Chapter 7

Investigation of Fractional Diabetes model

7.1 Introduction

In this work, we examine a diabetes mellitus model discussed by Muhammad et al. [136], by replacing integer ordered derivative with fractional ordered one and intended to solve the system of fractional ordered ordinary differential equations by novel approach. The prime parameters utilized in the described model are glucose level as $\mathbb{G}(t)$, insulin level as $\mathbb{I}(t)$, and insulin level in plasma as $\mathbb{X}(t)$. The following set of equations stands as the model:

$$\begin{aligned}
&\tfrac{d\mathbb{G}(t)}{dt} = -m_1\mathbb{G}(t) + m_2\mathbb{I}(t) + m_1\mathbb{G}_b,\\
&\tfrac{d\mathbb{X}(t)}{dt} = -m_2\mathbb{X}(t) + m_3\mathbb{I}(t) - m_3\mathbb{I}_b + m_6\mathbb{I}_b,\\
&\tfrac{d\mathbb{I}(t)}{dt} = -m_3\mathbb{I}(t) + m_4\mathbb{G}(t) + m_4m_5 - m_6\mathbb{I}(t) + m_6\mathbb{I}_b,\\
&\text{where } \mathbb{G}(0), \mathbb{X}(0), \mathbb{I}(0) \geq 0.
\end{aligned} \tag{7.1}$$

In this investigation, we are taking Type-1 diabetes patients into consideration. People with Type-1 diabetes may have a constant glucose monitor (CGM) to help them manage their blood glucose levels. The surveillance device receives the data from the sensing device, which uses a tiny needle to measure the level of glucose in the tissue fluids [134]. Additionally, its connection with insulin pumps for constant subcutaneous infusion enabled the creation of algorithms that provides insulin doses based on CGM data to reduce the frequency of critical situations. fractional order PID, an excellent adaptive algorithm, is used as the controller design [52] (see figure-1).

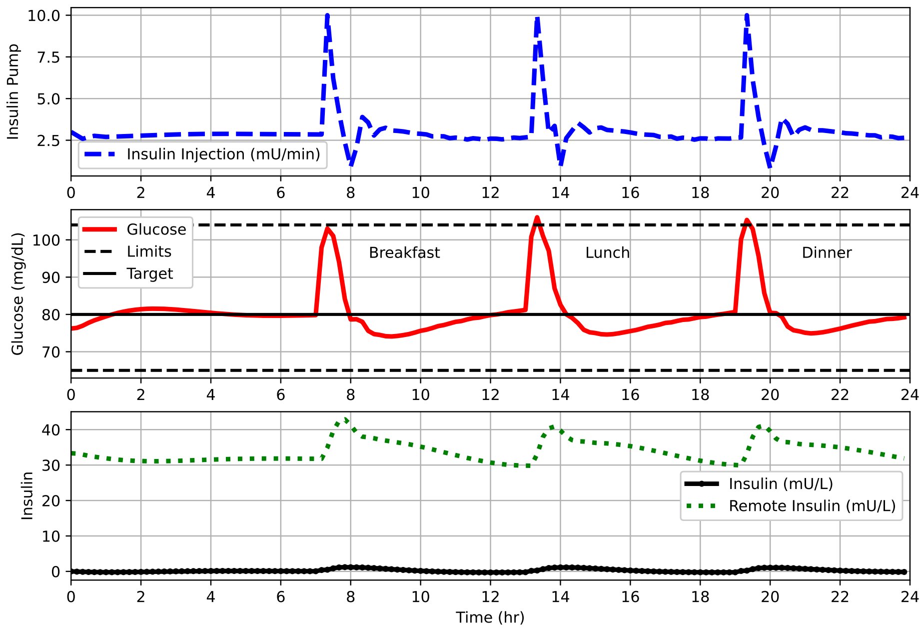

Figure 7.1: Simulation of Type-1 Diabetic Blood Glucose using PID controller

In the last few decades, fractional differential equations (FDEs) have been used to explore the huge percentage of linear and nonlinear real-world situations like [90, 131, 92, 151, 91, 158]. The fact that FDEs exhibit novel features for these issues due to their non-local property, which is one of the most significant benefits of using them instead of integer ordered differential equations. Recently, Ali et al. [15] explained the fractional ordered dynamics Model of the Zika virus with a mutation, Zhang et al. [170] reported using a fractional PI controller for active vibration control of a nuclear power plant's conventional pipe system, Saad [129] illustrated medium-voltage electrical cables for research nuclear reactors using Machine learning in the fractional Fourier domain, Raza et al. [126] presented a fractional model of inclined magnetic force for the water-based Casson nanofluid and kerosene oil, Simelane [143] outlined a saturated incidence FDE model for the hepatitis B virus, Jafari [72] built a fractional-order HIV/AIDS pandemic model, with free terminal time optimum control.

One of the key parameters of FDEs is the approximate solution that can be found for several nonlinear differential equations. There are several analytical and numerical methods for obtaining the approximate solutions for FDEs, (see [111, 167, 51]). The Laplace transform (LT) in combination with the Adomian decomposition method (ADM) [4, 160, 161], often known as the ADLTM [71], is likely the method that is very effective and convenient. ADLTM is one of the well-established methods to deal with Initial value problems (IVPs) for FODEs and FPDEs [53, 166, 54].

The current study's objective is to look at the diabetes model by considering the fractional ordered differential equations with ABC derivative [24]. And, to discuss the existence and uniqueness of the solution of model using Banach fixed point theory. The governing equations of the diabetes model with ABC derivative are defined as

$$\begin{aligned}
{}_{0}^{ABC}D_t^{\alpha}\mathbb{G}(t) &= -m_1\mathbb{G}(t) + m_2\mathbb{I}(t) + m_1\mathbb{G}_b, \\
{}_{0}^{ABC}D_t^{\alpha}\mathbb{X}(t) &= -m_2\mathbb{X}(t) + m_3\mathbb{I}(t) - m_3\mathbb{I}_b + m_6\mathbb{I}_b, \\
{}_{0}^{ABC}D_t^{\alpha}\mathbb{I}(t) &= -m_3\mathbb{I}(t) + m_4\mathbb{G}(t) + m_4 m_5 - m_6\mathbb{I}(t) + m_6\mathbb{I}_b,
\end{aligned} \tag{7.2}$$

with initial conditions,

$$\mathbb{G}(0) = \mathbb{G}_0, \mathbb{X}(0) = \mathbb{X}_0, \mathbb{I}(0) = \mathbb{I}_0\,(\geq 0); \tag{7.3}$$

wherein the rate of insulin-free uptake in adipose and muscular tissue is m_1, the basal pre-injection value of plasma glucose is $\mathbb{G}_b$, the rate of reduction in tissue glucose absorption capacity is m_2, the baseline value of plasma insulin before an injection is $\mathbb{I}_b$, the increase in $\mathbb{I}_b$'s capacity to absorb glucose without insulin is m_3, after a glucose dose and its concentration, the value of pancreatic cells discharged is m_4, the glucose threshold value is m_5, the rate at which insulin in plasma decreases is m_6.

Table 7.1: The details of the parameters utilized in the model, with complete descriptions are given.

Parameters	Normal Person	Diabetic Person	Reference Units
m_1	0.0317	0	min^{-1}
m_2	0.0123	0.17	min^{-1}
m_3	4.92E-06	5.30E-06	$min^{-2}(\mu U/ml)$
m_4	0.0039	0.0042	$min^{-2}(mg/dl)^{-2}$
m_5	79.053	90.25	mg/dl
m_6	0.2659	0.0264	min^{-1}
$\mathbb{G}_b$	80	80	mg/dl
$\mathbb{I}_b$	7	15	$\mu U/ml$

7.2 Preliminaries

Lemma 7.1. [43] Let

$$\begin{aligned} &{}_{0}^{ABC}D_t^{\alpha}\psi(t) = f(t), \text{ where } f(t) \in C\left([0,T]\right), \ 0 < \alpha \leq 1, \\ &\psi(0) = \psi_0, \end{aligned} \tag{7.4}$$

be the given FDE.

Then the solution of (7.4) is given as

$$\psi(t) = \psi_0 + \frac{(1-\alpha)}{\Lambda(\alpha)}f(t) + \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau. \tag{7.5}$$

Lemma 7.2. [144, 16] (using Banach's Fixed Point Theorem) Suppose, $S = C[0,T]$ is the Banach space of all continuous real-valued functions defined on $[0,T]$. Let B be another Banach space formed as $B = S \times S \times S$, where the norm on B is given as

$$\|\psi\| = \|\mathbb{G},\mathbb{X},\mathbb{I}\| = \underset{t\in[0,T]}{Sup}\left[|\mathbb{G}(t)| + |\mathbb{X}(t)| + |\mathbb{I}(t)|\right], \text{ where } \psi \in \text{B and } \mathbb{G},\mathbb{X},\mathbb{I} \in \text{S}. \tag{7.6}$$

Consider $A \subset B$, where A is convex. Let X, Y be defined on A, such that

1. For all $u \in A, X(u) + Y(u) \in A$;

2. X is a contraction mapping;

3. Y is continuous and compact.

Then, we have at least one fixed point $u \in A$, i.e., $X(u) + Y(u) = u$.

Lemma 7.3. [144] (Arzela-Ascoli Theorem) If B is a compact metric space, then a closed subspace A of $C[B, R]$ is compact if and only if it is bounded and equicontinuous.

