Chapter 3

Distributional chaos in a sequence on uniform spaces

Distributional chaos is a stronger form of chaos than Devaney and Li-Yorke chaos; while Devaney and Li-Yorke focus on the qualitative behavior of orbits, distributional chaos looks deeper into the statistical distribution of distances between points over time. Note that, there are examples justifying that Li-Yorke chaos and Devaney chaos need not imply any version of distributional chaos [48, 40]. Also, Oprocha gave an example of a map which is weakly mixing but not distributionally chaotic [40]. In 2007, Wang et al. introduced a generalized version of distributional chaos by considering the distribution function of distance between trajectories in terms of a given sequence of time, popularly known as distributional chaos in a sequence [65]. It is proved that both Devaney chaos and weakly mixing imply distributional chaos in sequence for self-maps defined on metric spaces [37, 38]. Moreover, for continuous self-maps defined on the intervals, it is proved that Li-Yorke chaos and distributional chaos in a sequence are equivalent.

In this chapter, we introduce and study the topological notion of distributional chaos in a sequence defined for uniformly continuous self-map on Hausdorff uniform space. In Section 3.1, we describe an alternate definition for Li-Yorke scrambled pair, and study the relation between notions of weakly mixing and Li-Yorke chaos for uniformly continuous self-map defined on a second countable Baire Hausdorff uniform space without isolated points. In Section 3.2, we define and study the notion of topological distributional chaos in a sequence for uniformly continuous self-maps defined on Hausdorff uniform spaces. In Section 3.3, we prove the equivalence of Li-Yorke chaos and topological distributional chaos in a sequence for uniformly continuous self-maps defined on second countable Baire Hausdorff uniform space without isolated points. As a consequence, we obtain that Devaney chaos implies topological distributional chaos in a sequence for maps on uniform spaces. Results presented in this chapter are published in the Turkish Journal of Mathematics [70].

Throughout this chapter, by a dynamical system, we mean a pair (X, f), where $(X, \mathcal{U})$ is a Hausdorff uniform space without isolated points and $f : X \to X$ is a uniformly continuous map. We denote the product map $f \times f$ on $X \times X$ by F and by $F^i(x, y)$ we mean $(f^i(x), f^i(y))$.

3.1 Li-Yorke chaos on uniform spaces

In this section, we study the topological notion of Li-Yorke chaos for a uniformly continuous self-map defined on a Hausdorff uniform space. In [6], the author introduced the notion of Li-Yorke chaos for group actions on uniform spaces. If we consider, the $\mathbb{Z}$-action induced by a uniformly continuous self-map f defined on Hausdorff uniform space $(X, \mathcal{U})$, then the set of proximal pairs, the set of asymptotic pairs, and the set of distal pairs with respect to f are denoted by PR, AR, and DR, respectively, and are defined as follows:

$$\begin{aligned} PR &= \{(x, y) \in X \times X \mid \forall E \in \mathcal{U}, \exists\, i \in \mathbb{N} \text{ such that } F^i(x, y) \in E\}, \\ AR &= \{(x, y) \in X \times X \mid \forall E \in \mathcal{U}, \exists\, k \in \mathbb{N} \text{ such that } F^i(x, y) \in E, \forall i \geq k\}, \\ DR &= X \times X \backslash PR \\ &= \{(x, y) \in X \times X \mid \exists E \in \mathcal{U} \text{ such that } F^i(x, y) \notin E, \forall i \in \mathbb{N}\}. \end{aligned}$$

A subset S of a uniform space X is said to be a *Li-Yorke scrambled set for f* if for any pair of distinct elements x, $y \in S$, $(x, y) \in PR \backslash AR$. A map f is said to be

Li-Yorke chaotic if there exists an uncountable Li-Yorke scrambled set for f.

