

Chapter 6

Fractional-order COVID-19 model

6.1 Introduction

In December 2019, world witnessed the historic epidemic (chinese conspiracy?) in terms of spreading of deadly SARS-COV-2 virus (Severe Acute Respiratory Syndrome Corona Virus 2) and its mutations, globally. Millions of people have lost their lives [127], claiming the lives of over 6,927,378 persons while infecting over 765,903,278 individuals in over 180 nations. As per the WHO report, globally 13,350,487,934 vaccine doses have been given till May 8, 2023 (See Figures 1 and 2). This virus can be found in the human body embryological tracheal organ cultures that have been obtained from an adult's respiratory tract [78].

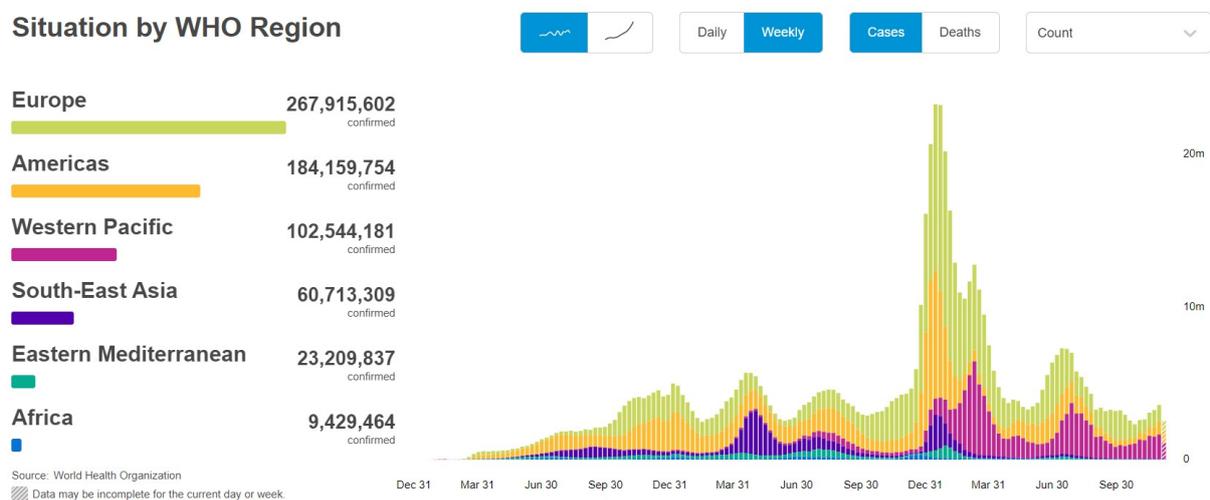


Figure 6.1: 765,903,278 positive COVID-19 cases reported to WHO globally as of May 10, 2023. Data available at <https://covid19.who.int/>

Situation by WHO Region

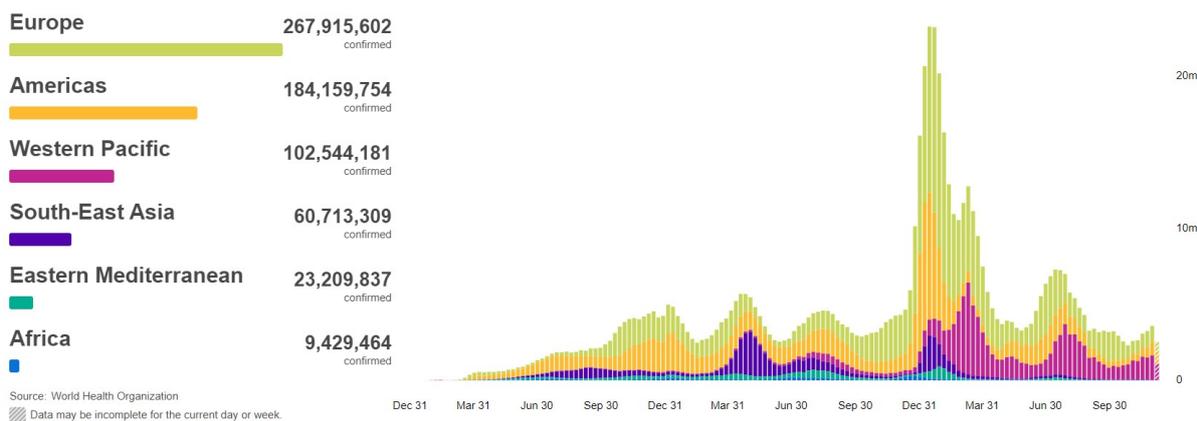


Figure 6.2: As of May 10, 2023, there were 765,903,278 cases reported of COVID-19 worldwide, including 6,927,378 losses. A total of 13,350,487,934 doses of vaccine have been provided as of May 8, 2023. Data available at <https://covid19.who.int/>

Name	Cases - cumulative total	Cases - newly reported in last 7 days	Deaths - cumulative total	Deaths - newly reported in last 7 days	Total vaccine doses administered per 100 population	Persons fully vaccinated with last dose of primary series per 100 population	Persons Boosted per 100 population
Global	647,972,911	3,347,311	6,642,832	9,507	166.89	64.19	30.23
United States of America	98,525,870	453,401	1,077,129	2,762	195.57	68.16	34.15
India	44,675,609	1,170	530,663	10	159.38	68.91	16.16
France	37,660,476	411,058	156,600	711	227.4	77.48	60.47
Germany	36,946,574	206,495	159,737	130	230.6	76.36	62.52
Brazil	35,751,411	253,630	691,449	872	232.26	79.23	50.01

Figure 6.3: Situation of COVID-19 by Region, Country, Territory and Area wise reported to WHO. Data available at <https://covid19.who.int/table>

India Situation

44,675,609

confirmed cases

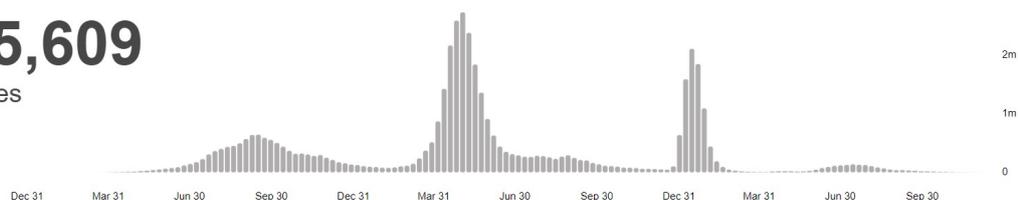


Figure 6.4: In India, from January 3, 2020, to May 10, 2023, there have been 44,972,800 confirmed cases of COVID-19 reported to WHO. Data available at <https://covid19.who.int/region/searo/country/in>

530,663

deaths

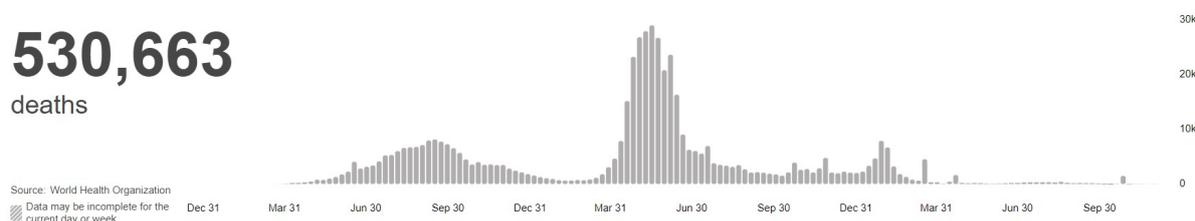


Figure 6.5: In India, from January 3, 2020 to May 10, 2023, WHO received reports of 44,972,800 positive samples of COVID-19, along with 531,707 losses. As of May 1, 2023, a total of 2,206,670,688 vaccine doses have been delivered. Data available at <https://covid19.who.int/region/searo/country/in>

One of the key characteristics of FDEs is more appropriate and realistic answers to models, developed for real world situations. There are many different methods having analytical and numerical approaches for retrieving the solutions for FDEs. The Laplace transform (LT) in combination with the Homotopy Perturbation Method (HPM) is one of the most powerful, effective, and accurate method [64, 65] popularly known as HPLTM [88]. HPLTM is an efficient method for solving initial value problems (IVPs) for the system of FDEs. By using the Laplace transform, the FDE under consideration is incorporated into algebraic equations, called subsidiary equations. The subsidiary equation's solution can be obtained using standard algebraic methods, and then it is transformed back to the main solution.

Numerous investigations on the mathematical modeling of the COVID-19 pandemic have been published as a result of the working of many authors during contemporary circum-

stances [115, 47, 119]. In 2021, Ali et al. [16] studied COVID-19 with four compartments — the healthy or susceptible population $\mathbb{S}(t)$, the exposed class $\mathbb{E}(t)$, the infected population $\mathbb{I}(t)$, and the recovered class $\mathbb{R}(t)$ at time t having first ordered derivatives. In the present work, we considered his model by implementing fractional-order ABC derivative and it deals with following set of equation (where $\alpha, (0 < \alpha \leq 1)$ is the order of the derivative) as

$$\begin{aligned}
 {}_0^{ABC}D_t^\alpha \mathbb{S}(t) &= a - \beta_1 \mathbb{S}(t)\mathbb{I}(t) - (\mu_1 + \mu_2)\mathbb{S}(t), \\
 {}_0^{ABC}D_t^\alpha \mathbb{E}(t) &= \beta_1 \mathbb{S}(t)\mathbb{I}(t) - (\mu_1 + \mu_3 + \beta_3 + \beta_2)\mathbb{E}(t), \\
 {}_0^{ABC}D_t^\alpha \mathbb{I}(t) &= \beta_2 \mathbb{E}(t) - (\mu_1 + \beta_4 + \mu_4)\mathbb{I}(t), \\
 {}_0^{ABC}D_t^\alpha \mathbb{R}(t) &= \beta_3 \mathbb{E}(t) + \beta_4 \mathbb{I}(t) - \mu_1 \mathbb{R}(t),
 \end{aligned}
 \tag{6.1}$$

with initial conditions

$$\mathbb{S}(0) = \mathbb{S}_0, \mathbb{E}(0) = \mathbb{E}_0, \mathbb{I}(0) = \mathbb{I}_0, \mathbb{R}(0) = \mathbb{R}_0.
 \tag{6.2}$$