7.3 Qualitative Analysis

For the considered model, we construct a function

$$\begin{aligned} f_1(t, \mathbb{G}(t), \mathbb{X}(t), \mathbb{I}(t)) &= -m_1\mathbb{G}(t) + m_2\mathbb{I}(t) + m_1\mathbb{G}_b, \\ f_2(t, \mathbb{G}(t), \mathbb{X}(t), \mathbb{I}(t)) &= -m_2\mathbb{X}(t) + m_3\mathbb{I}(t) - m_3\mathbb{I}_b + m_6\mathbb{I}_b, \\ f_3(t, \mathbb{G}(t), \mathbb{X}(t), \mathbb{I}(t)) &= -m_3\mathbb{I}(t) + m_4\mathbb{G}(t) + m_4 m_5 - m_6\mathbb{I}(t) + m_6\mathbb{I}_b. \end{aligned} \tag{7.7}$$

Also, we generalize FDE as

$$ {}_0^{ABC}D_t^{\alpha}\psi(t) = \omega(t, \psi(t)), \tag{7.8}$$

where $0 < \alpha \leq 1$, $t \in [0, T]$, $\psi(0) = \psi_0$.
As mentioned in Lemma – 7.1, the solution of (7.8) is

$$\psi(t) = \psi_0 + \frac{(1-\alpha)}{\Lambda(\alpha)}\omega(t, \psi(t)) + \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\omega(\tau, \psi(\tau))\, d\tau, \tag{7.9}$$

$$\text{where } \psi(t) = \begin{bmatrix} \mathbb{G}(t) \\ \mathbb{X}(t) \\ \mathbb{I}(t) \end{bmatrix}, \psi_0 = \begin{bmatrix} \mathbb{G}_0 \\ \mathbb{X}_0 \\ \mathbb{I}_0 \end{bmatrix}, \omega(t, \psi(t)) = \begin{bmatrix} f_1(t, \mathbb{G}(t), \mathbb{X}(t), \mathbb{I}(t)) \\ f_2(t, \mathbb{G}(t), \mathbb{X}(t), \mathbb{I}(t)) \\ f_3(t, \mathbb{G}(t), \mathbb{X}(t), \mathbb{I}(t)) \end{bmatrix}. \tag{7.10}$$

Using (7.9) and (7.10), let us define the operators X and Y as:

$$\begin{aligned} X(\psi) &= \psi_0 + \tfrac{(1-\alpha)}{\Lambda(\alpha)}\omega(t, \psi(t)), \\ Y(\psi) &= \tfrac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\omega(\tau, \psi(\tau))\, d\tau. \end{aligned} \tag{7.11}$$

7.4 Existence and Uniqueness

To establish the existence and uniqueness for the solution of (7.8), consider the hypothesis – 1 and Lipschitz condition – 2 given as below:

Hypotheses 1. For some real C_ω and D_ω,

$$|\omega\left(t, \psi(t)\right)| \leq C_\omega \left\|\psi\right\| + D_\omega. \tag{7.12}$$

Hypotheses 2. Lipschitz Condition

$$\text{For}\, \mathrm{L}_\omega > 0\, \text{and}\, \forall \psi, \overline{\psi} \in \mathrm{B},\ \left|\omega(\mathrm{t}, \psi) - \omega(\mathrm{t}, \overline{\psi})\right| \leq \mathrm{L}_\omega \left\|\psi - \overline{\psi}\right\|. \tag{7.13}$$

Theorem 7.1. If hypotheses – 1 and 2 are satisfied, then (7.8) has at least one solution in the form of (7.9), provided $\frac{(1-\alpha)}{\Lambda(\alpha)} L_\omega < 1$.

Proof. The theorem's proof is separated into two sections.

1. Let $\overline{\psi} \in A$, where $A = \{\psi \in B,\ \|\psi\|\ \leq \rho\,, \rho > 0\}$ is a closed and convex.
   From (7.11), we get

$$\begin{aligned} \left\|X(\psi) - X(\overline{\psi})\right\| &= \frac{(1-\alpha)}{\Lambda(\alpha)} \sup_{t\in[0,T]} \left|\omega\left(t, \psi(t)\right) - \omega\left(t, \overline{\psi}(t)\right)\right|, \\ &\leq \frac{(1-\alpha)}{\Lambda(\alpha)} L_\omega \left\|\psi - \overline{\psi}\right\|. \end{aligned} \tag{7.14}$$

   Therefore, we have X as contraction mapping.

2. We prove that Y is relatively compact. That is, we show that Y is equi-continuous and bounded.

Now, Y will be continuous on A, if we have ω continuous over $C([0,T))$.

$$\begin{aligned}\|Y(\psi)\| &= \sup_{t\in[0,T]}\left|\frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\omega\left(\tau,\psi(\tau)\right)d\tau\right|,\\ &\leq \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\left|\omega\left(\tau,\psi(\tau)\right)\right|d\tau,\\ &\leq \frac{T^\alpha}{\Gamma(\alpha)\Lambda(\alpha)}\left[C_\omega\|\psi\|+D_\omega\right],\\ &\leq \frac{T^\alpha}{\Gamma(\alpha)\Lambda(\alpha)}\left[C_\omega\rho+D_\omega\right]. \end{aligned} \tag{7.15}$$

Therefore, Y is bounded.

Moreover, let $t_1 > t_2$, where $t_1, t_2 \in [0,T]$, are arbitrary.

$$\begin{aligned}\left|Y\left(\psi(t_1)\right)-Y\left(\psi(t_2)\right)\right| &= \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)}\left|\int_0^{t_1}(t_1-\tau)^{\alpha-1}\omega\left(\tau,\psi(\tau)\right)d\tau\right.\\ &\quad\left.-\int_0^{t_2}(t_2-\tau)^{\alpha-1}\omega\left(\tau,\psi(\tau)\right)d\tau\right|,\\ &\leq \frac{|t_1^\alpha-t_2^\alpha|}{\Gamma(\alpha)\Lambda(\alpha)}\left|C_\omega\rho+D_\omega\right|. \end{aligned} \tag{7.16}$$

Therefore, $t_1 \to t_2 \Rightarrow |t_1^\alpha - t_2^\alpha| \to 0$.

Thus, from (7.16), we get

$$\left|Y\left(\psi(t_1)\right)-Y\left(\psi(t_2)\right)\right| \to 0.$$

Therefore, (1) and (2) proves that Y is bounded and uniformly continuous. By the Arzela-Ascoli theorem given as Lemma – 7.3, we have Y relatively compact and entirely continuous.

Hence, from Lemma – 7.2, there is at least one solution for system (7.8).

Theorem 7.2. System (7.8) has unique solution given as (7.9), under the hypotheses – 1 & 2 and $\Delta < 1$, where $\Delta = \frac{(1-\alpha)L_\omega}{\Lambda(\alpha)} + \frac{T^\alpha L_\omega}{\Gamma(\alpha)\Lambda(\alpha)}$.

Proof. Let $F : B \to B$ defined as

$$F\left[\psi(t)\right] = \psi_0 + \frac{(1-\alpha)}{\Lambda(\alpha)}\omega\left(t,\psi(t)\right) + \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\omega\left(\tau,\psi(\tau)\right)d\tau, \tag{7.17}$$

Let $\psi, \overline{\psi} \in B$, we have

$$
\begin{aligned}
\left\|F(\psi) - F(\overline{\psi})\right\| \leq & \frac{(1-\alpha)}{\Lambda(\alpha)} \sup_{t\in[0,T]} \left|\omega(t,\psi(t)) - \omega\left(t,\overline{\psi}(t)\right)\right| \\
& + \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)} \sup_{t\in[0,T]} \left| \int_0^t (t-\tau)^{\alpha-1} \omega(\tau,\psi(\tau)) d\tau - \int_0^t (t-\tau)^{\alpha-1} \omega\left(\tau,\overline{\psi}(\tau)\right) d\tau \right|, \\
\leq & \left(\frac{(1-\alpha)L_\omega}{\Lambda(\alpha)} + \frac{T^\alpha L_\omega}{\Gamma(\alpha)\Lambda(\alpha)} \right) \|\psi - \overline{\psi}\|, \\
\leq & \Delta \|\psi - \overline{\psi}\|.
\end{aligned} \tag{7.18}
$$

Thus, F is contraction.

Moreover, by Lemma - 7.2, since B is Banach space, we can say that F gives a unique fixed point. That is, for some $\psi \in B, F(\psi(t)) = \psi(t)$.

Thus,

$$
\psi(t) = \psi_0 + \frac{(1-\alpha)}{\Lambda(\alpha)} \omega(t,\psi(t)) + \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(\tau,\psi(\tau))\, d\tau, \tag{7.19}
$$

gives a unique solution to (7.8).

7.5 Stability Analysis

For any physical problem, stability analysis is among the most significant aspects for the solution of the differential equations. The same logic applies to the solution of FDEs. In FDEs, Ulam-Hyers (UH) stability condition is one of the most significant factors for the stability of solutions. It was offered by Ulam in 1940 and was further explored by Hyers. Later on, Rassias gave a more generalized form for the same stability condition, which came to be known as UHR stability condition [147].

Now, we have $S = C[0,T]$ the Banach space of all continuous real-valued functions defined on $[0,T]$. Then we know that $B = S \times S \times S$, is also a Banach space, where for $\phi \in B, ||\phi||$ is the supremum norm. Consider the positive real-valued function $F_\omega : [0,T] \to R^+$ and for $\xi > 0$, we consider Ulam's Stability postulates as follows:

$$\left|{}_{0}^{ABC}D_t^{\alpha}\phi(t)-\omega\left(t,\phi(t)\right)\right|\leq\xi, \tag{7.20}$$

$$\left|{}_{0}^{ABC}D_t^{\alpha}\phi(t)-\omega\left(t,\phi(t)\right)\right|\leq\xi F_{\omega}(t), \tag{7.21}$$

$$\left|{}_{0}^{ABC}D_t^{\alpha}\phi(t)-\omega\left(t,\phi(t)\right)\right|\leq F_{\omega}(t), \tag{7.22}$$

where $\forall t\in\Omega=[0,T]$ and $\xi=\max(\xi_i)$, for $i=1,2,3$.