For an increasing sequence $\{p_i\}$ of positive integers we define proximal relation $PR(f, \{p_i\})$, and asymptotic relation $AR(f, \{p_i\})$ with respect to sequence $\{p_i\}$, respectively as follows:

$$PR(f, \{p_i\}) = \{(x, y) \in X \times X \mid \forall E \in \mathcal{U}, \exists\, i \in \mathbb{N} \text{ such that } F^{p_i}(x, y) \in E\},$$
$$AR(f, \{p_i\}) = \{(x, y) \in X \times X \mid \forall E \in \mathcal{U}, \exists\, k \in \mathbb{N} \text{ such that } F^{p_i}(x, y) \in E, \forall i \geq k\}.$$

The set $X \times X \backslash PR(f, \{p_i\})$ is denoted by $DR(f, \{p_i\})$ and is called the distal relation with respect to sequence $\{p_i\}$.

In the above defined terminologies, we can say that a pair $(x, y) \in X \times X$ is a Li-Yorke scrambled pair if there exist increasing sequences $\{m_i\}$, $\{n_i\}$ such that

$$(x, y) \in DR(f, \{m_i\}) \cap AR(f, \{n_i\}).$$

We recall that, a map f is said to be topologically transitive if for any nonempty open subsets U and V of X, there exists an $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. If the product map $f \times f$ is topologically transitive, then we say that the map f is weakly mixing. Proof of the following lemma is similar to the result proved in [66] for metric spaces. We present the proof here for sake of completion.

Lemma 3.1.1. *Let X be a Hausdorff topological space and f be a continuous self-map on X. If $(Y, f|_Y)$ is transitive subsystem of (X, f) and $x \in X$ is such that $O(x)$ is dense in Y, then for each open set U in Y, the set $\{t \mid f^t(x) \in U\}$ is not bounded above.*

Proof. Let U be an open set in Y and $T_0 > 0$ be given, we shall show that there exists a $t > T_0$ such that $f^t(x) \in U$. For $x \in X$ and $T > 0$, we write

$$S(x, T) = \{f^t(x) \mid 0 \leq t \leq T\}.$$

First, we assume that $S(x, T_0) \subset Y$. Then the set $V = Y - S(x, T_0)$ is nonempty and open in Y. Moreover by transitivity of $f|_Y$, there exist $v \in V$ and $t_v > 0$ such that $f^{t_v}(v) \in U$. If $v \in O(x)$, then there exists $t_0 > T_0$ such that $v = f^{t_0}(x)$. Let

$t = t_0 + t_v$, then $t > T_0$, and we have $f^t(x) = f^{t_v}(f^{t_0}(x)) = f^{t_v}(v) \in U$. Also, if $v \notin O(x)$ then $f^{t_i}(x) \to v$ for some $t_i \to \infty$, which implies that $f^{t_v}(f^{t_i}(x)) \to f^{t_v}(v)$. Thus, in either case we have $f^t(x) \in U$ for some $t > T_0$.

Secondly, assume that $Y \subset S(x, T_0)$. Let $y \in U$, then $y \in O(x)$, that is $y = f^{t_y}(x)$ for some $t_y > 0$. Note that $f^t(y) \in Y$, for each $t > T_0$. Since $Y \subset S(x, T_0)$, it follows that there exist a periodic positive semi-orbit $P \subset S(x, T_0)$ such that $f^t(y) \in P$ for some $t \geq 0$. Denote $\bar{t} = \min\{t \geq 0 \mid f^t(y) \in P\}$, and define $B\left(y, \frac{\bar{t}}{2}\right) = \left\{f^t(y) \mid t \geq \frac{\bar{t}}{2}\right\}$. Let $V_1 = Y - S(x, \frac{\bar{t}}{2} + t_y)$ and $V_2 = Y - B(y, \frac{\bar{t}}{2})$. If $\bar{t} > 0$, then V_1 and V_2 are nonempty open sets of Y. For any $q \in V_1$, $q = f^{t_1}(x)$ where $t_1 > \frac{\bar{t}}{2} + t_y$, therefore $q = f^{t_q}(y)$, where $t_q = t_1 - t_y > \frac{\bar{t}}{2}$. Then for any $t > 0$, $f^t(q) = f^{t+t_q}(y) \in B(y, \frac{\bar{t}}{2})$, since $t + t_q > \frac{\bar{t}}{2}$. By definition of V_2, $f^t(q) \notin V_2$, for all $t > 0$. Thus, $f^t(V_1) \cap V_2 = \emptyset$, for all $t > 0$, which contradicts the transitivity of $f|_Y$. Hence $\bar{t} = 0$ and $f^0(y) = y \in P$. As P is periodic, clearly there is $t > t_0$ such that $f^t(x) = y \in U$. $\square$

