_l]$, for some x_k, $x_l \in \dot{F}$. By 4.1, there exists a point $x \in X$ such that $x \in V[x_k]$ and $f^n(x) \in V[x_l]$. Then $(x, x_k) \in V$ and $(x_k, y_1) \in V$ imply that $(x, y_1) \in V \circ V \subset E$. Also, $(f^n(x), x_l) \in V$ and $(x_l, y) \in V$ imply that $(f^n(x), y) = (f^n(x), f^n(y_2)) \in V \circ V \subset E$. Therefore, f has TQWSP. □

Theorem 4.4.9. *Let f be a uniformly continuous surjective self-map on a totally bounded uniform space $(X, \mathcal{U})$. Then the following statements are equivalent:*

(i) f is mixing;

(ii) f has TQWSP*;*

(iii) f has TSWSP*.*

Proof. By Lemma 4.4.8, f has TQWSP if and only if f is mixing. Also f has TSWSP implies f has TQWSP, and hence f is mixing. Thus we only need to show that if f has TQWSP, then f has TSWSP.

Suppose that f has TQWSP. For any $E \in \mathcal{U}$, let M be a positive integer as in the definition of TQWSP. Consider any two points $x, y \in X$ and a sequence $0 \leq a_1 \leq b_1 < a_2 \leq b_2$ with $a_2 - b_1 \geq M$. Choose x_1, y_1 such that $x_1 = f^{a_1}(x)$ and $y_1 = f^{a_1}(y)$, and let $n = a_2 - a_1 \geq a_2 - b_1 \geq M$. Since f has TQWSP, there exists $z \in X$ such that $(z, x_1) \in E$ and $(f^n(z), f^n(y_1)) \in E$. Further, f is surjective implies that there exists $z_1 \in X$ such that $f^{a_1}(z_1) = z$. Thus $(z, x_1) \in E$ implies $(f^{a_1}(z_1), f^{a_1}(x)) \in E$ and $(f^n(z), f^n(y_1)) \in E$ implies $(f^{n+a_1}(z_1), f^{n+a_1}(y)) = (f^{a_2}(z_1), f^{a_2}(y)) \in E$. Hence f has TSWSP. □

In [72, Theorem 3.1], the authors constructed an example of a system which has quasi-weak specification property but does not have specification property. From the example, it is clear that the notion of TQWSP is weaker than TSP. We define the notion of topological periodic quasi-weak specification property in the following manner:

Definition 4.4.10. A map f is said to have *topological periodic quasi-weak specification property* (briefly TPQWSP) if for every entourage $E \in \mathcal{U}$, there exists a positive integer M such that for any two points x_1, x_2 in X, any integer $n \geq M$ and any $p > 2M$, there exists a point $x \in X$ such that $f^p(x) = x$, $(x, x_1) \in E$, and $(f^n(x), f^n(x_2)) \in E$.

In [54], authors proved that self-homeomorphisms on uniform spaces having TSP admits Devaney chaos. We prove here that for uniformly continuous surjective self-maps on totally bounded uniform spaces, TPQWSP implies Devaney chaos. Clearly, TPQWSP implies TQWSP and hence we have the following result.

Proposition 4.4.11. *Let f be a uniformly continuous surjective self-map on a totally bounded uniform space $(X, \mathcal{U})$. If f has* TPQWSP *then f is mixing.*

Proposition 4.4.12. *Let f be a uniformly continuous surjective self-map on a uniform space $(X,\mathcal{U})$. If f has* TPQWSP *then f has dense set of periodic points.*

Proof. Let $\dot{U}$ be any nonempty open subset of X and let $x \in \dot{U}$. Choose an entourage $E \in \mathcal{U}$ such that $E[x] \subset \dot{U}$. For $E \in \mathcal{U}$, let M be a positive integer as in the definition of TPQWSP. Then for points $x = x_1$, x_2 in X, for any $n \geq M$, and any $p > 2M$, there exists a point $z \in X$ such that $z = f^p(z)$, $(z,x) \in E$ and $(f^n(z), f^n(x_2)) \in E$. But $(z,x) \in E$ implies $z \in E[x] \subset \dot{U}$. Thus, every open set $\dot{U}$ of X contains a periodic point. Hence, the set of periodic points of f is dense in X. □

Corollary 4.4.13. *Let f be a self-homeomorphism on a totally bounded Hausdorff uniform space $(X,\mathcal{U})$. If f has* TPQWSP*, then f is Devaney chaotic.*

Proof. From 4.4.11, we have that f is topologically mixing and hence transitive. Further by 4.4.12, f admits dense set of periodic points. By [61, Theorem 1], such an f is sensitive. Hence f is Devaney chaotic. □

For a uniformly continuous surjective self-map f on a uniform space $(X,\mathcal{U})$, we can also define the notions of topological quasi-weak specification point and topological semi-weak specification point in a similar manner.

Definition 4.4.14. A point $x \in X$ is said to be a *topological quasi-weak specification point* of f if for every $E \in \mathcal{U}$ there exists a positive integer M_x such that for $x_1 = x$, $x_2 \in X$ and any $n \geq M_x$, there exists a point $y \in X$ such that $(y, x_1) \in E$ and $(f^n(y), f^n(x_2)) \in E$.

Definition 4.4.15. A point $x \in X$ is said to be a *topological semi-weak specification point* of f if for every $E \in \mathcal{U}$ there exists a positive integer M_x such that for $x_1 = x$, $x_2 \in X$ and any sequence of integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2$ with $a_2 - b_1 \geq M_x$ there exists a point $y \in X$ such that $(f^i(y), f^i(x_j)) \in E$, for some integer i with $a_j \leq i \leq b_j$, for each positive integer $j = 1, 2$.

We denote the set of all topological quasi-weak specification points of f, topological semi-weak specification points of f by $\mathrm{TQWSP}(f)$, $\mathrm{TSWSP}(f)$, respectively. Following results can be proved using the arguments similar to those given in proof of Lemma 4.4.8 and Theorem 4.4.9, respectively.

Lemma 4.4.16. *Let f be a uniformly continuous surjective self-map on a totally bounded uniform space $(X, \mathcal{U})$. Then $x \in \mathrm{TQWSP}(f)$ if and only if $x \in \mathrm{M}(f)$.*

Lemma 4.4.17. *Let f be a uniformly continuous surjective self-map on a totally bounded uniform space $(X, \mathcal{U})$. Then for a point $x \in X$,*

$$x \in \mathrm{M}(f) \Leftrightarrow x \in \mathrm{TQWSP}(f) \Leftrightarrow x \in \mathrm{TSWSP}(f).$$