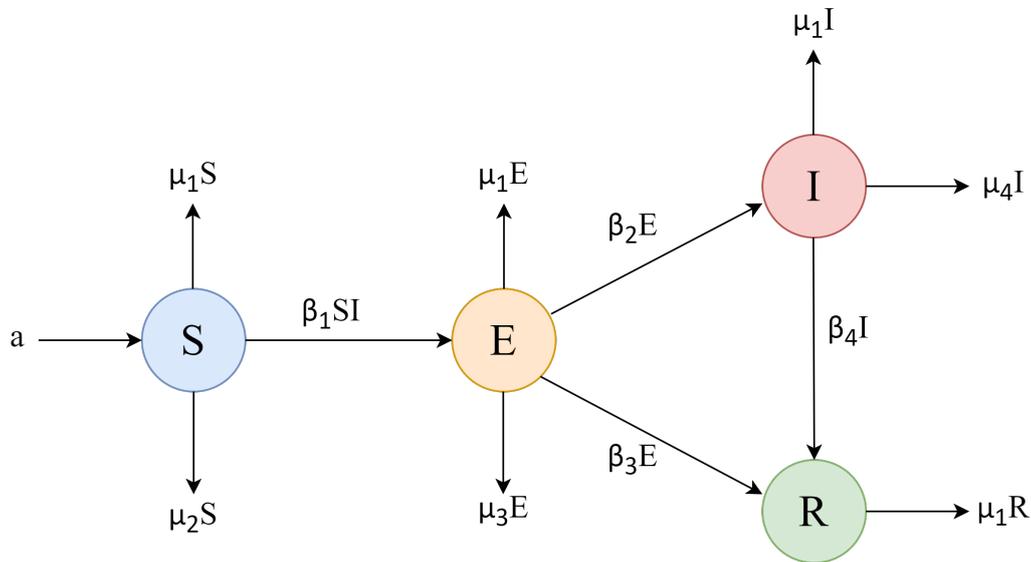


Figure 6.6: Diagram of Covid-19 Model

Table 6.1: The table provides full descriptions of the parameters framed in the model (1).

Notation	Value	Descriptions
\mathbb{S}_0	8065518	Susceptible Human Individuals (HI)
\mathbb{E}_0	200000	Exposed HI
\mathbb{I}_0	28234	Infectious HI
\mathbb{R}_0	0	Recovered HI
β_1	0.03	Rate of transmission from $\mathbb{S}(t)$ to $\mathbb{E}(t)$ HI
β_2	0.25	Rate of transmission from $\mathbb{S}(t)$ to $\mathbb{I}(t)$ HI
β_3	0.28	Recovery rate of $\mathbb{E}(t)$ HI
β_4	0.12	Recovery rate of $\mathbb{I}(t)$ HI
a	0.0018	Birth Rate
μ_1	0.01	Natural Death Rate
μ_2	0.26	$\mathbb{S}(t)$ HI Death rate due to Infection
μ_3	0.21	$\mathbb{E}(t)$ HI Death rate due to Infection
μ_4	0.35	$\mathbb{I}(t)$ HI Death rate due to Infection

6.2 Preliminaries

Lemma 6.1. [43] The solution of FDE as

$$\begin{aligned} {}_0^{ABC}D_t^\alpha \psi(t) &= f(t), \text{ where } f(t) \in C([0, T]), 0 < \alpha < 1, \\ \psi(0) &= \psi_0, \end{aligned} \quad (6.3)$$

is given by

$$\psi(t) = \psi_0 + \frac{(1-\alpha)}{\Gamma(\alpha)} f(t) + \frac{\alpha}{\Gamma(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \quad (6.4)$$

Lemma 6.2. [144, 56, 16] (using Banach's Fixed Point Theorem) Let $V = C[0, T]$ represent the Banach space of continuously defined real valued functions on $[0, T]$. Define a Banach space $B = V \times V \times V \times V$, with norm

$$\|\psi\| = \|\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{R}\| = \sup_{t \in [0, T]} [|\mathbb{S}(t)| + |\mathbb{E}(t)| + |\mathbb{I}(t)| + |\mathbb{R}(t)|], \quad (6.5)$$

where $\psi \in B$ and $\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{R} \in V$.

Moreover, consider a convex subset A of Banach space B . If the operators X, Y on A are such that they satisfy following three conditions, then there exists atleast one fixed point $u \in A$ for $X + Y$ i.e.,

$$(X + Y)(u) = X(u) + Y(u) = u.$$

i) $X(u) + Y(u) \in A, \forall u \in A$;

ii) X is a contraction;

iii) Y is continuous and compact,

then $X(u) + Y(u) = u$ gives at least one solution. **Lemma 6.3.** [144] (Arzela-Ascoli Theorem) If B is a compact metric space, then a closed subspace A of $C[B, R]$ is compact if and only if it is bounded and equicontinuous.

6.3 Qualitative Analysis

For considered model, we construct function

$$\begin{aligned} f_1(t, \mathbb{S}(t), \mathbb{E}(t), \mathbb{I}(t), \mathbb{R}(t)) &= a - \beta_1 \mathbb{S}(t) \mathbb{I}(t) - (\mu_1 + \mu_2) \mathbb{S}(t), \\ f_2(t, \mathbb{S}(t), \mathbb{E}(t), \mathbb{I}(t), \mathbb{R}(t)) &= \beta_1 \mathbb{S}(t) \mathbb{I}(t) - (\mu_1 + \mu_3 + \beta_3 + \beta_2) \mathbb{E}(t), \\ f_3(t, \mathbb{S}(t), \mathbb{E}(t), \mathbb{I}(t), \mathbb{R}(t)) &= \beta_2 \mathbb{E}(t) - (\mu_1 + \mu_4 + \beta_4) \mathbb{I}(t), \\ f_4(t, \mathbb{S}(t), \mathbb{E}(t), \mathbb{I}(t), \mathbb{R}(t)) &= \beta_3 \mathbb{E}(t) + \beta_4 \mathbb{I}(t) - \mu_1 \mathbb{R}(t). \end{aligned} \tag{6.6}$$

Also generalizing FDE (6.6) in ABC sense as

$$\begin{aligned} {}_0^{ABC} D_t^\alpha \psi(t) &= \omega(t, \psi(t)), \quad 0 < \alpha \leq 1, \quad t \in [0, T], \\ \psi(0) &= \psi_0. \end{aligned} \tag{6.7}$$

As mentioned in Lemma - 6.1, system (6.1) i.e. (6.7) yields

$$\psi(t) = \psi_0 + \frac{(1 - \alpha)}{\Lambda(\alpha)} \omega(t, \psi(t)) + \frac{\alpha}{\Gamma(\alpha) \Lambda(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau, \tag{6.8}$$

$$\text{where } \psi(t) = \begin{bmatrix} \mathbb{S}(t) \\ \mathbb{E}(t) \\ \mathbb{I}(t) \\ \mathbb{R}(t) \end{bmatrix}, \psi_0 = \begin{bmatrix} \mathbb{S}_0 \\ \mathbb{E}_0 \\ \mathbb{I}_0 \\ \mathbb{R}_0 \end{bmatrix}, \omega(t, \psi(t)) = \begin{bmatrix} f_1(t, \mathbb{S}(t), \mathbb{E}(t), \mathbb{I}(t), \mathbb{R}(t)) \\ f_2(t, \mathbb{S}(t), \mathbb{E}(t), \mathbb{I}(t), \mathbb{R}(t)) \\ f_3(t, \mathbb{S}(t), \mathbb{E}(t), \mathbb{I}(t), \mathbb{R}(t)) \\ f_4(t, \mathbb{S}(t), \mathbb{E}(t), \mathbb{I}(t), \mathbb{R}(t)) \end{bmatrix}. \quad (6.9)$$

Using equations (6.8) and (6.9), the operators X and Y defined as:

$$\begin{aligned} X(\psi) &= \psi_0 + \frac{(1-\alpha)}{\Lambda(\alpha)} \omega(t, \psi(t)), \\ Y(\psi) &= \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau. \end{aligned} \quad (6.10)$$

Let us consider the following Axioms to prove Existence and Uniqueness of system (6.2):

Hypotheses 1. There exist any C_ω and D_ω such that

$$|\omega(t, \psi(t))| \leq C_\omega \|\psi\| + D_\omega; \quad (6.11)$$

Hypotheses 2. There exist any $L_\omega > 0$ such that

$$\forall \psi, \bar{\psi} \in B, |\omega(t, \psi) - \omega(t, \bar{\psi})| \leq L_\omega \|\psi - \bar{\psi}\|. \quad (6.12)$$

Theorem 6.1. Considering the Hypotheses - 1 and 2, System (6.7) has at least one solution as, (6.8) if $\frac{(1-\alpha)}{\Lambda(\alpha)} L_\omega < 1$.

Proof. The entire theorem will be proven in two sections.

(i) Let consider $\bar{\psi} \in A$, where $A = \{\psi \in B, \|\psi\| \leq \rho, \rho > 0\}$ is a closed and convex set.

For $X(\psi)$, from equation (6.10),

$$\begin{aligned} \|X(\psi) - X(\bar{\psi})\| &= \frac{(1-\alpha)}{\Lambda(\alpha)} \sup_{t \in [0, T]} |\omega(t, \psi(t)) - \omega(t, \bar{\psi}(t))|, \\ &\leq \frac{(1-\alpha)}{\Lambda(\alpha)} L_\omega \|\psi - \bar{\psi}\|. \end{aligned} \quad (6.13)$$

Hence, X is a contraction.

(ii) Next, we will illustrate that the operator Y be relatively compact. It means Y should

be equicontinuous and bounded.