Theorem 3.1.2. *Let $(X, \mathcal{U})$ be a second countable Baire Hausdorff uniform space without isolated points and let f be a uniformly continuous self-map defined on X. If f is weakly mixing, then f is Li-Yorke chaotic.*

Proof. Since X is a second countable space, $X \times X$ has a countable open base, say $\{G_n\}_{n=1}^{\infty}$. Consider the set $D = \bigcap_{n=1}^{\infty} \bigcup_{t \in \mathbb{N}} F^{-t}(G_n)$. Note that for each $n \in \mathbb{N}$, the set $\bigcup_{t \in \mathbb{N}} F^{-t}(G_n)$ is open in $X \times X$. Now, f is weakly mixing implies that $f \times f$ is transitive and hence $\bigcup_{t \in \mathbb{N}} F^{-t}(G_n)$ is dense in $X \times X$. Thus, D is a countable intersection of dense sets in $X \times X$. Further by choice of D, orbit of any pair in D is dense in $X \times X$. Select any x_0, $y_0 \in X$, with $x_0 \neq y_0$.

For any $(x, y) \in D$ with $x \neq y$, it follows from Lemma 3.1.1 that any open set containing (x_0, x_0) contains infinite number of points of type $F^t(x, y)$. Therefore there exists an increasing sequence $\{n_i\}$ such that $F^{n_i}(x, y) \to (x_0, x_0)$ as $i \to \infty$. Then for any $E \in \mathcal{U}$, $F^{n_i}(x, y) \in E$, for all but finitely many i's. Thus, $(x, y) \in AR(f, \{n_i\})$.

Using similar argument, there exists an increasing sequence $\{m_i\}$ such that $F^{m_i}(x, y) \to (x_0, y_0)$ as $i \to \infty$. Since $x_0 \neq y_0$, there exists an entourage $E \in \mathcal{U}$ such that $(x_0, y_0) \notin E$. Also, since $F^{m_i}(x, y) \to (x_0, y_0)$, there exists a positive integer $k \in \mathbb{N}$ such that $F^{m_i}(x, y) \notin E$, for all $i > k$. Consider the sequence $m_i^{'} = m_{i+k}$, $i \in \mathbb{N}$. Then $F^{m_i^{'}}(x, y) \notin E$, for each $i \in \mathbb{N}$. Thus, $(x, y) \in DR(f, \{m_i^{'}\})$. Therefore, $(x, y) \in DR(f, \{m_i^{'}\}) \cap AR(f, \{n_i\})$. Hence, (x, y) is a Li-Yorke scrambled pair. Since $(x, y) \in D$ with $x \neq y$ is arbitrary, it follows that f is Li-Yorke chaotic. $\square$

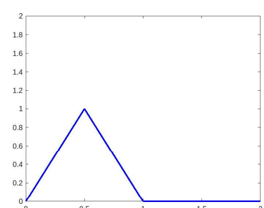
The following example justifies that Theorem 3.1.2 need not be true if the underlying space is not Hausdorff.

Example 3.1.3. Let $f : S^1 \to S^1$ be defined by $f(e^{i\theta}) = e^{2i\theta}$, where S^1 is equipped with the co-finite topology. Then S^1 with co-finite topology is not Hausdorff. Clearly, f is mixing and hence weakly mixing. Note that, for any two distinct points x, $y \in S^1$, the pair $(x, y) \in AR$ and hence the pair is not Li-Yorke scrambled, which proves that f is not Li-Yorke chaotic.

The following example justifies that the notion of weakly mixing is stronger than Li-Yorke chaos.