Definition 7.1. The given fractional diabetes model (7.2) is Ulam-Hyers stable, if for $\xi>0$ and every $\phi\in B$ satisfying (7.20), there exists a real number $F_\omega>0$ and a solution $\psi(\in B)$ of (7.2), such that

$$|\phi(t)-\psi(t)|\leq\xi C_{\omega},\ t\in\Omega, \tag{7.23}$$

where $\xi=\max(\xi_i)$ and $C_\omega=\max(C_{\omega_i})$, for $i=1,2,3$.

Definition 7.2. The given fractional diabetes model (7.2) is Generalized Ulam-Hyers stable, if for given $F_\omega:R^+\cup\{0\}\to R^+\cup\{0\}$, with $F_\omega(0)=0$, and for all $\xi>0\,\&\,\phi\in B$ in (7.21), we get

$$|\phi(t)-\psi(t)|\leq F_{\omega}(\xi),\ t\in\Omega, \tag{7.24}$$

where $\xi=\max(\xi_i)$ and $F_\omega=\max(F_{\omega_i})$, $i=1,2,3$.

Remark 7.1. For given $\phi\in B$, which satisfies (7.20), we get $\vartheta\in B$, such that the following holds:

1. $|\vartheta(t)|\leq\xi,\vartheta=\max(\vartheta_i)$, $i=1,2,3$;

2. ${}_{0}^{ABC}D_t^{\alpha}\phi(t)=\omega\left(t,\phi(t)\right)+\vartheta(t)$, $\forall t\in\Omega$.

Lemma 7.4. Given $0<\alpha\leq1$ and $\phi\in B$ satisfying condition (7.20), it also satisfies

$$\left|\phi(t)-\phi_0-\frac{1-\alpha}{\Lambda(\alpha)}\omega\left(t,\phi(t)\right)-\frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\omega\left(t,\phi(\tau)\right)d\tau\right| \leq\left(\frac{1-\alpha}{\Lambda(\alpha)}-\frac{T^\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\right)\xi. \tag{7.25}$$

Proof. Since ϕ satisfies (7.20), we get by remark – 7.1 (2),

$$
\begin{aligned}
{}_{0}^{ABC}D_{t}^{\alpha}\phi(t) &= \omega\left(t, \phi(t)\right) + \vartheta(t),\ t \in \Omega, \\
\phi(0) &= \phi_0 \geq 0.
\end{aligned} \tag{7.26}
$$

Thus, solution to (7.26) is

$$
\begin{aligned}
\phi(t) &= \phi_0 + \frac{1-\alpha}{\Lambda(\alpha)}\omega\left(t, \phi(t)\right) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\omega\left(t, \phi(\tau)\right)d\tau, \\
&\leq \frac{1-\alpha}{\Lambda(\alpha)}\vartheta(t) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\vartheta(\tau)d\tau.
\end{aligned} \tag{7.27}
$$

Again, from Remark – 7.1, we have

$$
\begin{aligned}
&\left|\phi(t) - \phi_0 - \frac{1-\alpha}{\Lambda(\alpha)}\omega\left(t, \phi(t)\right) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\omega\left(t, \phi(\tau)\right)d\tau\right| \\
&\quad \leq \left|\frac{1-\alpha}{\Lambda(\alpha)}\vartheta(t) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\vartheta(\tau)d\tau\right|, \\
&\quad \leq \frac{1-\alpha}{\Lambda(\alpha)}\left|\vartheta(t)\right| - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left|\vartheta(\tau)\right|d\tau, \\
&\quad \leq \left(\frac{1-\alpha}{\Lambda(\alpha)} - \frac{T^\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\right)\xi.
\end{aligned} \tag{7.28}
$$

Hence, the proof holds.

Theorem 7.3. Let ω be the continuous real-valued mapping defined on $[0, T] \times B$, and $\psi(t) \in B$. Then taking hypothesis – 2 and Theorem – 7.1 under consideration, the fractional diabetes model given by (7.2) is Ulam-Hyers stable on $[0, T]$.

Proof. For given $\xi > 0$, suppose we have $\phi \in B$, such that (7.20) holds. Let $\psi(t)$ be the outcome, as given in the form (7.9), for the system (7.8), as

$$
{}_{0}^{ABC}D_{t}^{\alpha}\psi(t) = \omega\left(t, \phi(t)\right),\ t \in \Omega,\ \text{with } \psi(0) = \psi_0, \tag{7.29}
$$

where

$$
\psi(t) = \psi_0 + \frac{1-\alpha}{\Lambda(\alpha)}\omega\left(t, \psi(t)\right) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\omega\left(t, \psi(\tau)\right)d\tau. \tag{7.30}
$$

Then, using Lemma – 7.4 and hypothesis – 2, we have

$$
\begin{aligned}
|\phi(t)-\psi(t)| &\leq \left| \begin{array}{l} \frac{1-\alpha}{\Lambda(\alpha)}\omega\left(t,\phi(t)\right)+\frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t\left(t-\tau\right)^{\alpha-1}\omega\left(t,\phi(\tau)\right)d\tau \\ -\frac{1-\alpha}{\Lambda(\alpha)}\omega\left(t,\psi(t)\right)-\frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t\left(t-\tau\right)^{\alpha-1}\omega\left(t,\psi(\tau)\right)d\tau \end{array} \right|, \\
&\leq \left(\frac{1-\alpha}{\Lambda(\alpha)}-\frac{T^\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\right)\xi+\frac{\alpha L_\omega}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t\left(t-\tau\right)^{\alpha-1}\left|\phi(\tau)-\psi(\tau)\right|d\tau, \\
&\leq \left(\frac{1-\alpha}{\Lambda(\alpha)}-\frac{T^\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\right)\xi+\frac{T^\alpha L_\omega}{\Lambda(\alpha)\Gamma(\alpha)}\left|\phi(\tau)-\psi(\tau)\right|. \qquad (7.31)
\end{aligned}
$$

As a result

$$
|\phi(t)-\psi(t)| \leq \xi C_\omega \;\; \text{where} \;\; C_\omega=\frac{(1-\alpha)\Gamma(\alpha)+T^\alpha}{\Lambda(\alpha)\Gamma(\alpha)-T^\alpha L_\omega}. \qquad (7.32)
$$

Corollary 7.1. For $F_\omega(\xi)=\xi C_\omega$ in Theorem – 7.3, where $F_\omega(0)=0$, the fractional diabetes model (7.2) is Generalized Ulam-Hyers stable.

Definition 7.3. If $F_\omega \in C(\Omega, R^+)$, be the positive real-valued continuous mapping defined on $\Omega \equiv [0,T]$. Then the fractional diabetes model (7.2) is Ulam-Hyers Rassias (UHR) stable, if for given, $\xi > 0$, and $\phi \in B$ satisfying (7.21), there exists a constant $K_{F_\omega} > 0$, such that for $\psi \in B$, we have

$$
|\phi(t)-\psi(t)| \leq K_{F_\omega}\xi F_\omega(t),\; t\in\Omega, \qquad (7.33)
$$

where $\xi=\max(\xi_i)$, $F_\omega=\max(F_{\omega_i})$, and $K_{F_\omega}=\max(K_{F_{\omega_i}})$, for $i=1,2,3$.

Definition 7.4. The fractional diabetes system (7.2) is considered to be Generalized Ulam-Hyers-Rassias (GUHR) Stable if for every $\psi \in B$, satisfying (7.22), there exist a real constant $K_{F_\omega} > 0$ and a mapping $F_\omega \in C\left(\Omega, R^+\right)$, such that we get a solution $\psi \in B$ of (7.2), satisfying

$$
|\phi(t)-\psi(t)| \leq K_{F_\omega}F_\omega(t),\; t\in\Omega, \qquad (7.34)
$$

where $F_\omega=\max(F_{\omega_i})$ and $K_{F_\omega}=\max(K_{F_{\omega_i}})$, for $i=1,2,3$.

Remark 7.2. The mapping $\phi \in B$ will be the solution of (7.21), if we get a mapping $\theta \in B$, which satisfies

1. $|\theta(t)| \leq \xi, \theta = \max(\theta_i),\ i = 1, 2, 3;$

2. ${}_0^{ABC}D_t^{\alpha}\phi(t) = \omega\left(t, \phi(t)\right) + \theta(t),\ \forall t \in \Omega.$

Hypothesis 3. For a given $F_\omega \in B$, there exists $\lambda_{F_\omega} > 0$, such that

$$ {}_0^{ABC}I_t^{\alpha}F_\omega \leq \lambda_{F_\omega}F_\omega(t), \forall t \in \Omega. \tag{7.35}$$

Lemma 7.5. Let $0 \leq \alpha \leq 1$. If $\phi \in B$ satisfies (7.21), we get $\phi \in B$ satisfying

$$\begin{aligned} &\left|\phi(t) - \phi_0 - \frac{1-\alpha}{\Lambda(\alpha)}\omega\left(t, \phi(t)\right) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\omega\left(t, \phi(\tau)\right)d\tau\right| \\ &\quad \leq \xi\lambda_{F_\omega}F_\omega(t). \end{aligned} \tag{7.36}$$

Proof. Suppose $\phi \in B$ satisfies (7.21).
Therefore, considering Remark – 7.2, yields

$$ {}_0^{ABC}D_t^{\alpha}\phi(t) = \omega\left(t, \phi(t)\right) + \theta(t),\ t \in \Omega,\ \phi(0) = \phi_0. \tag{7.37}$$

The solution of (7.37) is given as

$$\begin{aligned} \phi(t) &= \phi_0 + \frac{1-\alpha}{\Lambda(\alpha)}\omega\left(t, \phi(t)\right) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\omega\left(t, \phi(\tau)\right)d\tau, \\ &\leq \frac{1-\alpha}{\Lambda(\alpha)}\theta(t) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\theta(\tau)d\tau. \end{aligned} \tag{7.38}$$

Again, by Remark – 7.2, we get

$$\left|\phi(t) - \phi_0 - \frac{1-\alpha}{\Lambda(\alpha)}\omega\left(t, \phi(t)\right) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\omega\left(t, \phi(\tau)\right) d\tau\right|$$
$$\leq \left|\frac{1-\alpha}{\Lambda(\alpha)}\theta(t) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\theta(\tau) d\tau\right|,$$
$$\leq \frac{1-\alpha}{\Lambda(\alpha)}\left|\theta(t)\right| - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left|\theta(\tau)\right| d\tau,$$
$$\leq \xi\lambda_{F_\omega}F_\omega(t). \tag{7.39}$$

Hence, proved.