It is obvious that ω is continuous then Y is also continuous for all $\psi \in A$.

$$\begin{aligned}
\|Y(\psi)\| &= \sup_{t \in [0, T]} \left| \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau \right|, \\
&\leq \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)} \int_0^T (T - \tau)^{\alpha-1} |\omega(\tau, \psi(\tau))| d\tau, \\
&\leq \frac{T^\alpha}{\Gamma(\alpha)\Lambda(\alpha)} [C_\omega \|\psi\| + D_\omega], \\
&\leq \frac{T^\alpha}{\Gamma(\alpha)\Lambda(\alpha)} [C_\omega \rho + D_\omega].
\end{aligned} \tag{6.14}$$

Thus, Y is bounded.

For equicontinuity, we take $t_1 > t_2$ for any $t_1, t_2 \in [0, T]$, such that

$$\begin{aligned}
|Y(\psi(t_1)) - Y(\psi(t_2))| &= \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)} \left| \int_0^{t_1} (t_1 - \tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau \right. \\
&\quad \left. - \int_0^{t_2} (t_2 - \tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau \right|, \\
&\leq \frac{|t_1^\alpha - t_2^\alpha|}{\Gamma(\alpha)\Lambda(\alpha)} |C_\omega \rho + D_\omega|.
\end{aligned} \tag{6.15}$$

As $t_1 \rightarrow t_2 \Rightarrow |t_1^\alpha - t_2^\alpha| \rightarrow 0$, from (6.15),

$$|Y(\psi(t_1)) - Y(\psi(t_2))| \rightarrow 0.$$

It proves that Y is continuous. Uniform continuity of Y is established in view of continuity and boundedness of Y .

From Lemma - 6.3 (Arzela-Ascoli theorem), we can say that Y is relatively compact and entirely continuous.

Thus, Theorem - 6.1 leads to the conclusion that the system (6.7) has at least one solution.

Theorem 6.2. Under the axioms (6.11) and (6.12) the system (6.7) has a unique solution given by (6.8) if $\Delta = \frac{(1-\alpha)L_\omega}{\Lambda(\alpha)} + \frac{T^\alpha L_\omega}{\Gamma(\alpha)\Lambda(\alpha)} < 1$.

Proof. For Banach space B , the operator $F : B \rightarrow B$ defined as

$$F[\psi(t)] = \psi_0 + \frac{(1-\alpha)}{\Lambda(\alpha)}\omega(t, \psi(t)) + \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau, \quad (6.16)$$

For $\psi, \bar{\psi} \in B$, we get

$$\begin{aligned} \|F(\psi) - F(\bar{\psi})\| &\leq \frac{(1-\alpha)}{\Lambda(\alpha)} \max_{t \in [0, T]} |\omega(t, \psi(t)) - \omega(t, \bar{\psi}(t))| \\ &\quad + \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)} \max_{t \in [0, T]} \left| \int_0^t (t-\tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau \right. \\ &\quad \left. - \int_0^t (t-\tau)^{\alpha-1} \omega(\tau, \bar{\psi}(\tau)) d\tau \right|, \\ &\leq \left(\frac{(1-\alpha)L_\omega}{\Lambda(\alpha)} + \frac{T^\alpha L_\omega}{\Gamma(\alpha)\Lambda(\alpha)} \right) \|\psi - \bar{\psi}\|, \\ &\leq \Delta \|\psi - \bar{\psi}\|. \end{aligned} \quad (6.17)$$

From (6.17), we can say F is contraction.

Using Lemma - 6.2, the operator F has an unique fixed point, that is $F(\psi(t)) = \psi(t)$. i.e.

$$\psi(t) = \psi_0 + \frac{(1-\alpha)}{\Lambda(\alpha)}\omega(t, \psi(t)) + \frac{\alpha}{\Gamma(\alpha)\Lambda(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau, \quad (6.18)$$

which is equation (6.8), as a unique solution of generalized FDE as equation (6.7). Thus, existence and uniqueness has been proved for our system (6.1).

6.4 Stability Analysis

The stability analysis of FDEs is one of the significant factor. There are various forms and types of stability, where Ulam-Hyers (UH) stability represents one of the significant types. This stability was proposed by Ulam in 1940 and further investigated by Hyers. Rassias extended this stability into a more general form known as Ulam-Hyers-Rassias (UHR) stability. The stability described above used in this article.

For $\phi \in B$, $\xi > 0$, and $\Omega = [0, T]$, \exists a positive real number such that a continuous

mapping $F_\omega : \Omega \rightarrow R^+$. Suppose that

$$|{}_0^{ABC}D_t^\alpha \phi(t) - \omega(t, \phi(t))| \leq \xi, \quad (6.19)$$

$$|{}_0^{ABC}D_t^\alpha \phi(t) - \omega(t, \phi(t))| \leq \xi F_\omega, \quad (6.20)$$

$$|{}_0^{ABC}D_t^\alpha \phi(t) - \omega(t, \phi(t))| \leq F_\omega, \quad (6.21)$$

where $\forall t \in \Omega$ and $\xi = \max(\xi_i)$, for $i = 1, 2, 3$.

Definition 6.1. The fractional COVID-19 model (6.1) is Ulam-Hyers (UH) Stable if \exists a real number $C_\omega > 0$ such that for each $\xi > 0$ and for every ϕ satisfying (6.19), \exists a solution $\psi \in B$ of (6.1) such that

$$|\phi(t) - \psi(t)| \leq \xi C_\omega, \quad t \in \Omega, \quad (6.22)$$

where $\xi = \max(\xi_i)$ and $C_\omega = \max(C_{\omega_i})$, for $i = 1, 2, 3$.

Definition 6.2. The fractional COVID-19 model (6.1) is Generalized Ulam-Hyers (GUH) Stable if \exists a mapping $F_\omega \in C(R^+, R^+)$ having $F_\omega(0) = 0$ such that for each $\xi > 0$ and for every result ϕ of (6.20), \exists a result $\psi \in B$ of (6.1) such that

$$|\phi(t) - \psi(t)| \leq F_\omega(\xi), \quad t \in \Omega, \quad (6.23)$$

where $\xi = \max(\xi_i)$ and $F_\omega = \max(F_{\omega_i})$, for $i = 1, 2, 3$.

Remark 6.1. If given $\phi \in B$ satisfies (6.19), then there exists $\vartheta \in B$, so that

$$(a) |\vartheta(t)| \leq \xi, \quad \vartheta = \max(\vartheta_i), \quad (i = 1, 2, 3)$$

$$(b) {}_0^{ABC}D_t^\alpha \phi(t) = \omega(t, \phi(t)) + \vartheta(t), \quad \forall t \in \Omega.$$

Lemma 6.4. For $0 \leq \alpha \leq 1$, if ϕ is a member of the Banach space B and satisfies (6.19), then ϕ satisfies

$$\begin{aligned} \left| \phi(t) - \phi_0 - \frac{1-\alpha}{\Lambda(\alpha)} \omega(t, \phi(t)) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \phi(\tau)) d\tau \right| \\ \leq \left(\frac{1-\alpha}{\Lambda(\alpha)} - \frac{T^\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \right) \xi. \end{aligned} \quad (6.24)$$

Proof. Let ϕ be a result of (6.19) and therefore from Remark 6.1(b), we get

$$\begin{aligned} {}_0^{ABC}D_t^\alpha \phi(t) &= \omega(t, \phi(t)) + \vartheta(t), \quad t \in \Omega, \\ \phi(0) &= \phi_0 \geq 0. \end{aligned} \quad (6.25)$$

Then the solution of above (6.25) can be estimated using following expression as

$$\begin{aligned} \phi(t) &= \phi_0 + \frac{1-\alpha}{\Lambda(\alpha)} \omega(t, \phi(t)) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \phi(\tau)) d\tau, \\ &\leq \frac{1-\alpha}{\Lambda(\alpha)} \vartheta(t) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \vartheta(\tau) d\tau. \end{aligned} \quad (6.26)$$

Now, using Remark - 6.1, we have

$$\begin{aligned} &\left| \phi(t) - \phi_0 - \frac{1-\alpha}{\Lambda(\alpha)} \omega(t, \phi(t)) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \phi(\tau)) d\tau \right| \\ &\leq \left| \frac{1-\alpha}{\Lambda(\alpha)} \vartheta(t) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \vartheta(\tau) d\tau \right|, \\ &\leq \frac{1-\alpha}{\Lambda(\alpha)} |\vartheta(t)| - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |\vartheta(\tau)| d\tau, \\ &\leq \left(\frac{1-\alpha}{\Lambda(\alpha)} - \frac{T^\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \right) \xi. \end{aligned} \quad (6.27)$$

Hence, the proved of Lemma - 6.4.

Theorem 6.3. Considering a real valued continuous map ω on $[0, T] \times B$ (or $\Omega \times B$), so that for every $\psi(t) \in B$, $\omega \in C(\Omega \times B, R)$, therefore under the assumption of Hypothesis - 2 and conclusion of Theorem - 6.1, fractional COVID-19 system (6.1) is UH stable on Ω .

Proof. Let $\xi > 0$ and ϕ (as member of Banach space B) are any responses of (6.19). Observe that ψ as a member of Banach space B be the unique result of the system (6.7), as

$$\begin{aligned} {}_0^{ABC}D_t^\alpha \psi(t) &= \omega(t, \phi(t)), \quad t \in \Omega, \\ \psi(0) &= \psi_0. \end{aligned} \quad (6.28)$$

where

$$\psi(t) = \psi_0 + \frac{1-\alpha}{\Lambda(\alpha)}\omega(t, \psi(t)) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \psi(\tau)) d\tau. \quad (6.29)$$

In the view of Lemma - 6.4. and Hypothesis - 2, we get

$$\begin{aligned} |\phi(t) - \psi(t)| &\leq \left| \begin{aligned} &\frac{1-\alpha}{\Lambda(\alpha)}\omega(t, \phi(t)) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \phi(\tau)) d\tau \\ &-\frac{1-\alpha}{\Lambda(\alpha)}\omega(t, \psi(t)) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \psi(\tau)) d\tau \end{aligned} \right|, \\ &\leq \left(\frac{1-\alpha}{\Lambda(\alpha)} - \frac{T^\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \right) \xi + \frac{\alpha L_\omega}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |\phi(\tau) - \psi(\tau)| d\tau, \\ &\leq \left(\frac{1-\alpha}{\Lambda(\alpha)} - \frac{T^\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \right) \xi + \frac{T^\alpha L_\omega}{\Lambda(\alpha)\Gamma(\alpha)} |\phi(\tau) - \psi(\tau)|. \end{aligned} \quad (6.30)$$

It follows that

$$|\phi(t) - \psi(t)| \leq \xi C_\omega, \quad (6.31)$$

where $C_\omega = \frac{(1-\alpha)\Gamma(\alpha)+T^\alpha}{\Lambda(\alpha)\Gamma(\alpha)-T^\alpha L_\omega}$.