Example 3.1.4. Let f be a continuous self-map on the interval [0, 2] given by:

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}; \\ 2(1-x), & \frac{1}{2} \leq x \leq 1; \\ 0, & 1 \leq x \leq 2. \end{cases}$$

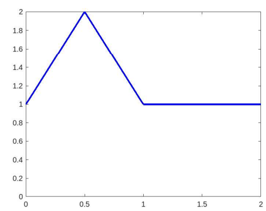


Then f is not transitive, and hence not weakly mixing. It is well-known that a periodic point of period 3 implies Li-Yorke chaos [36]. Note that, the point $x = \frac{2}{9}$ is a periodic point of f with period 3 and hence f is Li-Yorke chaotic.

Note that, there are dynamical systems which are neither Li-Yorke chaotic nor weakly mixing.

Example 3.1.5. Let f be a continuous self-map on the interval $[0,\, 2]$ given by:

$$f(x) = \begin{cases} 1+2x, & 0 \leq x \leq \frac{1}{2}; \\ 3-2x, & \frac{1}{2} \leq x \leq 1; \\ 1, & 1 \leq x \leq 2. \end{cases}$$



Then f is not transitive, and hence not a weakly mixing. Moreover, f is not Li-Yorke chaotic, as every pair of distinct points is asymptotic.

3.2 Topological distributional chaos in a sequence

In this section, we define and study the notion of topological distributional chaos in a sequence for uniformly continuous self-maps defined on Hausdorff uniform spaces.

For an increasing sequence $\{p_i\}$ of positive integers, an entourage $U \in \mathcal{U}$, and points $x,\, y \in X$, define the lower and upper distribution functions $F_{xy}(U,\, \{p_i\})$ and $F^*_{xy}(U,\, \{p_i\})$, respectively as follows:

$$F_{xy}(U,\, \{p_i\}) = \liminf_{n\to\infty} \frac{1}{n} \#\{0 \leq i < n \mid F^{p_i}(x,y) \in U\},$$
$$F^*_{xy}(U,\, \{p_i\}) = \limsup_{n\to\infty} \frac{1}{n} \#\{0 \leq i < n \mid F^{p_i}(x,y) \in U\},$$

where $\#A$ denotes the cardinality of the set A.

Definition 3.2.1. A subset D of X is said to be *topologically distributionally chaotic set* (or *scrambled set*) in an increasing sequence $\{p_i\}$, if for any $x, y \in D$ with $x \neq y$, we have

(i) $F_{xy}(U,\, \{p_i\}) = 0$, for some $U \in \mathcal{U}$, and

(ii) $F^*_{xy}(U,\, \{p_i\}) = 1$, for all $U \in \mathcal{U}$.

Such a pair (x, y) is called a topologically distributionally chaotic pair or a scrambled pair for f in a sequence $\{p_i\}$. We denote by $DCR(f, \{p_i\})$, the collection of all pairs $(x, y) \in X \times X$ such that (x, y) is a topologically distributionally chaotic pair for f in a sequence $\{p_i\}$, and call it the topologically distributionally chaotic relation with respect to a sequence $\{p_i\}$.

Definition 3.2.2. A map f is said to be *topologically distributionally chaotic in sequence* $\{p_i\}$ if f has an uncountable topologically distributionally scrambled set in an increasing sequence $\{p_i\}$.

Remark 3.2.3.

(i) If we consider the uniform space $(X, \mathcal{U})$, where (X, d) is a metric space and $\mathcal{U}$ is the uniformity generated by the family $\{d^{-1}[0, \epsilon) \mid \epsilon > 0\}$, then every entourage E contains $E_\epsilon = d^{-1}[0, \epsilon)$, for some $\epsilon > 0$, and any E_ϵ is an entourage. In this case, topological distributional chaos in a sequence coincides with the metric notion of distributional chaos in a sequence.

(ii) If $\{p_i\}$ is the sequence of positive integers, then f is topologically distributionally chaotic of type 1.

In the following proposition, we show that the notion of topological distributional chaos in a sequence is a dynamical property. Recall that, for uniformly continuous self-maps f and g on uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$, respectively, are said to be *topologically conjugate* if there exists a uniform homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$.