Theorem 7.4. Under the hypothesis – 2 for $\psi(t)$ as in (7.9), considering the mapping $\omega \in C(\Omega \times R, R)$, for every $\psi \in B$ we have fractional diabetes system (7.2) is Ulam-Hyers-Rassias stable over Ω.

Proof. Using lemma – 7.5, and hypothesis – 2 and 3, we have

$$\left|\phi(t) - \psi(t)\right| \leq \left|\begin{array}{l} \frac{1-\alpha}{\Lambda(\alpha)}\omega\left(t, \phi(t)\right) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\omega\left(t, \phi(\tau)\right) d\tau \\ -\frac{1-\alpha}{\Lambda(\alpha)}\omega\left(t, \psi(t)\right) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\omega\left(t, \psi(\tau)\right) d\tau \end{array}\right|,$$
$$\leq \left(\frac{1-\alpha}{\Lambda(\alpha)} - \frac{T^\alpha}{\Lambda(\alpha)\Gamma(\alpha)}\right)\xi + \frac{\alpha L_\omega}{\Lambda(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left|\phi(\tau) - \psi(\tau)\right| d\tau,$$
$$\leq \lambda_{F_\omega}F_\omega(t)\xi + \frac{T^\alpha L_\omega}{\Lambda(\alpha)\Gamma(\alpha)}\left|\phi(\tau) - \psi(\tau)\right|. \tag{7.40}$$

Thus, we have

$$\left|\phi(t) - \psi(t)\right| \leq K_{F_\omega}\xi F_\omega(t) \quad \text{where } K_{F_\omega} = \frac{\lambda_{F_\omega}}{1 - \frac{T^\alpha L_\omega}{\Lambda(\alpha)\Gamma(\alpha)}}. \tag{7.41}$$

Hence, the theorem is proved.

Corollary 7.2. If we have $\xi = 1$ in Theorem – 7.4, then system (7.2) is GUHR stable.

7.6 Working of Adomian Decomposition Laplace Transform Method

Consider an FDE with ABC operator as:

$$ {}_{0}^{ABC}D_t^{\alpha}\psi(t) + R\psi(t) + N\psi(t) = f(t),\ t > 0,\ n-1 < \alpha \leq n,\ n \in N, \tag{7.42}$$

such that

$$\psi(0) = C_0,\ \frac{d\psi(0)}{dt} = C_1,\ \frac{d^2\psi(0)}{dt^2} = C_2, ...,\ \frac{d^{n-1}\psi(0)}{dt^{n-1}} = C_{n-1}, \tag{7.43}$$

where R denotes linear terms, N is non-linear terms of $\psi(t)$, and $f(t)$ is continuous function.

Taking Laplace transform to (7.42),

$$\mathcal{L}\left\{{}_{0}^{ABC}D_t^{\alpha}\psi(t)\right\} = -\mathcal{L}\left\{R\psi(t)\right\} - \mathcal{L}\left\{N\psi(t)\right\} + \mathcal{L}\left\{f(t)\right\}, \tag{7.44}$$

using differentiation properties (1.6),

$$\begin{aligned}\frac{\Lambda(\alpha)}{(n-\alpha)s^{\alpha}+\alpha}\left[s^{\alpha}\mathcal{L}\left\{\psi(t)\right\} - \sum_{k=0}^{n-1} s^{\alpha-k-1}D_t^{k}\psi(0)\right] =& -\mathcal{L}\left\{R\psi(t)\right\} - \mathcal{L}\left\{N\psi(t)\right\} \\ &+ \mathcal{L}\left\{f(t)\right\},\end{aligned} \tag{7.45}$$

with normalized function considering $\Lambda(\alpha) = 1$, we get

$$\begin{aligned}\mathcal{L}\left\{\psi(t)\right\} =& \left[\frac{1}{s}\psi(0) + \frac{1}{s^2}\frac{d\psi(0)}{dt} + ... + \frac{1}{s^n}\frac{d^{n-1}\psi(0)}{dt^{n-1}}\right] \\ &+ \left(\frac{(n-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\left[-\mathcal{L}\left\{R\psi(t)\right\} - \mathcal{L}\left\{N\psi(t)\right\} + \mathcal{L}\left\{f(t)\right\}\right].\end{aligned} \tag{7.46}$$

Taking inverse Laplace transform (ILT) to (7.46),

$$\psi(t) = \omega(t) - \mathcal{L}^{-1}\left[\left(\frac{(n-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\mathcal{L}\left\{R\psi(t)\right\} + \left(\frac{(n-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\mathcal{L}\left\{N\psi(t)\right\}\right], \tag{7.47}$$

where $\omega(t)$ denotes ILT of first and last terms of (7.46).

Applying Adomian decomposition method [4] to (7.47) gives

$$\begin{aligned}\sum_{i=0}^{\infty}\psi_i(t) =&\omega(t)-\mathcal{L}^{-1}\left[\left(\frac{(n-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\mathcal{L}\left\{R\left(\sum_{i=0}^{\infty}\psi_i(t)\right)\right\}\right.\\ &\left.+\left(\frac{(n-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\mathcal{L}\left\{N\left(\sum_{i=0}^{\infty}\psi_i(t)\right)\right\}\right], \end{aligned} \tag{7.48}$$

where nonlinear terms of (7.48) using Adomian polynomial [160],

$$\mathrm{A}_n=\frac{1}{n}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n}\lambda^i\psi_i(t)\right)\right],\quad \lambda=0, n=0,1,2,... \tag{7.49}$$

Substituting (7.49) into (7.48), and comparing the both sides, we get

$$\begin{aligned}&\psi_0(t)=\omega(t),\\ &\psi_1(t)=-\mathcal{L}^{-1}\left[\left(\tfrac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\mathcal{L}\left\{R\psi_0(t)\right\}+\left(\tfrac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\mathcal{L}\left\{A_0\right\}\right],\\ &\psi_2(t)=-\mathcal{L}^{-1}\left[\left(\tfrac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\mathcal{L}\left\{R\psi_1(t)\right\}+\left(\tfrac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\mathcal{L}\left\{A_1\right\}\right],\\ &\vdots\end{aligned} \tag{7.50}$$

Hence the equation (7.42) gives the solution as

$$\psi(t)=\sum_{i=0}^{\infty}\psi_i(t)=\psi_0(t)+\psi_1(t)+\psi_2(t)+... \tag{7.51}$$

7.7 Solution of Diabetes Model

This section provides the semi-analytic approach ADLTM solution to the diabetes model (7.2) with initial conditions (7.3).

Operating Laplace transform to the system of differential equations (7.2), we have

$$\begin{aligned}&\mathcal{L}\left\{{}_0^{ABC}D_t^{\alpha}\mathbb{G}(t)\right\}=\mathcal{L}\left\{-m_1\mathbb{G}(t)+m_2\mathbb{I}(t)+m_1\mathbb{G}_b\right\},\\ &\mathcal{L}\left\{{}_0^{ABC}D_t^{\alpha}\mathbb{X}(t)\right\}=\mathcal{L}\left\{-m_2\mathbb{X}(t)+m_3\mathbb{I}(t)-m_3\mathbb{I}_b+m_6\mathbb{I}_b\right\},\\ &\mathcal{L}\left\{{}_0^{ABC}D_t^{\alpha}\mathbb{I}(t)\right\}=\mathcal{L}\left\{-m_3\mathbb{I}(t)+m_4\mathbb{G}(t)+m_4m_5-m_6\mathbb{I}(t)+m_6\mathbb{I}_b\right\},\end{aligned} \tag{7.52}$$

using differentiation property (1.6), we have

$$\begin{aligned}
\frac{1}{(1-\alpha)\left(s^{\alpha}+\frac{\alpha}{1-\alpha}\right)}\left[s^{\alpha}\mathcal{L}\left\{\mathbb{G}(t)\right\}-s^{\alpha-1}\mathbb{G}(0)\right] =&\mathcal{L}\left\{-m_1\mathbb{G}(t)+m_2\mathbb{I}(t)+m_1\mathbb{G}_b\right\},\\
\frac{1}{(1-\alpha)\left(s^{\alpha}+\frac{\alpha}{1-\alpha}\right)}\left[s^{\alpha}\mathcal{L}\left\{\mathbb{X}(t)\right\}-s^{\alpha-1}\mathbb{X}(0)\right] =&\mathcal{L}\left\{-m_2\mathbb{X}(t)+m_3\mathbb{I}(t)-m_3\mathbb{I}_b+m_6\mathbb{I}_b\right\},\\
\frac{1}{(1-\alpha)\left(s^{\alpha}+\frac{\alpha}{1-\alpha}\right)}\left[s^{\alpha}\mathcal{L}\left\{\mathbb{I}(t)\right\}-s^{\alpha-1}\mathbb{I}(0)\right] =&\mathcal{L}\left\{-m_3\mathbb{I}(t)+m_4\mathbb{G}(t)+m_4m_5\right.\\
&\left.-\,m_6\mathbb{I}(t)+m_6\mathbb{I}_b\right\},
\end{aligned}\tag{7.53}$$