Hence, Theorem is proved.

Corollary 6.1. In the above Theorem - 6.3, considering $F_\omega(\xi) = \xi C_\omega$, so that $F_\omega(0) = 0$, we have the fractional COVID-19 system (6.1) is GUH stable.

Definition 6.3. The fractional COVID-19 model (6.1) is Ulam-Hyers-Rassias (UHR) Stable if \exists a mapping $F_\omega \in C(\Omega, R^+)$ and a real constant $K_{F_\omega} > 0$ such that for each $\xi > 0$ and for every result ϕ of (6.20), \exists a result $\psi \in B$ of (6.1) such that

$$|\phi(t) - \psi(t)| \leq K_{F_\omega} \xi F_\omega(t), t \in \Omega, \quad (6.32)$$

where $\xi = \max(\xi_i)$, $F_\omega = \max(F_{\omega_i})$, and $K_{F_\omega} = \max(K_{F_{\omega_i}})$, for $i = 1, 2, 3$.

Definition 6.4. The fractional COVID-19 model (6.1) is Generalized Ulam-Hyers-Rassias (GUHR) Stable if \exists a mapping $F_\omega \in C(\Omega, R^+)$ and a real constant $K_{F_\omega} > 0$

such that for every result ϕ of (6.21), \exists a result $\psi \in B$ of (6.1) such that

$$|\phi(t) - \psi(t)| \leq K_{F_\omega} F_\omega(t), t \in \Omega, \quad (6.33)$$

where $F_\omega = \max(F_{\omega_i})$ and $K_{F_\omega} = \max(K_{F_{\omega_i}})$, for $i = 1, 2, 3$.

Remark 6.2. If mapping $\phi \in B$ satisfies (6.20), then there exist a mapping $\theta \in B$ such that

- (a) $|\theta(t)| \leq \xi, \theta = \max(\theta_i), (i = 1, 2, 3)$
- (b) ${}_0^{ABC}D_t^\alpha \phi(t) = \omega(t, \phi(t)) + \theta(t), \forall t \in \Omega.$

Hypothesis 3.

For a given $F_\omega \in B$ there exist $\lambda_{F_\omega} > 0$ such that

$${}_0^{ABC}I_t^\alpha F_\omega \leq \lambda_{F_\omega} F_\omega(t), t \in \Omega. \quad (6.34)$$

Lemma 6.5. For $0 \leq \alpha \leq 1$, if $\phi \in B$ is a solution of (6.20), then ϕ satisfies following inequality:

$$\left| \phi(t) - \phi_0 - \frac{1-\alpha}{\Lambda(\alpha)} \omega(t, \phi(t)) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \phi(\tau)) d\tau \right| \leq \xi \lambda_{F_\omega} F_\omega(t). \quad (6.35)$$

Proof. Using Remark 6.2(b) and assuming that ϕ is a solution of (6.20), then

$$\begin{aligned} {}_0^{ABC}D_t^\alpha \phi(t) &= \omega(t, \phi(t)) + \theta(t), t \in \Omega, \\ \phi(0) &= \phi_0. \end{aligned} \quad (6.36)$$

Following that, the estimated solution of (6.36) can be written as

$$\begin{aligned} \phi(t) &= \phi_0 + \frac{1-\alpha}{\Lambda(\alpha)} \omega(t, \phi(t)) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \phi(\tau)) d\tau, \\ &\leq \frac{1-\alpha}{\Lambda(\alpha)} \theta(t) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \theta(\tau) d\tau. \end{aligned} \quad (6.37)$$

Now, using the facts of Remark - 6.2, we have

$$\begin{aligned}
& \left| \phi(t) - \phi_0 - \frac{1-\alpha}{\Lambda(\alpha)} \omega(t, \phi(t)) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \phi(\tau)) d\tau \right| \\
& \leq \left| \frac{1-\alpha}{\Lambda(\alpha)} \theta(t) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \theta(\tau) d\tau \right|, \\
& \leq \frac{1-\alpha}{\Lambda(\alpha)} |\theta(t)| - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |\theta(\tau)| d\tau, \\
& \leq \xi \lambda_{F_\omega} F_\omega(t). \tag{6.38}
\end{aligned}$$

Hence, Lemma is proved.

Theorem 6.4. Suppose that a mapping $\omega \in C(\Omega \times R, R)$ such that for every $\psi \in B$, under the assumption of Hypotheses - 2 and $\psi(t)$ as in (6.8), the fractional COVID-19 system (6.1) is UHR stable on Ω .

Proof. In the view of Lemma - 6.5, Hypothesis (2) and (3), we get

$$\begin{aligned}
|\phi(t) - \psi(t)| & \leq \left| \begin{aligned} & \frac{1-\alpha}{\Lambda(\alpha)} \omega(t, \phi(t)) + \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \phi(\tau)) d\tau \\ & - \frac{1-\alpha}{\Lambda(\alpha)} \omega(t, \psi(t)) - \frac{\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \psi(\tau)) d\tau \end{aligned} \right|, \\
& \leq \left(\frac{1-\alpha}{\Lambda(\alpha)} - \frac{T^\alpha}{\Lambda(\alpha)\Gamma(\alpha)} \right) \xi \\
& \quad + \frac{\alpha L_\omega}{\Lambda(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |\phi(\tau) - \psi(\tau)| d\tau, \\
& \leq \lambda_{F_\omega} F_\omega(t) \xi + \frac{T^\alpha L_\omega}{\Lambda(\alpha)\Gamma(\alpha)} |\phi(\tau) - \psi(\tau)|. \tag{6.39}
\end{aligned}$$

It follows that

$$|\phi(t) - \psi(t)| \leq K_{F_\omega} \xi F_\omega(t), \tag{6.40}$$

where $K_{F_\omega} = \frac{\lambda_{F_\omega}}{1 - \frac{T^\alpha L_\omega}{\Lambda(\alpha)\Gamma(\alpha)}}$.

Hence, Theorem is proved.

Corollary 6.2. Setting $\xi = 1$ into Theorem - 6.4, then fractional COVID-19 System (6.1) is GUHR stable.

Equilibrium Points

Let us consider the Exposed and the Infected population of our system (6.1), as

$$\begin{aligned} {}_0^{ABC}D_t^\alpha \mathbb{E}(t) &= \beta_1 \mathbb{S}(t) \mathbb{I}(t) - (\mu_1 + \mu_3 + \beta_3 + \beta_2) \mathbb{E}(t), \\ {}_0^{ABC}D_t^\alpha \mathbb{I}(t) &= \beta_2 \mathbb{E}(t) - (\mu_1 + \beta_4 + \mu_4) \mathbb{I}(t). \end{aligned} \quad (6.41)$$

To find the Reproductive number R_0 , we will have to find the largest eigenvalue of matrix $[AB^{-1}]$, and it is

$$R_0 = \frac{\beta_1 \mathbb{S}^0(t) \beta_2}{(\mu_4 + \beta_4 + \mu_1) (\mu_3 + \beta_3 + \beta_2 + \mu_1)}, \quad (6.42)$$

where $A = \begin{bmatrix} 0 & \beta_1 \mathbb{S}(t) \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \mu_3 + \beta_3 + \beta_2 + \mu_1 & 0 \\ -\beta_2 & \mu_4 + \beta_4 + \mu_1 \end{bmatrix}$.

Here, we are dividing whole population in two kind of equilibrium system,

(i) Disease-Free equilibrium model, at initial time

$$\mathbb{E}_0 (\mathbb{S}_0, \mathbb{E}_0, \mathbb{I}_0, \mathbb{R}_0) = \mathbb{E}_0 \left(\frac{a}{\mu_1 + \mu_2}, 0, 0, 0 \right), \quad (6.43)$$

(ii) Endemic equilibrium model, considered as $\mathbb{E}^*(\mathbb{S}^*, \mathbb{E}^*, \mathbb{I}^*, \mathbb{R}^*)$,

$$\text{where } \mathbb{S}^* = \frac{a}{\beta_1 \mathbb{I}^* + \mu_1 + \mu_2}, \mathbb{E}^* = \mathbb{I}^* \frac{(\mu_1 + \beta_4 + \mu_4)}{\beta_2}, \text{ and } \mathbb{R}^* = \mathbb{I}^* \left(\frac{\beta_3(\mu_1 + \beta_4 + \mu_4) + \beta_4 \beta_2}{\beta_2 \mu_1} \right).$$

Theorem 6.6. If $R_0 < 1$, Disease-Free equilibrium point $\mathbb{E}_0 = (\mathbb{S}_0, 0, 0, 0)$ of system (6.1) is locally asymptotically stable .