Proposition 3.2.4. *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be Hausdorff uniform spaces. Suppose that $f : X \to X$ and $g : Y \to Y$ are topologically conjugate. Then f is topologically distributionally chaotic in a sequence $\{p_i\}$ implies g is topologically distributionally chaotic in a sequence $\{p_i\}$.*

Proof. Since $f : X \to X$ and $g : Y \to Y$ are topologically conjugate, there exists a uniform homeomorphism $h : X \to Y$ such that $g \circ h = h \circ f$. For points $x_1, x_2 \in X$, let $y_1 = h(x_1), y_2 = h(x_2)$. Now for any entourage $U \in \mathcal{U}$, $h : X \to Y$ is a uniform homeomorphism implies $V = H(U) \in \mathcal{V}$, where $H = h \times h$. Then for $G = g \times g$, we have

$$
\begin{aligned}
G^{(n)}_{y_1y_2}(V,\{p_i\}) &= \frac{1}{n}\#\{0 \le i < n \mid G^{p_i}(y_1,y_2) \in V\}\\
&= \frac{1}{n}\#\{0 \le i < n \mid (g^{p_i}h(x_1), g^{p_i}h(x_2)) \in V\}\\
&= \frac{1}{n}\#\{0 \le i < n \mid (hf^{p_i}(x_1), hf^{p_i}(x_2)) \in V\}\\
&= \frac{1}{n}\#\{0 \le i < n \mid H(f^{p_i}(x_1), f^{p_i}(x_2)) \in H(U)\}\\
&= \frac{1}{n}\#\{0 \le i < n \mid F^{p_i}(x_1,x_2) \in U\}\\
&= F^{(n)}_{x_1x_2}(U,\{p_i\}). \qquad (3.1)
\end{aligned}
$$

From (3.1), it follows that f is topologically distributionally chaotic in a sequence $\{p_i\}$ implies that g is topologically distributionally chaotic in a sequence $\{p_i\}$. □

Lemma 3.2.5. *Let $(X,\mathcal{U})$ be a Hausdorff uniform space and let f be an uniformly continuous self-map on X. If $\{m_i\}$ and $\{n_i\}$ are increasing sequences of positive integers, then there exists an increasing sequence $\{p_i\}$ of positive integers such that*

$$DR(f,\{m_i\}) \cap AR(f,\{n_i\}) \subset DCR(f,\{p_i\}).$$

Proof. Let $b_1 = 2$, and $b_i = 2^{b_1+b_2+\cdots+b_{i-1}}$ for $i > 1$. Then $\{b_i\}$ is an increasing sequence of positive integers. Let

$$
p_i = \begin{cases} m'_i, & \text{if } i \le b_1 \text{ or } \sum_{j=1}^{2k} b_j < i \le \sum_{j=1}^{2k+1} b_j,\ k \in \mathbb{N},\\ n'_i, & \text{otherwise,} \end{cases}
$$

where $\{m'_i\}$ and $\{n'_i\}$ are subsequences of $\{m_i\}$ and $\{n_i\}$, respectively, with $m'_l > n'_j$, $n'_l > m'_j$, for any $l > j$. Then $\{p_i\}$ is an increasing sequence of positive integers.

Let $(x,y) \in DR(f,\{m_i\}) \cap AR(f,\{n_i\})$. Then $(x,y) \in DR(f,\{m_i\})$ implies there exists an entourage $U \in \mathcal{U}$ such that $F^{m_i}(x,y) \notin U$, for all $i \in \mathbb{N}$. Thus,

$$\begin{aligned}
F_{xy}(U,\,\{p_i\}) &= \liminf_{n\to\infty} \frac{1}{n}\#\{0 \le i < n \mid F^{p_i}(x,y) \in U\} \\
&\le \lim_{i\to\infty} \frac{1}{j_i}\#\{0 \le k < j_i \mid F^{p_k}(x,y) \in U\} \quad \Big(\text{where } j_i = \sum_{h=1}^{2i+1} b_h\Big) \\
&\le \lim_{i\to\infty} \frac{b_1 + b_2 + \cdots + b_{2i}}{j_i} \\
&= \lim_{i\to\infty} \frac{b_1 + b_2 + \cdots + b_{2i}}{b_1 + b_2 + \cdots + b_{2i} + 2^{b_1+b_2+\cdots+b_{2i}}} = 0.
\end{aligned}$$