considering the initial conditions (7.3) and taking inverse Laplace transform

$$\begin{aligned}
&\mathbb{G}(t)=\mathbb{G}_0+\mathcal{L}^{-1}\left[\tfrac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\mathcal{L}\left\{-m_1\mathbb{G}(t)+m_2\mathbb{I}(t)+m_1\mathbb{G}_b\right\}\right],\\
&\mathbb{X}(t)=\mathbb{X}_0+\mathcal{L}^{-1}\left[\tfrac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\mathcal{L}\left\{-m_2\mathbb{X}(t)+m_3\mathbb{I}(t)-m_3\mathbb{I}_b+m_6\mathbb{I}_b\right\}\right],\\
&\mathbb{I}(t)=\mathbb{I}_0+\mathcal{L}^{-1}\left[\tfrac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\mathcal{L}\left\{-m_3\mathbb{I}(t)+m_4\mathbb{G}(t)+m_4m_5-m_6\mathbb{I}(t)+m_6\mathbb{I}_b\right\}\right],
\end{aligned}\tag{7.54}$$

now we apply Adomian decomposition method [4] to equations (7.54), which yields

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathbb{G}_n(t) =&\mathbb{G}_0+\mathcal{L}^{-1}\left[\frac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\mathcal{L}\left\{-m_1\sum_{n=0}^{\infty}\mathbb{G}_n(t)+m_2\sum_{n=0}^{\infty}\mathbb{I}_n(t)+m_1\mathbb{G}_b\right\}\right],\\
\sum_{n=0}^{\infty}\mathbb{X}_n(t) =&\mathbb{X}_0+\mathcal{L}^{-1}\left[\frac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\mathcal{L}\left\{-m_2\sum_{n=0}^{\infty}\mathbb{X}_n(t)+m_3\sum_{n=0}^{\infty}\mathbb{I}_n(t)\right.\right.\\
&\left.\left.-\,m_3\mathbb{I}_b+m_6\mathbb{I}_b\right\}\right],\\
\sum_{n=0}^{\infty}\mathbb{I}_n(t) =&\mathbb{I}_0+\mathcal{L}^{-1}\left[\frac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\mathcal{L}\left\{-m_3\sum_{n=0}^{\infty}\mathbb{I}_n(t)+m_4\sum_{n=0}^{\infty}\mathbb{G}_n(t)\right.\right.\\
&\left.\left.+\,m_4m_5-m_6\sum_{n=0}^{\infty}\mathbb{I}_n(t)+m_6\mathbb{I}_b\right\}\right].
\end{aligned}\tag{7.55}$$

Comparing series terms of (7.55) and taking the values of initial conditions as in (7.3) from [136], we have

$$\mathbb{G}_0(t)=\mathbb{G}_0=100,\mathbb{X}_0(t)=\mathbb{X}_0=0,\mathbb{I}_0(t)=\mathbb{I}_0=7,$$

and considering the values of the parameters from the table - 7.1 as given in section - 7.1, taken from [136] for the normal person as below, and its simplification yields the answer what we seek:

$$
\begin{aligned}
\mathbb{G}_1(t) &= \mathcal{L}^{-1}\left[\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\mathcal{L}\left\{-m_1\mathbb{G}_0(t)+m_2\mathbb{I}_0(t)+m_1\mathbb{G}_b\right\}\right],\\
&= -0.548\left(1-\alpha+\alpha\frac{t^\alpha}{\Gamma(1+\alpha)}\right),\\
\mathbb{X}_1(t) &= \mathcal{L}^{-1}\left[\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\mathcal{L}\left\{-m_2\mathbb{X}_0(t)+m_3\mathbb{I}_0(t)-m_3\mathbb{I}_b+m_6\mathbb{I}_b\right\}\right],\\
&= 1.849\left(1-\alpha+\alpha\frac{t^\alpha}{\Gamma(1+\alpha)}\right),\\
\mathbb{I}_1(t) &= \mathcal{L}^{-1}\left[\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\mathcal{L}\left\{-m_3\mathbb{I}_0(t)+m_4\mathbb{G}_0(t)+m_4m_5-m_6\mathbb{I}_0(t)+m_6\mathbb{I}_b\right\}\right],\\
&= 0.698\left(1-\alpha+\alpha\frac{t^\alpha}{\Gamma(1+\alpha)}\right),\\
\mathbb{G}_2(t) &= \mathcal{L}^{-1}\left[\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\mathcal{L}\left\{-m_1\mathbb{G}_1(t)+m_2\mathbb{G}_1(t)+m_1\mathbb{G}_b\right\}\right],\\
&= 2.536\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right)\\
&\quad+0.026\left((1-\alpha)^2+2\alpha(1-\alpha)\frac{t^\alpha}{\Gamma(1-\alpha)}+\alpha^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right),\\
\mathbb{X}_2(t) &= \mathcal{L}^{-1}\left[\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\mathcal{L}\left\{-m_2\mathbb{X}_1(t)+m_3\mathbb{I}_1(t)-m_3\mathbb{I}_b+m_6\mathbb{I}_b\right\}\right],\\
&= 1.861\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right)\\
&\quad-0.023\left((1-\alpha)^2+2\alpha(1-\alpha)\frac{t^\alpha}{\Gamma(1-\alpha)}+\alpha^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right),\\
\mathbb{I}_2(t) &= \mathcal{L}^{-1}\left[\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\mathcal{L}\left\{-m_3\mathbb{I}_1(t)+m_4\mathbb{G}_1(t)+m_4m_5-m_6\mathbb{I}_1(t)+m_6\mathbb{I}_b\right\}\right],\\
&= 2.17\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right)\\
&\quad-0.183\left((1-\alpha)^2+2\alpha(1-\alpha)\frac{t^\alpha}{\Gamma(1-\alpha)}+\alpha^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right),\\
&\vdots\\
\mathbb{G}_n(t) &= \mathcal{L}^{-1}\left[\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\mathcal{L}\left\{-m_1\mathbb{G}_{n-1}(t)+m_2\mathbb{I}_{n-1}(t)+m_1\mathbb{G}_b\right\}\right],\\
\mathbb{X}_n(t) &= \mathcal{L}^{-1}\left[\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\mathcal{L}\left\{-m_2\mathbb{X}_{n-1}(t)+m_3\mathbb{I}_{n-1}(t)-m_3\mathbb{I}_b+m_6\mathbb{I}_b\right\}\right],\\
\mathbb{I}_n(t) &= \mathcal{L}^{-1}\left[\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\mathcal{L}\left\{-m_3\mathbb{I}_{n-1}(t)+m_4\mathbb{G}_{n-1}(t)+m_4m_5\right.\right.\\
&\quad\left.\left.-\ m_6\mathbb{I}_{n-1}(t)+m_6\mathbb{I}_b\right\}\right].
\end{aligned}
\tag{7.56}
$$

Thus, the approximate solution of (7.2) can be obtained as

$$\begin{aligned}
\mathbb{G}(t) &= 100 - 0.548\left(1-\alpha+\alpha\frac{t^\alpha}{\Gamma(1+\alpha)}\right) + 2.536\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right) \\
&\quad + 0.026\left((1-\alpha)^2 + 2\alpha(1-\alpha)\frac{t^\alpha}{\Gamma(1-\alpha)} + \alpha^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) + ... \\
\mathbb{X}(t) &= 0 + 1.849\left(1-\alpha+\alpha\frac{t^\alpha}{\Gamma(1+\alpha)}\right) + 1.861\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right) \\
&\quad - 0.023\left((1-\alpha)^2 + 2\alpha(1-\alpha)\frac{t^\alpha}{\Gamma(1-\alpha)} + \alpha^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) + ... \\
\mathbb{I}(t) &= 7 + 0.698\left(1-\alpha+\alpha\frac{t^\alpha}{\Gamma(1+\alpha)}\right) + 2.17\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right) \\
&\quad - 0.183\left((1-\alpha)^2 + 2\alpha(1-\alpha)\frac{t^\alpha}{\Gamma(1-\alpha)} + \alpha^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) + ...
\end{aligned} \tag{7.57}$$

Table 7.2: The approximate solution of diabetes model for Normal Person with several fractional order α.

Order	The solution of Diabetes Model
$\alpha = 1$	$\mathbb{G}(t) = 100 - 0.5479t + 1.280978590t^2 + ...$ $\mathbb{X}(t) = 1.86130000t + 0.9191875025t^2 + ...$ $\mathbb{I}(t) = 7 + 0.69827226t + 0.9908979305t^2 + ...$
$\alpha = 0.9$	$\mathbb{G}(t) = 99.97082957 - 0.0332281588t^{0.9} + 1.237814922t^{1.8} + ...$ $\mathbb{X}(t) = 0.2045137500 + 2.085827375t^{0.9} + 0.8882146950t^{1.8} + ...$ $\mathbb{I}(t) = 7.089645185 + 1.024332802t^{0.9} + 0.9575087789t^{1.8} + ...$
$\alpha = 0.8$	$\mathbb{G}(t) = 99.99289829 + 0.4096123521t^{0.8} + 1.146911323t^{1.6} + ...$ $\mathbb{X}(t) = 0.4457950002 + 2.230358813t^{0.8} + 0.8229853047t^{1.6} + ...$ $\mathbb{I}(t) = 7.218926286 + 1.280667025t^{0.8} + 0.8871905166t^{1.6} + ...$
$\alpha = 0.7$	$\mathbb{G}(t) = 100.0662061 + 0.7621202906t^{0.7} + 1.010618256t^{1.4} + ...$ $\mathbb{X}(t) = 0.7238437504 + 2.283666134t^{0.7} + 0.7251859467t^{1.4} + ...$ $\mathbb{I}(t) = 7.387843306 + 1.453982530t^{0.7} + 0.7817613401t^{1.4} + ...$
$\alpha = 0.6$	$\mathbb{G}(t) = 100.1907531 + 1.008376014t^{0.6} + 0.8370870388t^{1.2} + ...$ $\mathbb{X}(t) = 1.038660001 + 2.237454571t^{0.6} + 0.6006657339t^{1.2} + ...$ $\mathbb{I}(t) = 7.596396242 + 1.533521915t^{0.6} + 0.6475266809t^{1.2} + ...$

In a similar fashion deriving the results for diabetic person we get details as described in below:

Table 7.3: The approximate solution of diabetes model for diabetic person with several fractional order α.