Proof. The Jacobian matrix for equilibrium point \mathbb{E}^0 of our model (6.1) can be defined as

$$J_1^{|\mathbb{E}^0|} = \begin{bmatrix} -(\mu_1 + \mu_2) & 0 & -\beta_1 \mathbb{S}_0 & 0 \\ 0 & -(\mu_1 + \mu_3 + \beta_3 + \beta_2) & \beta_1 \mathbb{S}_0 & 0 \\ 0 & \beta_2 & -(\mu_1 + \beta_4 + \mu_4) & 0 \\ 0 & \beta_3 & \beta_4 & -\mu_1 \end{bmatrix}. \quad (6.44)$$

In matrix (6.44), eigenvalues $\lambda_1 = -(\mu_1 + \mu_2)$ and $\lambda_2 = -\mu_1$ are negative. To find remaining eigenvalues, performing row operations

$$J_2^{|0|} = \begin{bmatrix} -(\mu_1 + \mu_3 + \beta_3 + \beta_2) & \beta_1 \mathbb{S}_0 \\ 0 & -(\mu_1 + \beta_4 + \mu_4)(\mu_1 + \mu_3 + \beta_3 + \beta_2) + \beta_2 \beta_1 \mathbb{S}_0 \end{bmatrix}, \quad (6.45)$$

From matrix (6.45),

$$\lambda_3 = -(\mu_1 + \mu_3 + \beta_3 + \beta_2) \text{ and } \lambda_4 = -(\mu_1 + \beta_4 + \mu_4)(\mu_1 + \mu_3 + \beta_3 + \beta_2) + \beta_2 \beta_1 \mathbb{S}_0. \quad (6.46)$$

Since $\lambda_4 < 0$,

$$\begin{aligned} &\Rightarrow -(\mu_1 + \beta_4 + \mu_4)(\mu_1 + \mu_3 + \beta_3 + \beta_2) + \beta_2 \beta_1 \mathbb{S}_0 < 0, \\ &\Rightarrow \frac{\beta_2 \beta_1 \mathbb{S}_0}{(\mu_1 + \beta_4 + \mu_4)(\mu_1 + \mu_3 + \beta_3 + \beta_2)} < 1, \\ &\Rightarrow R_0 < 1. \end{aligned} \quad (6.47)$$

Thus, The theorem is Proved.

Theorem 6.7. If $\mathbb{R}_0 > 1$, Disease-endemic equilibrium point $\mathbb{E}^*(\mathbb{S}^*, \mathbb{E}^*, \mathbb{I}^*, \mathbb{R}^*)$ of system (6.1) is locally asymptotically stable .

Proof. The Jacobian matrix for equilibrium point \mathbb{E}^* of our model (6.1) can be defined as

$$J_1^{*|} = \begin{bmatrix} -(\beta_1 \mathbb{I}^* + \mu_1 + \mu_2) & 0 & -\beta_1 \mathbb{S}^* & 0 \\ \beta_1 \mathbb{I}^* & -(\mu_1 + \mu_3 + \beta_3 + \beta_2) & \beta_1 \mathbb{S}^* & 0 \\ 0 & \beta_2 & -(\mu_1 + \beta_4 + \mu_4) & 0 \\ 0 & \beta_3 & \beta_4 & -\mu_1 \end{bmatrix}. \quad (6.48)$$

From (6.48), one of the eigenvalue is $\lambda_1 = -\mu_1$. Lets find the remaining eigenvalues by

performing row operations, we get

$$J_2^{*|} = \begin{bmatrix} -(\beta_1\mathbb{I}^* + \mu_1 + \mu_2) & 0 & -\beta_1\mathbb{S}^* \\ \beta_1\mathbb{I}^* & -(\mu_1 + \mu_3 + \beta_3 + \beta_2) & \beta_1\mathbb{S}^* \\ 0 & \beta_2 & -(\mu_1 + \beta_4 + \mu_4) \end{bmatrix}, \quad (6.49)$$

Now by performing a row operation $(\beta_1\mathbb{I}^* + \mu_1 + \mu_2)\mathbb{R}_2 + (\beta_1\mathbb{I}^*)\mathbb{R}_1 \rightarrow \mathbb{R}_2$

$$J_3^{*|} = \begin{bmatrix} -(\beta_1\mathbb{I}^* + \mu_1 + \mu_2) & 0 & -\beta_1\mathbb{S}^* \\ 0 & -(\mu_1 + \mu_3 + \beta_3 + \beta_2)(\beta_1\mathbb{I}^* + \mu_1 + \mu_2) & \beta_1\mathbb{S}^*(\beta_1\mathbb{I}^* + \mu_1 + \mu_2) - \beta_1^2\mathbb{S}^*\mathbb{I}^* \\ 0 & \beta_2 & -(\mu_1 + \beta_4 + \mu_4) \end{bmatrix}, \quad (6.50)$$

Another eigenvalue of matrix in (6.50) is $\lambda_2 = -(\beta_1\mathbb{I}^* + \mu_1 + \mu_2)$. Performing some more row operations

$$J_4^{*|} = \begin{bmatrix} -(\mu_1 + \mu_3 + \beta_3 + \beta_2)(\beta_1\mathbb{I}^* + \mu_1 + \mu_2) & \beta_1\mathbb{S}^*(\beta_1\mathbb{I}^* + \mu_1 + \mu_2) - \beta_1^2\mathbb{S}^*\mathbb{I}^* \\ \beta_2 & -(\mu_1 + \beta_4 + \mu_4) \end{bmatrix}, \quad (6.51)$$

Further, row operation $(\beta_2)\mathbb{R}_1 + (\mu_1 + \mu_3 + \beta_3 + \beta_2)(\beta_1\mathbb{I}^* + \mu_1 + \mu_2)\mathbb{R}_2 \rightarrow \mathbb{R}_2$

$$J_5^{*|} = \begin{bmatrix} -(\mu_1 + \mu_3 + \beta_3 + \beta_2)(\beta_1\mathbb{I}^* + \mu_1 + \mu_2) & \beta_1\mathbb{S}^*(\beta_1\mathbb{I}^* + \mu_1 + \mu_2) - \beta_1^2\mathbb{S}^*\mathbb{I}^* \\ 0 & \beta_2\beta_1\mathbb{S}^*(\beta_1\mathbb{I}^* + \mu_1 + \mu_2) - \beta_2\beta_1^2\mathbb{S}^*\mathbb{I}^* \\ & -(\mu_1 + \beta_4 + \mu_4)(\mu_1 + \mu_3 + \beta_3 + \beta_2)(\beta_1\mathbb{I}^* + \mu_1 + \mu_2) \end{bmatrix}, \quad (6.52)$$

Eventually, the last two eigenvalues are $\lambda_3 = -(\mu_1 + \mu_3 + \beta_3 + \beta_2)(\beta_1\mathbb{I}^* + \mu_1 + \mu_2)$ and $\lambda_4 = \beta_2\beta_1\mathbb{S}^*(\beta_1\mathbb{I}^* + \mu_1 + \mu_2) - \beta_2\beta_1^2\mathbb{S}^*\mathbb{I}^* - (\mu_1 + \beta_4 + \mu_4)(\mu_1 + \mu_3 + \beta_3 + \beta_2)(\beta_1\mathbb{I}^* + \mu_1 + \mu_2)$.

For $\lambda_4 < 0$,

$$\begin{aligned}
&\Rightarrow \beta_2 \beta_1 \mathbb{S}^* (\beta_1 \mathbb{I}^* + \mu_1 + \mu_2) - \beta_2 \beta_1^2 \mathbb{S}^* \mathbb{I}^* \\
&\quad - (\mu_1 + \beta_4 + \mu_4) (\mu_1 + \mu_3 + \beta_3 + \beta_2) (\beta_1 \mathbb{I}^* + \mu_1 + \mu_2) < 0, \\
&\Rightarrow \frac{\beta_2 \beta_1}{(\mu_1 + \beta_4 + \mu_4) (\mu_1 + \mu_3 + \beta_3 + \beta_2)} \mathbb{S}^* (\beta_1 \mathbb{I}^* + \mu_1 + \mu_2) < \beta_2 \beta_1^2 \mathbb{S}^* \mathbb{I}^* + (\beta_1 \mathbb{I}^* + \mu_1 + \mu_2), \\
&\Rightarrow \frac{\beta_2 \beta_1}{(\mu_1 + \beta_4 + \mu_4) (\mu_1 + \mu_3 + \beta_3 + \beta_2)} < \frac{\beta_2 \beta_1^2 \mathbb{I}^*}{(\beta_1 \mathbb{I}^* + \mu_1 + \mu_2)} + \frac{1}{\mathbb{S}^*}, \\
&\Rightarrow 1 < \mathbb{R}_0 < \frac{\beta_2 \beta_1 (\mathbb{I}^* + \mu_1 + \mu_2)}{\mathbb{S}^* (\beta_1 \mathbb{I}^* + \mu_1 + \mu_2)}. \tag{6.53}
\end{aligned}$$

Hence, theorem is proved.

6.5 Working of Homotopy Perturbation Laplace Transform Method

Consider a generalized FDE with ABC operator as:

$${}_0^{ABC} D_t^\alpha \psi(t) + R\psi(t) + N\psi(t) = f(t), \quad t > 0, \quad n - 1 < \alpha \leq n, \quad n \in \mathbb{N}, \tag{6.54}$$

such that

$$\psi(0) = C_0, \quad \frac{d\psi(0)}{dt} = C_1, \quad \frac{d^2\psi(0)}{dt^2} = C_2, \dots, \quad \frac{d^{n-1}\psi(0)}{dt^{n-1}} = C_{n-1}, \tag{6.55}$$

where $(D_t \equiv \frac{d}{dt})$ is fractional differential operator, R is linear terms, N is non-linear terms of $\psi(t)$, and $f(t)$ is a continuous function.

Operating Laplace transform to Eq. (6.54),

$$\mathcal{L} \{ {}_0^{ABC} D_t^\alpha \psi(t) \} = -\mathcal{L} \{ R\psi(t) \} - \mathcal{L} \{ N\psi(t) \} + \mathcal{L} \{ f(t) \}, \tag{6.56}$$

using differentiation properties (1.6),

$$\frac{\Lambda(\alpha)}{(n-\alpha)s^\alpha + \alpha} \left[s^\alpha \mathcal{L}\{\psi(t)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} D_t^k \psi(0) \right] = -\mathcal{L}\{R\psi(t)\} - \mathcal{L}\{N\psi(t)\} + \mathcal{L}\{f(t)\}, \quad (6.57)$$

with normalized function considering $\Lambda(\alpha) = 1$, we get

$$\begin{aligned} \mathcal{L}\{\psi(t)\} &= \left[\frac{1}{s} \psi(0) + \frac{1}{s^2} \frac{d\psi(0)}{dt} + \dots + \frac{1}{s^n} \frac{d^{n-1}\psi(0)}{dt^{n-1}} \right] \\ &+ \left(\frac{(n-\alpha)s^\alpha + \alpha}{s^\alpha} \right) [-\mathcal{L}\{R\psi(t)\} - \mathcal{L}\{N\psi(t)\} + \mathcal{L}\{f(t)\}], \end{aligned} \quad (6.58)$$

taking Inverse Laplace (ILT) to Eq. (6.58),

$$\psi(t) = \omega(t) - \mathcal{L}^{-1} \left[\left(\frac{(n-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L}\{R\psi(t)\} + \left(\frac{(n-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L}\{N\psi(t)\} \right], \quad (6.59)$$

where $\omega(t)$ is ILT of first and last terms of Eq. (6.58).