Further, $(x,y) \in AR(f,\{n_i\})$ implies that for any $U \in \mathcal{U}$ there exists a positive integer $N > 0$ such that $F^{n_i}(x,y) \in U$, for all $i > N$. Thus,

$$\begin{aligned}
F^*_{xy}(U,\,\{p_i\}) &= \limsup_{n\to\infty} \frac{1}{n}\#\{0 \le i < n \mid F^{p_i}(x,y) \in U\} \\
&\ge \lim_{i\to\infty} \frac{1}{l_i}\#\{0 \le j < l_i \mid F^{p_j}(x,y) \in U\} \quad \Big(\text{where } l_i = \sum_{h=1}^{2i} b_h\Big) \\
&\ge \lim_{i\to\infty} \frac{b_{2i}}{l_i} \\
&= \lim_{i\to\infty} \frac{2^{b_1+b_2+\cdots+b_{2i-1}}}{b_1 + b_2 + \cdots + b_{2i-1} + 2^{b_1+b_2+\cdots+b_{2i-1}}} = 1.
\end{aligned}$$

Hence, $(x,y) \in DCR(f,\{p_i\})$. □

3.3 Equivalence between Li-Yorke chaos and topological distributional chaos in a sequence

In this section, we study the relationship between the topological notions of Li-Yorke chaos and distributional chaos in a sequence for uniformly continuous self-maps defined on Hausdorff uniform space without isolated points.

In recent years, many authors have used the families of subsets of positive integers to study properties of dynamical systems. We recall that, a Furstenberg family $\mathcal{F}$ is a family, consisting of some subsets of the set of positive integers, which is hereditary upwards, that is, if $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ then $F_2 \in \mathcal{F}$. A class of

Furstenberg families can also be defined by considering the upper density with respect to a sequence. For strictly increasing sequence $Q = \{n_i\}$ of positive integers and $P \subseteq \mathbb{N}$, the upper density of P with respect to Q is given by

$$\bar{d}(P \mid Q) = \limsup_{k\to\infty} \frac{\#\{P \cap \{n_1, n_2, \ldots, n_k\}\}}{k},$$

where $\#A$ denotes the cardinality of the set A.

For every $a \in [0, 1]$, define

$$\overline{\mathcal{M}}_Q(a) = \{P \subseteq \mathbb{N} \mid P \cap Q \text{ is infinite and } \bar{d}(P \mid Q) \geq a\}.$$

Note that $\overline{\mathcal{M}}_Q(a)$ is a Furstenberg family. For $x \in X$ and $A \subseteq X$, define

$$N(x, A) = \{n \in \mathbb{N} \mid f^n(x) \in A\} \text{ and}$$
$$\overline{\mathcal{M}}_Q(a, A) = \{x \in X \mid N(x, A) \in \overline{\mathcal{M}}_Q(a)\}.$$

We can now rephrase the definition of a topologically distributionally scrambled pair in a sequence as follows: Let $(X, \mathcal{U})$ be a Hausdorff uniform space, f a uniformly continuous self-map on X and $Q = \{p_i\}$ an increasing sequence of positive integers. Then $(x, y) \in X \times X$ is a topologically distributionally scrambled pair in sequence Q if

(i) for some $U \in \mathcal{U}$, $(x, y) \in \overline{\mathcal{M}}_Q(1, X \times X \setminus \overline{U})$, and

(ii) for any $U \in \mathcal{U}$, $(x, y) \in \overline{\mathcal{M}}_Q(1, U)$,

where $\overline{U}$ denotes the closure of U in $X \times X$.

The following lemma can be proved along the lines of [34, Lemma 3.2].

Lemma 3.3.1. *Let X be a topological space, f a continuous self-map on X, Q strictly increasing sequences of positive integers and $a \in [0, 1]$. Then for any nonempty open subset W of X, $\overline{\mathcal{M}}_Q(a, W)$ is a G_δ set.*

The following result follows from Lemma 3.3.1.