Order	The solution of Diabetes Model
$\alpha = 1$	$\mathbb{G}(t) = 240 - 2.55t - 0.2223209850t^2 + \ldots$ $\mathbb{X}(t) = 0.396t + 0.1643039254t^2 + \ldots$ $\mathbb{I}(t) = 15 + 1.3869705t + 0.3745683140t^2 + \ldots$
$\alpha = 0.9$	$\mathbb{G}(t) = 239.7405536 - 2.469453037t^{0.9} - 0.2148296896t^{1.8} + \ldots$ $\mathbb{X}(t) = 0.4288607851 + 0.4320692204t^{0.9} + 0.1587675644t^{1.8} + \ldots$ $\mathbb{I}(t) = 15.14618842 + 1.438102703t^{0.9} + 0.3619469149t^{1.8} + \ldots$
$\alpha = 0.8$	$\mathbb{G}(t) = 239.4722143 - 2.343057178t^{0.8} - 0.1990528625t^{1.6} + \ldots$ $\mathbb{X}(t) = 0.9234431404 + 0.4530404387t^{0.8} + 0.1471078706t^{1.6} + \ldots$ $\mathbb{I}(t) = 15.30735957 + 1.448704780t^{0.8} + 0.3353659804t^{1.6} + \ldots$
$\alpha = 0.7$	$\mathbb{G}(t) = 239.1949822 - 2.170003937t^{0.7} - 0.1753984400t^{1.4} + \ldots$ $\mathbb{X}(t) = 0.1483747066 + 0.4569641182t^{0.7} + 0.1296263248t^{1.4} + \ldots$ $\mathbb{I}(t) = 15.48351345 + 1.414772106t^{0.7} + 0.2955128052t^{1.4} + \ldots$
$\alpha = 0.6$	$\mathbb{G}(t) = 238.9088573 - 1.951201115t^{0.6} - 0.1452811284t^{1.2} + \ldots$ $\mathbb{X}(t) = 0.2109772561 + 0.4424454138t^{0.6} + 0.1073684506t^{1.2} + \ldots$ $\mathbb{I}(t) = 15.67465006 + 1.333796764t^{0.6} + 0.2447708988t^{1.2} + \ldots$

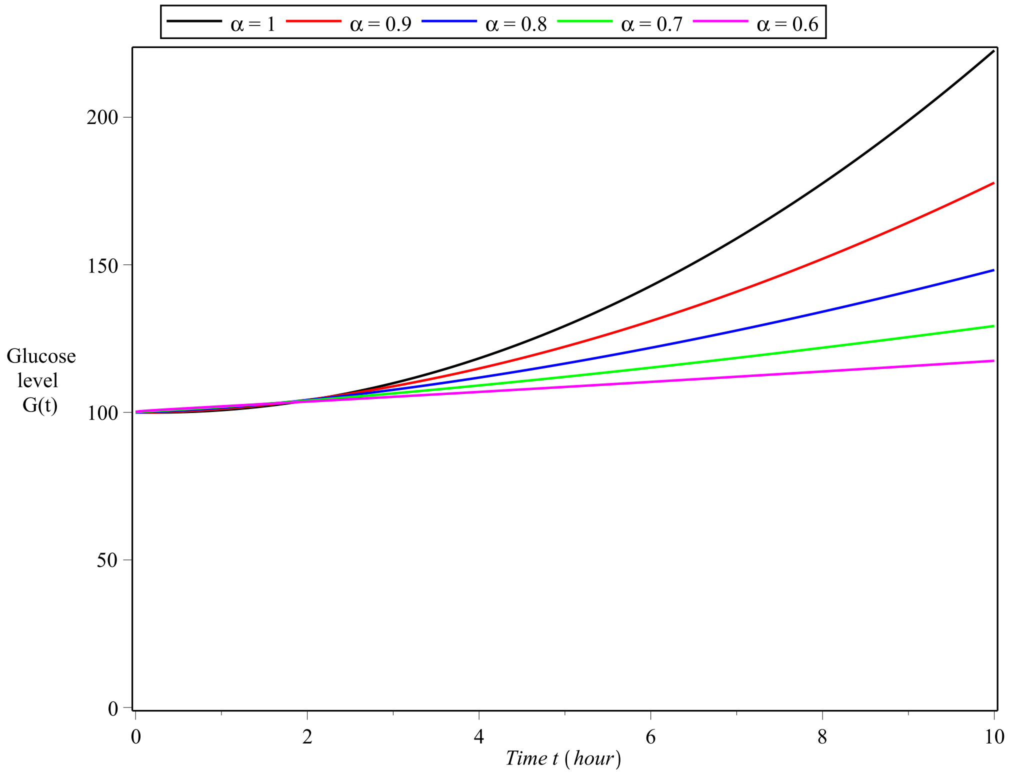

Figure 7.2: Glucose level $\mathbb{G}(t)$ of Normal person at various order α.

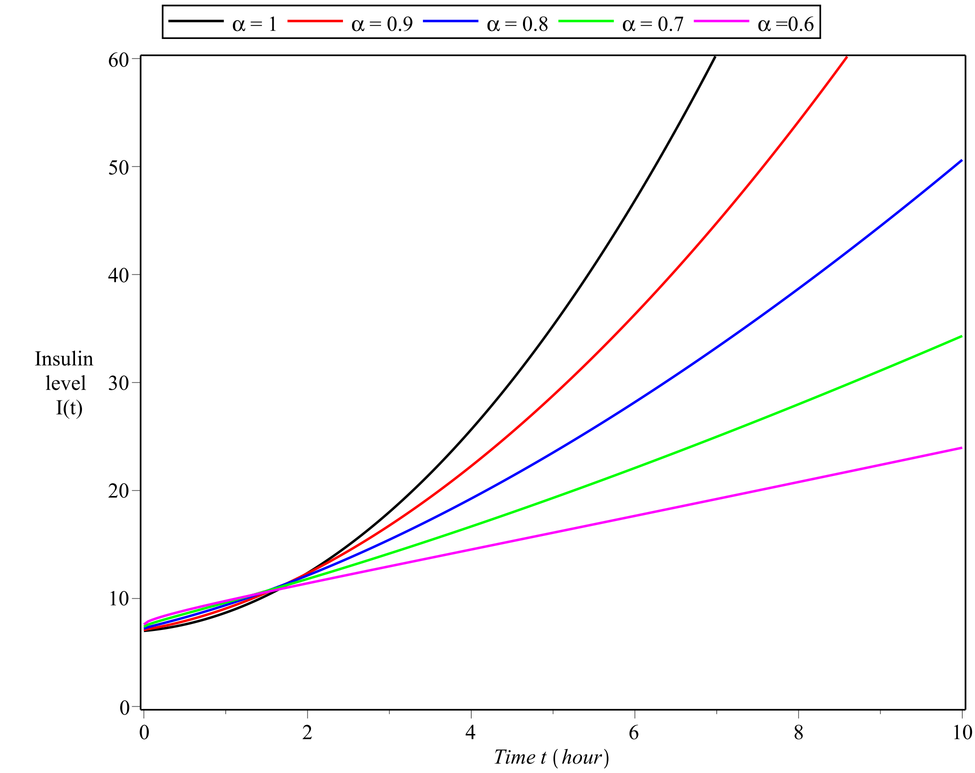

Figure 7.3: Insulin level $\mathbb{I}(t)$ of Normal person at various order α.

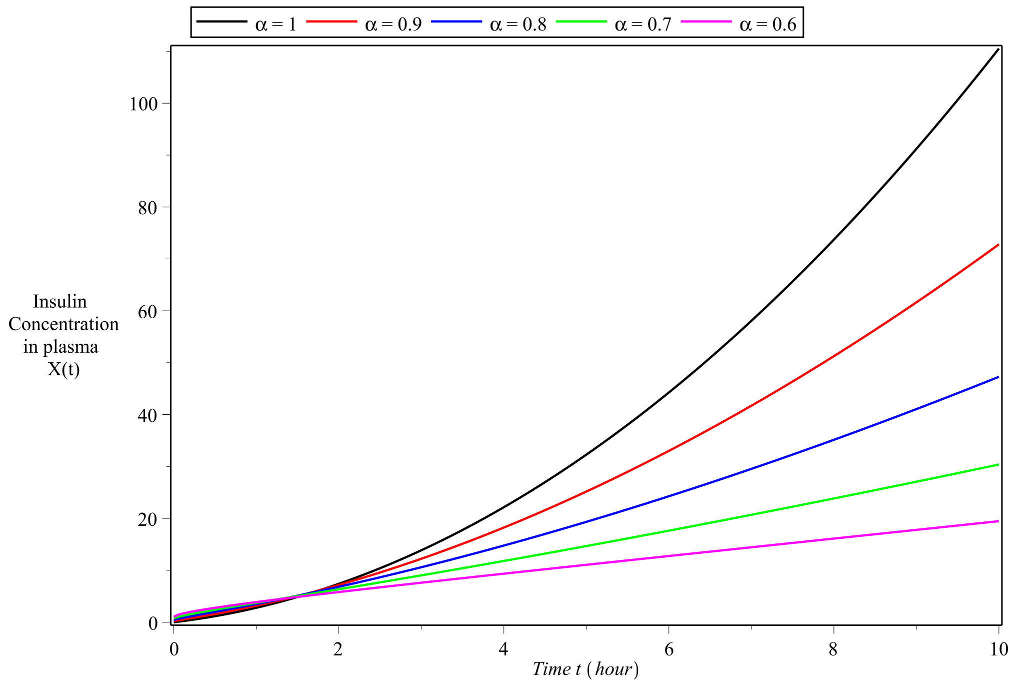

Figure 7.4: Insulin level in plasma $\mathbb{X}(t)$ of Normal person at various order α.