Applying HPM [64] to Eq. (6.59) gives

$$\begin{aligned} \sum_{i=0}^{\infty} p^i \psi_i(t) &= \omega(t) - p \mathcal{L}^{-1} \left[\left(\frac{(n-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \left\{ R \left(\sum_{i=0}^{\infty} p^i \psi_i(t) \right) \right\} \right. \\ &\left. + \left(\frac{(n-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \left\{ N \left(\sum_{i=0}^{\infty} p^i \psi_i(t) \right) \right\} \right], \end{aligned} \quad (6.60)$$

to determine non-linear terms of above Eq. (6.60), we use He's Polynomial [65],

$$N\psi(t) = \sum_{m=0}^{\infty} p^m H_m(\psi), \quad (6.61)$$

where $H_m(\psi_0, \psi_1, \psi_2, \dots, \psi_m) = \frac{1}{m!} \left[\frac{d^m}{dp^m} N \left(\sum_{i=0}^{\infty} p^i \psi_i(t) \right) \right]_{p=0}$, $m = 0, 1, 2, \dots$

Substituting Eq. (6.61) into (6.60), and comparing the coefficients of p^0, p^1, p^2, \dots , we

have

$$\begin{aligned}
p^0 : \psi_0(t) &= \omega(t), \\
p^1 : \psi_1(t) &= -\mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \{ R\psi_0(t) \} + \left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \{ H_0 \} \right], \\
p^2 : \psi_2(t) &= -\mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \{ R\psi_1(t) \} + \left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \{ H_1 \} \right], \\
&\vdots
\end{aligned} \tag{6.62}$$

The solution of Eq. (6.54) can be obtained as

$$\psi(t) = \sum_{i=0}^{\infty} p^i \psi_i(t) = \psi_0(t) + p^1 \psi_1(t) + p^2 \psi_2(t) + \dots \tag{6.63}$$

as $p \rightarrow 1$, gives

$$\psi(t) = \psi_0(t) + \psi_1(t) + \psi_2(t) + \dots \tag{6.64}$$

6.6 Approximate Solution of SEIR Model

In this section, we presenting the analytical approach to system (6.1) given by

$$\begin{aligned}
{}_0^{ABC} D_t^\alpha \mathbb{S}(t) &= a - \beta_1 \mathbb{S}(t) \mathbb{I}(t) - (\mu_1 + \mu_2) \mathbb{S}(t), \\
{}_0^{ABC} D_t^\alpha \mathbb{E}(t) &= \beta_1 \mathbb{S}(t) \mathbb{I}(t) - (\mu_1 + \mu_3 + \beta_3 + \beta_2) \mathbb{E}(t), \\
{}_0^{ABC} D_t^\alpha \mathbb{I}(t) &= \beta_2 \mathbb{E}(t) - (\mu_1 + \beta_4 + \mu_4) \mathbb{I}(t), \\
{}_0^{ABC} D_t^\alpha \mathbb{R}(t) &= \beta_3 \mathbb{E}(t) + \beta_4 \mathbb{I}(t) - \mu_1 \mathbb{R}(t),
\end{aligned} \tag{6.65}$$

Applying Laplace transform to the system of equation (6.65), we achieve

$$\begin{aligned}
\mathcal{L} \{ {}_0^{ABC} D_t^\alpha \mathbb{S}(t) \} &= \mathcal{L} \{ a - \beta_1 \mathbb{S}(t) \mathbb{I}(t) - (\mu_1 + \mu_2) \mathbb{S}(t) \}, \\
\mathcal{L} \{ {}_0^{ABC} D_t^\alpha \mathbb{E}(t) \} &= \mathcal{L} \{ \beta_1 \mathbb{S}(t) \mathbb{I}(t) - (\mu_1 + \mu_3 + \beta_3 + \beta_2) \mathbb{E}(t) \}, \\
\mathcal{L} \{ {}_0^{ABC} D_t^\alpha \mathbb{I}(t) \} &= \mathcal{L} \{ \beta_2 \mathbb{E}(t) - (\mu_1 + \beta_4 + \mu_4) \mathbb{I}(t) \}, \\
\mathcal{L} \{ {}_0^{ABC} D_t^\alpha \mathbb{R}(t) \} &= \mathcal{L} \{ \beta_3 \mathbb{E}(t) + \beta_4 \mathbb{I}(t) - \mu_1 \mathbb{R}(t) \},
\end{aligned} \tag{6.66}$$

using differentiation property (1.6) with $\Lambda(\alpha) = 1$ and $n = 1$, we get

$$\begin{aligned}
\frac{1}{(1-\alpha)(s^\alpha + \frac{\alpha}{1-\alpha})} [s^\alpha \mathcal{L}\{\mathbb{S}(t)\} - s^{\alpha-1}\mathbb{S}(0)] &= \mathcal{L}\{a - \beta_1\mathbb{S}(t)\mathbb{I}(t) - (\mu_1 + \mu_2)\mathbb{S}(t)\}, \\
\frac{1}{(1-\alpha)(s^\alpha + \frac{\alpha}{1-\alpha})} [s^\alpha \mathcal{L}\{\mathbb{E}(t)\} - s^{\alpha-1}\mathbb{E}(0)] &= \mathcal{L}\{\beta_1\mathbb{S}(t)\mathbb{I}(t) - (\mu_1 + \mu_3 + \beta_3 + \beta_2)\mathbb{E}(t)\}, \\
\frac{1}{(1-\alpha)(s^\alpha + \frac{\alpha}{1-\alpha})} [s^\alpha \mathcal{L}\{\mathbb{I}(t)\} - s^{\alpha-1}\mathbb{I}(0)] &= \mathcal{L}\{\beta_2\mathbb{E}(t) - (\mu_1 + \beta_4 + \mu_4)\mathbb{I}(t)\}, \\
\frac{1}{(1-\alpha)(s^\alpha + \frac{\alpha}{1-\alpha})} [s^\alpha \mathcal{L}\{\mathbb{R}(t)\} - s^{\alpha-1}\mathbb{R}(0)] &= \mathcal{L}\{\beta_3\mathbb{E}(t) + \beta_4\mathbb{I}(t) - \mu_1\mathbb{R}(t)\},
\end{aligned} \tag{6.67}$$

applying initial conditions (6.2) and taking inverse Laplace, we have

$$\begin{aligned}
\mathbb{S}(t) &= \mathbb{S}_0 + \mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L}\{a - \beta_1\mathbb{S}(t)\mathbb{I}(t) - (\mu_1 + \mu_2)\mathbb{S}(t)\} \right], \\
\mathbb{E}(t) &= \mathbb{E}_0 + \mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L}\{\beta_1\mathbb{S}(t)\mathbb{I}(t) - (\mu_1 + \mu_3 + \beta_3 + \beta_2)\mathbb{E}(t)\} \right], \\
\mathbb{I}(t) &= \mathbb{I}_0 + \mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L}\{\beta_2\mathbb{E}(t) - (\mu_1 + \beta_4 + \mu_4)\mathbb{I}(t)\} \right], \\
\mathbb{R}(t) &= \mathbb{R}_0 + \mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L}\{\beta_3\mathbb{E}(t) + \beta_4\mathbb{I}(t) - \mu_1\mathbb{R}(t)\} \right].
\end{aligned} \tag{6.68}$$

Now, we applying Homotopy Perturbation Method [62, 64] to (6.68), we get

$$\begin{aligned}
\sum_{i=0}^{\infty} p^i S_i(t) &= \mathbb{S}_0 + p\mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L}\left\{ a - \beta_1 \sum_{i=0}^{\infty} S_i(t)I_i(t) - (\mu_1 + \mu_2) \sum_{i=0}^{\infty} S_i(t) \right\} \right], \\
\sum_{i=0}^{\infty} p^i E_i(t) &= \mathbb{E}_0 + p\mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L}\left\{ \beta_1 \sum_{i=0}^{\infty} S_i(t)I_i(t) - (\mu_1 + \mu_3 + \beta_3 + \beta_2) \sum_{i=0}^{\infty} E_i(t) \right\} \right], \\
\sum_{i=0}^{\infty} p^i I_i(t) &= \mathbb{I}_0 + p\mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L}\left\{ \beta_2 \sum_{i=0}^{\infty} E_i(t) - (\mu_1 + \beta_4 + \mu_4) \sum_{i=0}^{\infty} I_i(t) \right\} \right], \\
\sum_{i=0}^{\infty} p^i R_i(t) &= \mathbb{R}_0 + p\mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L}\left\{ \beta_3 \sum_{i=0}^{\infty} E_i(t) + \beta_4 \sum_{i=0}^{\infty} I_i(t) - \mu_1 \sum_{i=0}^{\infty} R_i(t) \right\} \right].
\end{aligned} \tag{6.69}$$

In above Eq. (6.69), non-linear terms are decomposed using He's Polynomial H_i ,

$$\begin{aligned}
H_0 &= \mathbb{S}_0(t)\mathbb{I}_0(t), \\
H_1 &= \mathbb{S}_0(t)\mathbb{I}_1(t) + \mathbb{S}_1(t)\mathbb{I}_0(t), \\
H_2 &= \mathbb{S}_0(t)\mathbb{I}_2(t) + \mathbb{S}_1(t)\mathbb{I}_1(t) + \mathbb{S}_2(t)\mathbb{I}_0(t), \\
H_3 &= \mathbb{S}_0(t)\mathbb{I}_3(t) + \mathbb{S}_1(t)\mathbb{I}_2(t) + \mathbb{S}_2(t)\mathbb{I}_1(t) + \mathbb{S}_3(t)\mathbb{I}_0(t), \\
&\vdots
\end{aligned} \tag{6.70}$$