Lemma 3.3.2. *Let* $(X,\mathcal{U})$ *be a second countable Baire Hausdorff uniform space,* f *a uniformly continuous self-map on* X *and* Q *an increasing sequence of positive integers. Then the set of all topologically distributionally scrambled pairs in the sequence* Q *is a* G_δ *subset of* $X \times X$*.*

Lemma 3.3.3 [24]**.** *Let* $\{S_i\}$ *be a sequence of increasing sequences of positive integers. Then there exists an increasing sequence* Q *of positive integers such that* $\bar{d}(S_i \cap Q \mid Q) = 1$*, for all* $i \geq 1$*.*

Lemma 3.3.4. *Let* $(X,\mathcal{U})$ *be a Hausdorff uniform space and let* f *be a uniformly continuous self-map on* X*. If* S *is a countable Li-Yorke scrambled set, then there exist an increasing sequence* Q *of positive integers such that* S *is topologically distributionally scrambled set in the sequence* Q*.*

Proof. For any pair of distinct points $x, y \in S$, by definition there exist sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that $(x, y) \in DR(f, \{m_i\}) \cap AR(f, \{n_i\})$. Using Lemma 3.2.5, there exists an increasing sequence $\{p_i^{(x,y)}\}$ of positive integers such that $(x, y) \in DCR(f, \{p_i^{(x,y)}\})$. Hence by Lemma 3.3.3, there exists a sequence Q such that S is topologically distributionally scrambled set in the sequence Q. □

Using the arguments used by Huang and Ye to prove Lemma 3.1 in [27], we can obtain the following version of the lemma.

Lemma 3.3.5. *Let* $(X,\mathcal{U})$ *be a second countable Baire Hausdorff uniform space without isolated points. If* R *is a symmetric relation with the property that there is a dense* G_δ *subset* A *of* X *such that for each* $x \in A$*,* $R(x)$ *contains a dense* G_δ *subset, then there is an uncountable dense subset* B *of* X *such that* $B \times B \setminus \triangle \subset R$*, where* $R(x) = \{y \mid (x, y) \in R\}$*.*

Proof. Let $\mathcal{B}$ be the collection of all nonempty subsets of A such that for each $B \in \mathcal{B}$, $B \times B \setminus \triangle \subset R$. And let $\{U_n\}_{n=1}^{\infty}$ be a countable open base for topology on X. Now for any $x_1 \in U_1 \cap A$, as $A \cap R(x_1)$ contains a dense G_δ subset, there exists a point $x_2 \in U_2 \cap A \cap R(x_1)$, $x_1 \neq x_2$. Then $\{x_1, x_2\} \in \mathcal{B}$, and hence

$\mathcal{B} \neq \emptyset$. Similarly, as $A \cap R(x_1) \cap R(x_2)$ contains a dense G_δ subset, there exists a point $x_3 \in U_3 \cap A \cap R(x_1) \cap R(x_2)$, $x_1 \neq x_2 \neq x_3$, implying that $\{x_1, x_2, x_3\} \in \mathcal{B}$. Continuing this way, we can construct a dense subset $C = \{x_n \mid n \in \mathbb{N}\}$ such that $C \in \mathcal{B}$ and $x_n \in U_n$, for each $n \in \mathbb{N}$. By Zorn's Lemma there exists a maximal element B (under the usual set inclusion) of $\mathcal{B}$ such that $C \subset B$. Note that, $\overline{B} = X$.

We claim that B is uncountable. On contrary, assume that B is countable, say $B = \{y_n \mid n \in \mathbb{N}\}$. As $A \cap \left(\bigcap_{n=1}^{\infty} R(x_n)\right)$ contains a dense G_δ subset, we get $y \in A \cap \left(\bigcap_{n=1}^{\infty} R(x_n)\right)$ such that for each $n \in \mathbb{N}$, $y \neq x_n$ and $(y, x_n) \in R$. Then $B' = B \cup \{y\} \subset A$ and $B' \times B' \setminus \triangle \subset R$. This contradicts the maximality of B in $\mathcal{B}$. Thus, B is uncountable. □

Lemma 3.3.6 [68]**.** *Let $(X,\mathcal{U})$ be a second countable Baire Hausdorff uniform space without isolated points. If R is a symmetric relation on X which contains a dense G_δ subset $X \times X$, then there exists a dense G_δ subset A of X such that for any $x \in A$, there exists a dense G_δ subset A_x of X with $\{(x, y) \mid y \in A_x\} \subset R$.*

From Lemma 3.3.5 and Lemma 3.3.6, we have the following.