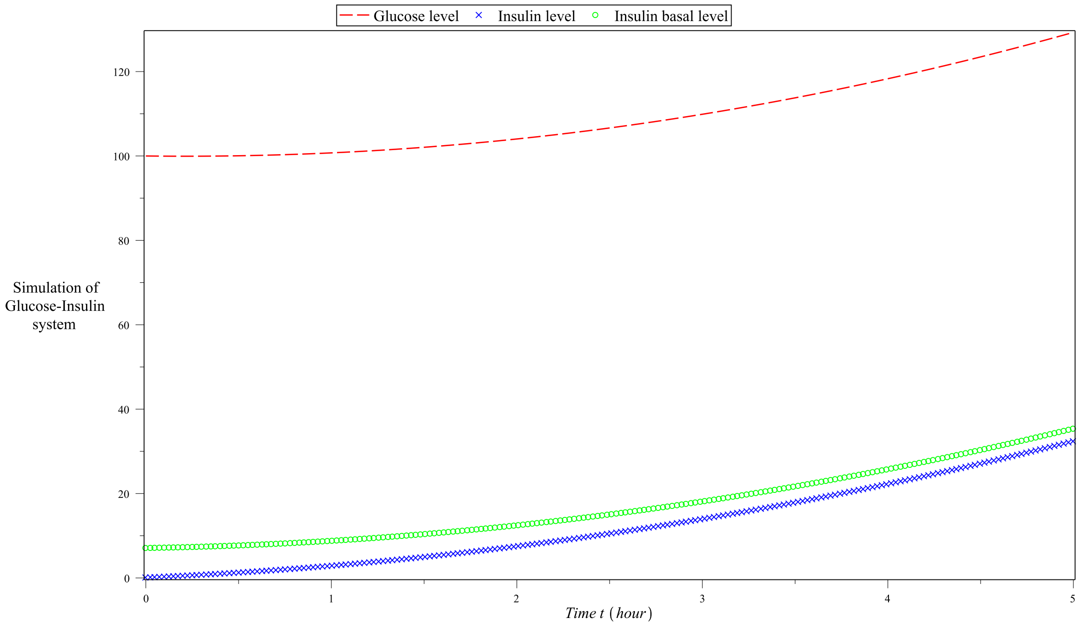

Figure 7.5: The Comparison of Glucose level $\mathbb{G}(t)$, Insulin level $\mathbb{I}(t)$, and Insulin level in plasma $\mathbb{X}(t)$ of Normal person is shown at $\alpha = 1$.

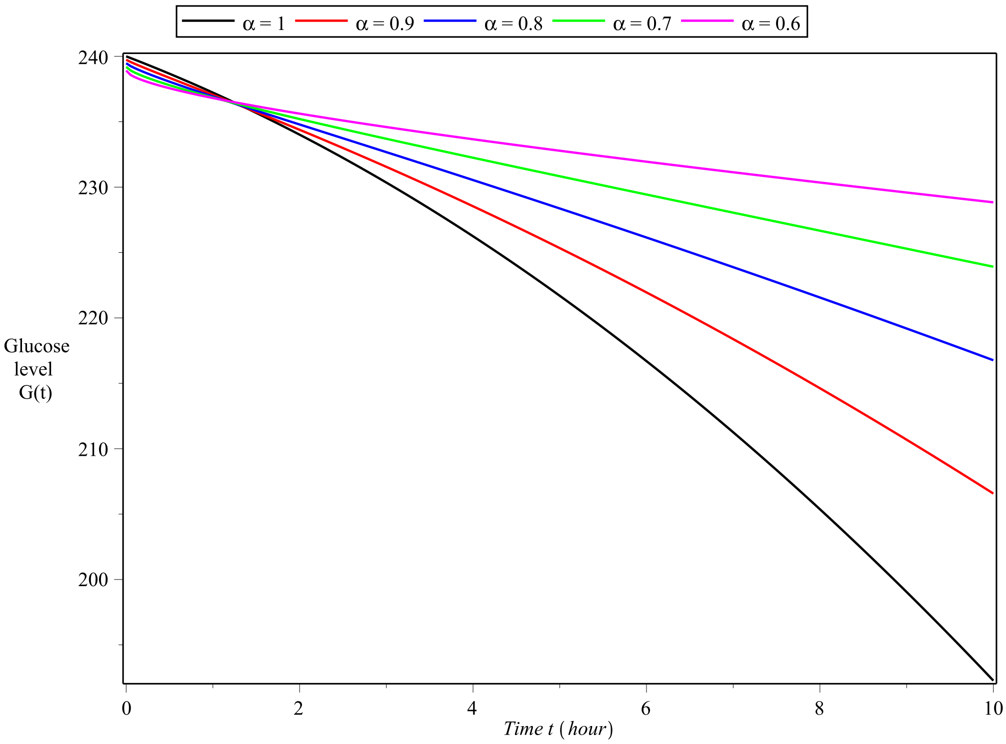

Figure 7.6: Glucose level $\mathbb{G}(t)$ of Type-1 Diabetic patient at various order α.

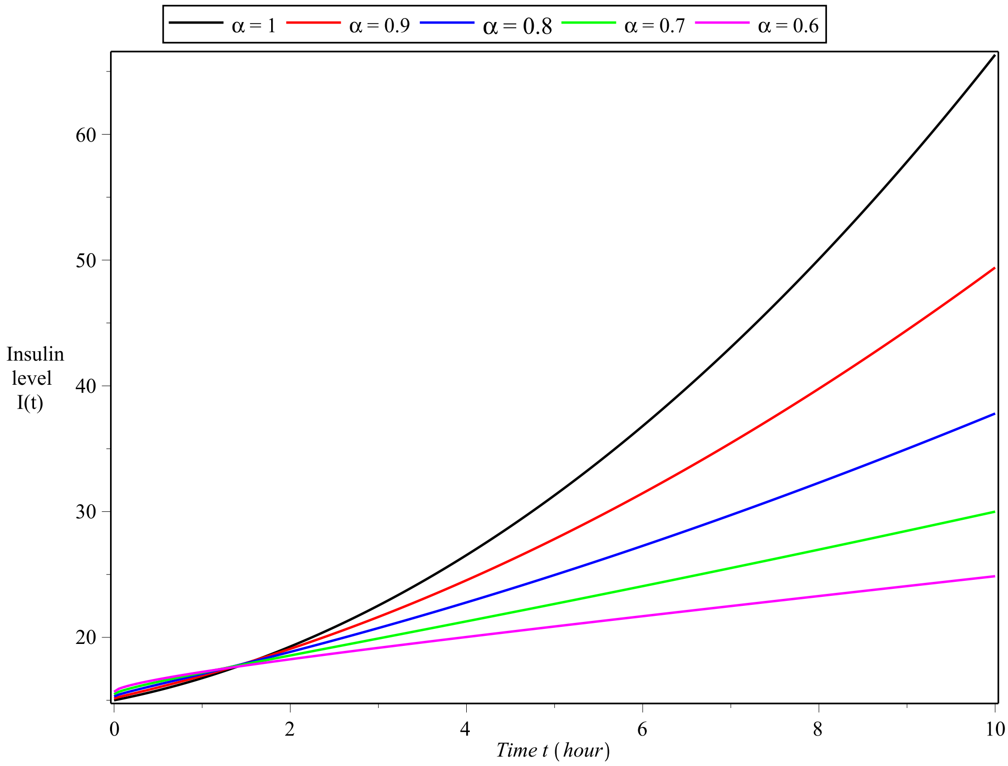

Figure 7.7: Insulin level $\mathbb{I}(t)$ of Type-1 Diabetic patient at various order α.