Comparing “p” terms of Eq. (6.69), we get

$$\begin{aligned}
p^0 : \mathbb{S}_0(t) &= \mathbb{S}_0, \\
\mathbb{E}_0(t) &= \mathbb{E}_0, \\
\mathbb{I}_0(t) &= \mathbb{I}_0, \\
\mathbb{R}_0(t) &= \mathbb{R}_0, \\
p^1 : \mathbb{S}_1(t) &= \mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \{ a - (\mu_1 + \mu_2)\mathbb{S}_0(t) - \beta_1 H_0 \} \right], \\
&= (a - (\mu_1 + \mu_2)\mathbb{S}_0 - \beta_1 \mathbb{S}_0 \mathbb{I}_0) \left(1 - \alpha + \alpha \frac{t^\alpha}{\Gamma(1+\alpha)} \right), \\
\mathbb{E}_1(t) &= \mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \{ -(\mu_1 + \mu_3 + \beta_3 + \beta_2)\mathbb{E}_0(t) + \beta_1 H_0 \} \right], \\
&= (-(\mu_1 + \mu_3 + \beta_3 + \beta_2)\mathbb{E}_0 + \beta_1 \mathbb{S}_0 \mathbb{I}_0) \left(1 - \alpha + \alpha \frac{t^\alpha}{\Gamma(1+\alpha)} \right), \\
\mathbb{I}_1(t) &= \mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \{ \beta_2 \mathbb{E}_0(t) - (\mu_1 + \beta_4 + \mu_4)\mathbb{I}_0(t) \} \right], \\
&= (\beta_2 \mathbb{E}_0(t) - (\mu_1 + \beta_4 + \mu_4)\mathbb{I}_0) \left(1 - \alpha + \alpha \frac{t^\alpha}{\Gamma(1+\alpha)} \right), \\
\mathbb{R}_1(t) &= \mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \{ \beta_3 \mathbb{E}_0(t) + \beta_4 \mathbb{I}_0(t) - \mu_1 \mathbb{R}_0(t) \} \right], \\
&= (\beta_3 \mathbb{E}_0 + \beta_4 \mathbb{I}_0 - \mu_1 \mathbb{R}_0) \left(1 - \alpha + \alpha \frac{t^\alpha}{\Gamma(1+\alpha)} \right),
\end{aligned}$$

$$\begin{aligned}
p^2 : \mathbb{S}_2(t) &= \mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \{ a - (\mu_1 + \mu_2)\mathbb{S}_1(t) - \beta_1 H_1 \} \right], \\
&= a \left(1 - \alpha + \alpha \frac{t^\alpha}{\Gamma(1+\alpha)} \right) - (a\beta_1 \mathbb{I}_0 + \beta_1 \beta_2 \mathbb{S}_0 \mathbb{E}_0 - \beta_1^2 \mathbb{S}_0 \mathbb{I}_0^2 + a(\mu_1 + \mu_2) \\
&\quad - (\mu_1 + \mu_2)^2 \mathbb{S}_0 - 2\beta_1(\mu_1 + \mu_2)\mathbb{S}_0 \mathbb{I}_0 - \beta_1(\mu_1 + \beta_4 + \mu_4)\mathbb{I}_0) \left((1-\alpha)^2 \right. \\
&\quad \left. + 2\alpha(1-\alpha) \frac{t^\alpha}{\Gamma(1+\alpha)} + \alpha^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right), \\
\mathbb{E}_2(t) &= \mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \{ -(\mu_1 + \mu_3 + \beta_3 + \beta_2)\mathbb{E}_1(t) + \beta_1 H_1 \} \right], \\
&= (\beta_1 \beta_2 \mathbb{S}_0 \mathbb{E}_0 - \beta_1 \mathbb{S}_0 \mathbb{I}_0 (\mu_1 + \beta_4 + \mu_4) + a\beta_1 \mathbb{I}_0 - \beta_1 \mathbb{S}_0 \mathbb{I}_0^2 - \beta_1 (\mu_1 + \mu_3 + \beta_3 + \beta_2) \mathbb{S}_0 \mathbb{I}_0 \\
&\quad - \beta_1 (\mu_1 + \mu_2) \mathbb{S}_0 \mathbb{I}_0 + (\mu_1 + \mu_3 + \beta_3 + \beta_2)^2 \mathbb{E}_0) \left((1-\alpha)^2 + 2\alpha(1-\alpha) \frac{t^\alpha}{\Gamma(1+\alpha)} \right. \\
&\quad \left. + \alpha^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right), \\
\mathbb{I}_2(t) &= \mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \{ \beta_2 \mathbb{E}_1(t) - (\mu_1 + \beta_4 + \mu_4)\mathbb{I}_1(t) \} \right], \\
&= (\beta_1 \beta_2 \mathbb{S}_0 \mathbb{I}_0 - \beta_2 (\mu_1 + \mu_3 + \beta_3 + \beta_2) \mathbb{E}_0 - \beta_2 (\mu_1 + \beta_4 + \mu_4) \mathbb{E}_0 \\
&\quad + (\mu_1 + \beta_4 + \mu_4)^2 \mathbb{I}_0) \left((1-\alpha)^2 + 2\alpha(1-\alpha) \frac{t^\alpha}{\Gamma(1+\alpha)} + \alpha^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right), \\
\mathbb{R}_2(t) &= \mathcal{L}^{-1} \left[\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L} \{ \beta_3 \mathbb{E}_1(t) + \beta_4 \mathbb{I}_1(t) - \mu_1 \mathbb{R}_1(t) \} \right], \\
&= (\beta_1 \beta_3 \mathbb{S}_0 \mathbb{I}_0 - \beta_3 (\mu_1 + \mu_3 + \beta_3 + \beta_2) \mathbb{E}_0 + \beta_4 \beta_2 \mathbb{E}_0 - \beta_4 (\mu_1 + \beta_4 + \mu_4) \mathbb{I}_0 - \mu_1 \beta_3 \mathbb{E}_0 \\
&\quad - \mu_1 \beta_4 \mathbb{I}_0 + \mu_1^2 \mathbb{R}_0) \left((1-\alpha)^2 + 2\alpha(1-\alpha) \frac{t^\alpha}{\Gamma(1+\alpha)} + \alpha^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right), \\
&\vdots
\end{aligned} \tag{6.71}$$

Thus the solution of Eq. (6.65) can be obtained using (6.64) as

$$\begin{aligned}
\mathbb{S}(t) &= \mathbb{S}_0(t) + \mathbb{S}_1(t) + \mathbb{S}_2(t) + \dots \\
&= \mathbb{S}_0 + (2a - (\mu_1 + \mu_2)\mathbb{S}_0 - \beta_1\mathbb{S}_0\mathbb{I}_0) \left(1 - \alpha + \alpha \frac{t^\alpha}{\Gamma(1 + \alpha)}\right) - (a\beta_1\mathbb{I}_0 + \beta_1\beta_2\mathbb{S}_0\mathbb{E}_0 - \beta_1^2\mathbb{S}_0\mathbb{I}_0^2 \\
&\quad + a(\mu_1 + \mu_2) - (\mu_1 + \mu_2)^2\mathbb{S}_0 - 2\beta_1(\mu_1 + \mu_2)\mathbb{S}_0\mathbb{I}_0 - \beta_1(\mu_1 + \beta_4 + \mu_4)\mathbb{I}_0) \left((1 - \alpha)^2\right. \\
&\quad \left. + 2\alpha(1 - \alpha) \frac{t^\alpha}{\Gamma(1 + \alpha)} + \alpha^2 \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)}\right) + \dots \\
\mathbb{E}(t) &= \mathbb{E}_0(t) + \mathbb{E}_1(t) + \mathbb{E}_2(t) + \dots \\
&= \mathbb{E}_0 + (-(\mu_1 + \mu_3 + \beta_3 + \beta_2)\mathbb{E}_0 + \beta_1\mathbb{S}_0\mathbb{I}_0) \left(1 - \alpha + \alpha \frac{t^\alpha}{\Gamma(1 + \alpha)}\right) + (\beta_1\beta_2\mathbb{S}_0\mathbb{E}_0 \\
&\quad - \beta_1\mathbb{S}_0\mathbb{I}_0(\mu_1 + \beta_4 + \mu_4) + a\beta_1\mathbb{I}_0 - \beta_1\mathbb{S}_0\mathbb{I}_0^2 - \beta_1(\mu_1 + \mu_2)\mathbb{S}_0\mathbb{I}_0 - \beta_1(\mu_1 + \mu_3 + \beta_3 + \beta_2)\mathbb{S}_0\mathbb{I}_0 \\
&\quad + (\mu_1 + \mu_3 + \beta_3 + \beta_2)^2\mathbb{E}_0) \left((1 - \alpha)^2 + 2\alpha(1 - \alpha) \frac{t^\alpha}{\Gamma(1 + \alpha)} + \alpha^2 \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)}\right) + \dots \\
\mathbb{I}(t) &= \mathbb{I}_0(t) + \mathbb{I}_1(t) + \mathbb{I}_2(t) + \dots \\
&= \mathbb{I}_0 + (\beta_2\mathbb{E}_0(t) - (\mu_1 + \beta_4 + \mu_4)\mathbb{I}_0) \left(1 - \alpha + \alpha \frac{t^\alpha}{\Gamma(1 + \alpha)}\right) + (\beta_1\beta_2\mathbb{S}_0\mathbb{I}_0 \\
&\quad - \beta_2(\mu_1 + \mu_3 + \beta_3 + \beta_2)\mathbb{E}_0 - \beta_2(\mu_1 + \beta_4 + \mu_4)\mathbb{E}_0 + (\mu_1 + \beta_4 + \mu_4)^2\mathbb{I}_0) \left((1 - \alpha)^2\right. \\
&\quad \left. + 2\alpha(1 - \alpha) \frac{t^\alpha}{\Gamma(1 + \alpha)} + \alpha^2 \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)}\right) + \dots \\
\mathbb{R}(t) &= \mathbb{R}_0(t) + \mathbb{R}_1(t) + \mathbb{R}_2(t) + \dots \\
&= \mathbb{R}_0 + (\beta_3\mathbb{E}_0 + \beta_4\mathbb{I}_0 - \mu_1\mathbb{R}_0) \left(1 - \alpha + \alpha \frac{t^\alpha}{\Gamma(1 + \alpha)}\right) + (\beta_1\beta_3\mathbb{S}_0\mathbb{I}_0 - \beta_3(\mu_1 + \mu_3 + \beta_3 + \beta_2)\mathbb{E}_0 \\
&\quad + \beta_4\beta_2\mathbb{E}_0 - \beta_4(\mu_1 + \beta_4 + \mu_4)\mathbb{I}_0 - \mu_1\beta_3\mathbb{E}_0 - \mu_1\beta_4\mathbb{I}_0 + \mu_1^2\mathbb{R}_0) \left((1 - \alpha)^2\right. \\
&\quad \left. + 2\alpha(1 - \alpha) \frac{t^\alpha}{\Gamma(1 + \alpha)} + \alpha^2 \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)}\right) + \dots \tag{6.72}
\end{aligned}$$