Lemma 3.3.7. *Let $(X,\mathcal{U})$ be a second countable Baire Hausdorff uniform space without isolated points. If R is a symmetric relation on X which contains a dense G_δ subset of $X \times X$. Then there is an uncountable dense subset B of X such that $B \times B \setminus \triangle \subset R$*

Theorem 3.3.8. *Let $(X,\mathcal{U})$ be a second countable Baire Hausdorff uniform space without isolated points and let f be a uniformly continuous self-map defined on X. Then f is chaotic in the sense of Li-Yorke if and only if f is topologically distributionally chaotic in a sequence.*

Proof. If f is chaotic in the sense of Li-Yorke, then by definition f has an uncountable scrambled set $D \subset X$. Since X is second countable, then so is D, hence we can choose a countable dense subset S of D. By Lemma 3.3.4, there exist an increasing sequence Q of positive integers such that S is topologically distributionally scrambled set in the sequence Q. Let E be the collection of all topologically distributionally scrambled pairs in the sequence Q, then by Lemma 3.3.2, E is a G_δ subset of $X \times X$. Since $S \times S \setminus \triangle \subset E$ and S is dense in D. By Lemma 3.3.7, there exists an uncountable dense set $K \subset D$ such that $K \times K \setminus \triangle \subset E$. Thus (X, f) is topologically distributionally chaotic in a sequence.

Conversely, if f is topologically distributionally chaotic in a sequence $Q = \{p_i\}$, then f has an uncountable distributionally scrambled set D in sequence $\{p_i\}$. Then for any $x, y \in D$ with $x \neq y$, we have $F^*_{xy}(U, \{p_i\}) = 1$, for all $U \in \mathcal{U}$, and $F_{xy}(U, \{p_i\}) = 0$, for some $U \in \mathcal{U}$. This implies that for each $U \in \mathcal{U}$, there exists some $j \in \mathbb{N}$ such that $F^{p_j}(x, y) \in U$. Thus $(x, y) \in PR$. Note that $(x, y) \notin AR$. For if $(x, y) \in AR$, then for any $U \in \mathcal{U}$, there exist a positive integer $N > 0$ such that $F^{p_i}(x, y) \in U$, for all $p_i > N$. This implies that $F_{xy}(U, \{p_i\}) = F^*_{xy}(U, \{p_i\}) = 1$, for each $U \in \mathcal{U}$, which contradicts that f is topologically distributionally chaotic in a sequence $\{p_i\}$. Therefore, $(x, y) \in PR \backslash AR$, for all $(x, y) \in D$. Thus, f has an uncountable Li-Yorke scrambled set. Hence, f is Li-Yorke chaotic. □

From Theorem 3.1.2, we have that if f is weakly mixing then f is Li-Yorke chaotic. Thus from Theorem 3.3.8, we have the following.

Corollary 3.3.9. *Let $(X, \mathcal{U})$ be a second countable Baire Hausdorff uniform space without isolated points and f a uniformly continuous self-map defined on X. If f is weakly mixing then f is topologically distributionally chaotic in a sequence.*

From [6, Theorem 1.2], we have that if f is Devaney chaotic then f is Li-Yorke chaotic. Therefore from Theorem 3.3.8, we have the following.

Corollary 3.3.10. *Let $(X, \mathcal{U})$ be a second countable Baire Hausdorff uniform space without isolated points and f a uniform self-homeomorphism on X. If f is Devaney chaotic then f is topologically distributionally chaotic in a sequence.*

As a consequence of the results obtained, for a uniform self-homeomorphism f defined on a second countable Baire Hausdorff uniform space X without isolated points, we have the following implications:

$$\begin{array}{ccl} \text{Weakly mixing} & \Rightarrow & \text{Li-Yorke chaos} \\ & & \quad\Updownarrow \\ \text{Devaney chaos} & \Rightarrow & \text{Topological distributional chaos in a sequence} \end{array}$$