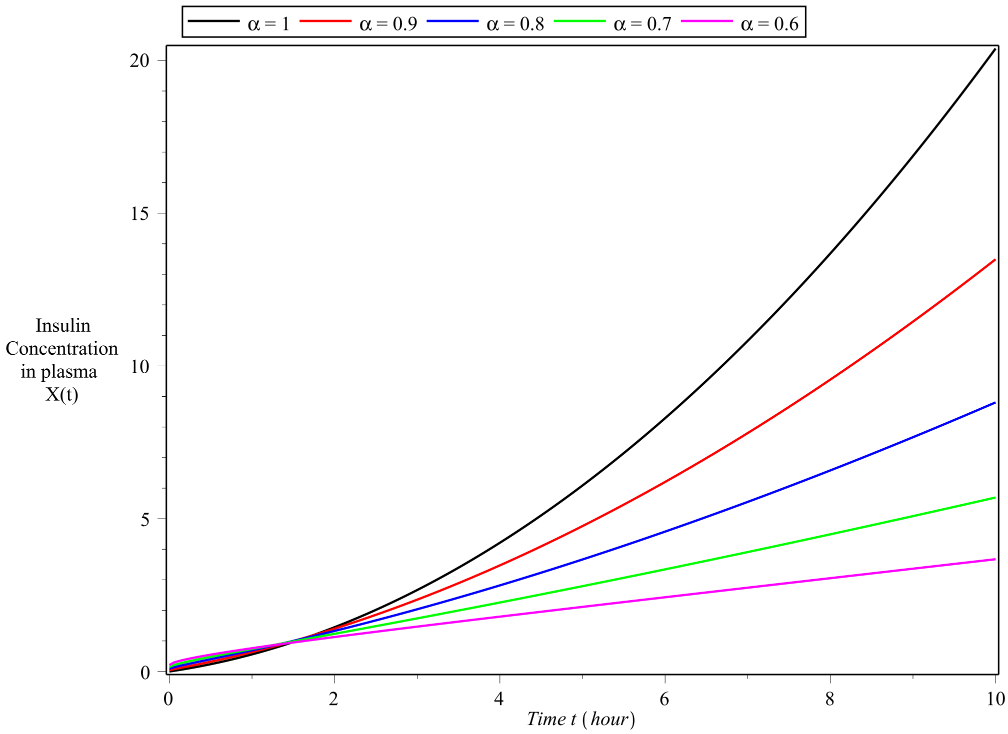

Figure 7.8: Insulin level in plasma $\mathbb{X}(t)$ of Type-1 Diabetic patient at various order α.

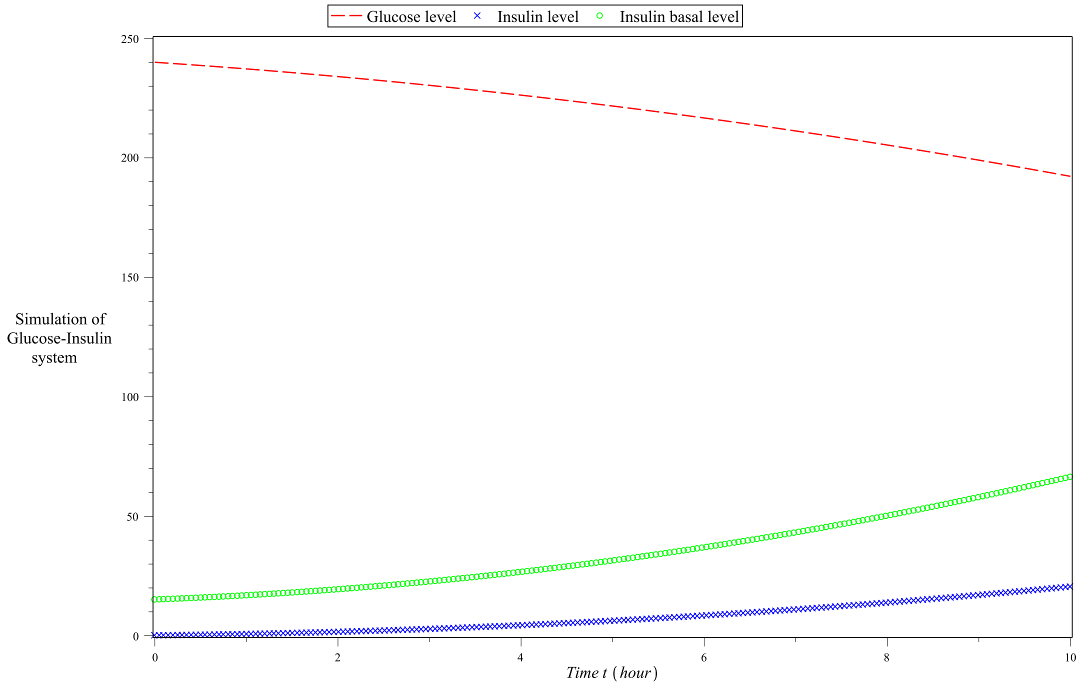

Figure 7.9: The Comparison of Glucose level $\mathbb{G}(t)$, Insulin level $\mathbb{I}(t)$, and Insulin level in plasma $\mathbb{X}(t)$ of Type-1 Diabetic patient is shown at $\alpha = 1$.

Table 7.4: The Absolute error in ADLTM with the comparison of Modified homotopy analysis transform method (MHATM) [155] and Homotopy perturbation transform method (HPTM) [164] at $\alpha = 1$ for Normal person dynamic system.

Prime Parameters	t	| Exact-ADLTM |	| Exact-MHATM |	| Exact-HPTM |
$\mathbb{G}(t)$	2	2.85623E-07	2.19226E-05	1.09284E-05
	4	4.72386E-07	3.14359E-05	4.01337E-05
	6	7.86792E-07	5.02589E-04	4.32780E-04
	8	1.32745E-06	7.94189E-04	2.57343E-03
	10	9.02689E-06	3.94315E-03	8.82410E-03
$\mathbb{X}(t)$	2	2.32624E-08	1.19700E-06	3.17700E-06
	4	5.28570E-08	4.47059E-06	5.23486E-06
	6	6.27946E-08	7.19379E-06	1.58273E-05
	8	1.30581E-07	3.16379E-05	7.82057E-05
	10	3.90589E-06	7.37379E-05	1.35837E-04
$\mathbb{I}(t)$	2	2.80589E-07	1.37378E-06	1.36379E-06
	4	3.30378E-07	4.63269E-06	4.35294E-06
	6	8.13569E-07	8.62267E-06	1.52568E-05
	8	3.41367E-06	2.56689E-05	5.67268E-05
	10	1.89478E-05	6.22678E-05	3.54380E-04

Table 7.5: The Absolute error in ADLTM with the comparison of MHATM and HPTM at $\alpha = 1$ for Type–1 Diabetic patient system.

Prime Parameters	t	| Exact-ADLTM |	|Exact-MHATM|	|Exact-HPTM|
$\mathbb{G}(t)$	2	2.47732E-07	4.15800E-06	4.71536E-06
	4	9.00367E-07	1.85627E-05	3.88564E-05
	6	1.18263E-06	3.73692E-05	9.01132E-05
	8	4.29356E-06	9.03947E-05	5.99763E-04
	10	8.28063E-06	2.41740E-04	4.46975E-03
$\mathbb{X}(t)$	2	3.28594E-09	2.02750E-07	1.75435E-06
	4	2.48459E-08	1.53510E-06	6.27565E-06
	6	4.37926E-08	4.30855E-06	2.12065E-05
	8	5.50749E-07	2.22600E-05	7.22952E-05
	10	2.02742E-06	7.50837E-05	1.79765E-04
$\mathbb{I}(t)$	2	3.68247E-08	3.14930E-06	1.30375E-05
	4	9.74589E-08	8.75637E-06	6.31557E-05
	6	6.26893E-07	1.17406E-05	1.75135E-04
	8	6.90580E-07	3.85400E-05	4.27645E-04
	10	3.49890E-06	1.78077E-04	9.09237E-04

7.8 Results

The semi-analytical solution of the fractional Diabetes model has been presented by using the ABC derivative. Fixed point theory and the ADLTM technique are used to derive solutions. Tables –7.2 and 7.3 express the solution of the system with different fractional order α as a normal person and a Type-1 diabetic patient. The simulations show that the dynamics of the model are significantly affected by changes in the parameter. Figures $-7.2, 7.3$, and 7.4 show the bounded response for a common person using actual values for plasma concentrations of glucose, insulin, and insulin basal. Similarly figures $-7.6, 7.7$, and 7.8 reflect the solution for a Type-1 Diabetics patient. Figures -7.5 and 7.9 exhibit the whole functioning of the glucose-insulin system in a normal person and a person with Type-1 diabetes mellitus respectively. The Absolute error is calculated in the proposed scheme and other existing methods like the Modified Homotopy Analysis Transform Method (MHATM) and the Homotopy Perturbation Transform Method (HPTM) at fractional order $\alpha = 1$ for Normal person and Type-1 diabetes patient system.

7.9 Biological Interpretation

In the human glucose-insulin regulatory system, diverse metabolic issues can arise, including diabetes type-1 and type-2, hyperinsulinemia, hypoglycemia, etc. In this regard, the analysis and characterization of such a biological system is a necessity. It is well known that mathematical models are an excellent option to study and predict natural phenomena to some extent. In this way, fractional-ordered theory provides generalizations for derivatives and integrals to arbitrary orders giving us a framework to add memory properties and an additional dimension to the mathematical models to approximate real-world phenomena with higher accuracy. In this present work, we study the glucose and insulin governing mechanisms using a fractional-order version of a mathematical model. Applying the fractional-order ABC derivative, we can investigate different concentration rates among insulin level, glucose level, and insulin level in plasma. Additionally, the model incorporates two time-lags to represent the elapsed time of two processes, i.e., the delay in secrete insulin for a blood glucose increment and the lag to get a glucose reduction caused by raised insulin level.

7.10 Conclusions

This chapter discusses the theoretical and numerical analysis of the bio-medical diabetes model utilizing ABC fractional derivative for the investigation of various compartments for glucose-insulin therapy. It demonstrates the consistency, originality, and application of the model for the regulation of blood glucose concentration in a Type-1 diabetic and a normal individual. Through a study of the numerical outputs, the impact of various fractional orders is examined. Graphical representations show how parameters affect the percentage of diabetics who have time-related complications. Through numerical simulation, we can see that the ABC non-integer order derivative exhibits greater conversion efficiency. Tables – 7.4 and 7.5 conclude that the ADLTM gives better accuracy as compared to other existing methods like MHATM, and HPTM. This study's hypothesis has implications for treating diabetes-related problems and for medical professionals who treat diabetic patients.