The approximate solution of fractional COVID-19 model at $\alpha = 1$ is

$$\begin{aligned}
\mathbb{S}(t) &= 8065518 - 6.833832746 \times 10^9 t + 2.890709534 \times 10^{12} t^2 + \dots \\
\mathbb{E}(t) &= 200000 + 6.831505056 \times 10^9 t + 2.892348780 \times 10^{12} t^2 + \dots \\
\mathbb{I}(t) &= 28234 + 36447.68 t + 8.539293845 \times 10^8 t^2 + \dots \\
\mathbb{R}(t) &= 59388.08 t + 9.564125980 \times 10^8 t^2 + \dots
\end{aligned} \tag{6.73}$$

The approximate solution of fractional COVID-19 model at $\alpha = 0.99$ is

$$\begin{aligned}
\mathbb{S}(t) &= 5.178690972 \times 10^8 + 1.081594386 \times 10^{11}t^{0.99} + 2.885729730 \times 10^{12}t^{1.98} + \dots \\
\mathbb{E}(t) &= 6.469848067 \times 10^8 + 1.218101894 \times 10^{11}t^{0.99} + 2.887366153 \times 10^{12}t^{1.98} + \dots \\
\mathbb{I}(t) &= 1.993843537 \times 10^5 + 3.399401093 \times 10^7t^{0.99} + 8.524583268 \times 10^8t^{1.98} + \dots \\
\mathbb{R}(t) &= 1.918764004 \times 10^5 + 3.809221530 \times 10^7t^{0.99} + 9.547649933 \times 10^8t^{1.98} + \dots
\end{aligned}
\tag{6.74}$$

The approximate solution of fractional COVID-19 model at $\alpha = 0.96$ is

$$\begin{aligned}
\mathbb{S}(t) &= 8.984982715 \times 10^9 + 4.446819030 \times 10^{11}t^{0.96} + 2.864528019 \times 10^{12}t^{1.92} + \dots \\
\mathbb{E}(t) &= 9.528976300 \times 10^9 + 4.582733785 \times 10^{11}t^{0.96} + 2.866152420 \times 10^{12}t^{1.92} + \dots \\
\mathbb{I}(t) &= 2.762265937 \times 10^6 + 1.333667477 \times 10^8t^{0.96} + 8.461952403 \times 10^8t^{1.92} + \dots \\
\mathbb{R}(t) &= 3.062895837 \times 10^6 + 1.493907009 \times 10^8t^{0.96} + 9.477502507 \times 10^8t^{1.92} + \dots
\end{aligned}
\tag{6.75}$$

The approximate solution of fractional COVID-19 model at $\alpha = 0.93$ is

$$\begin{aligned}
\mathbb{S}(t) &= 2.785865066 \times 10^{10} + 7.675726699 \times 10^{11}t^{0.93} + 2.833768933 \times 10^{12}t^{1.86} + \dots \\
\mathbb{E}(t) &= 2.882342340 \times 10^{10} + 7.810811701 \times 10^{11}t^{0.93} + 2.835375890 \times 10^{12}t^{1.86} + \dots \\
\mathbb{I}(t) &= 8.399293306 \times 10^6 + 2.287102150 \times 10^8t^{0.93} + 8.371088594 \times 10^8t^{1.86} + \dots \\
\mathbb{R}(t) &= 9.377000626 \times 10^6 + 2.561763277 \times 10^8t^{0.93} + 9.375733797 \times 10^8t^{1.86} + \dots
\end{aligned}
\tag{6.76}$$

The approximate solution of fractional COVID-19 model at $\alpha = 0.90$ is

$$\begin{aligned}
\mathbb{S}(t) &= 5.713887291 \times 10^{10} + 1.075630833 \times 10^{12}t^{0.90} + 2.793304608 \times 10^{12}t^{1.80} + \dots \\
\mathbb{E}(t) &= 5.853032612 \times 10^{10} + 1.089032154 \times 10^{12}t^{0.90} + 2.794888619 \times 10^{12}t^{1.80} + \dots \\
\mathbb{I}(t) &= 1.711046646 \times 10^7 + 3.196696859 \times 10^8t^{0.90} + 8.251555050 \times 10^8t^{1.80} + \dots \\
\mathbb{R}(t) &= 1.913419077 \times 10^7 + 3.580517972 \times 10^8t^{0.90} + 9.241854591 \times 10^8t^{1.80} + \dots
\end{aligned}
\tag{6.77}$$

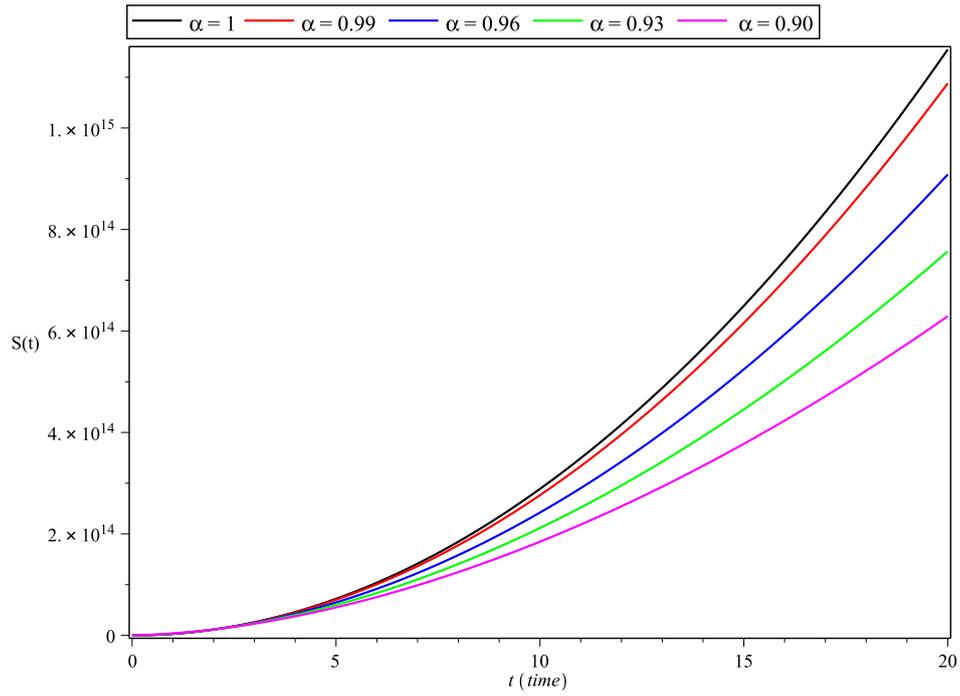


Figure 6.7: The behavior of Susceptible Population $S(t)$ at various order α .

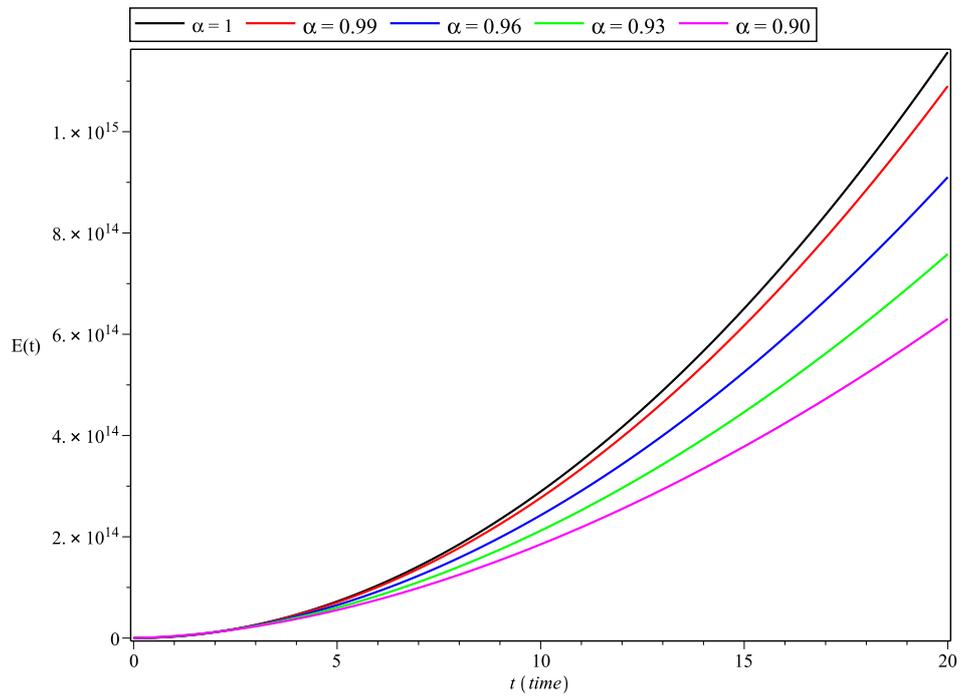


Figure 6.8: The behavior of Exposed Population $E(t)$ at various order α .

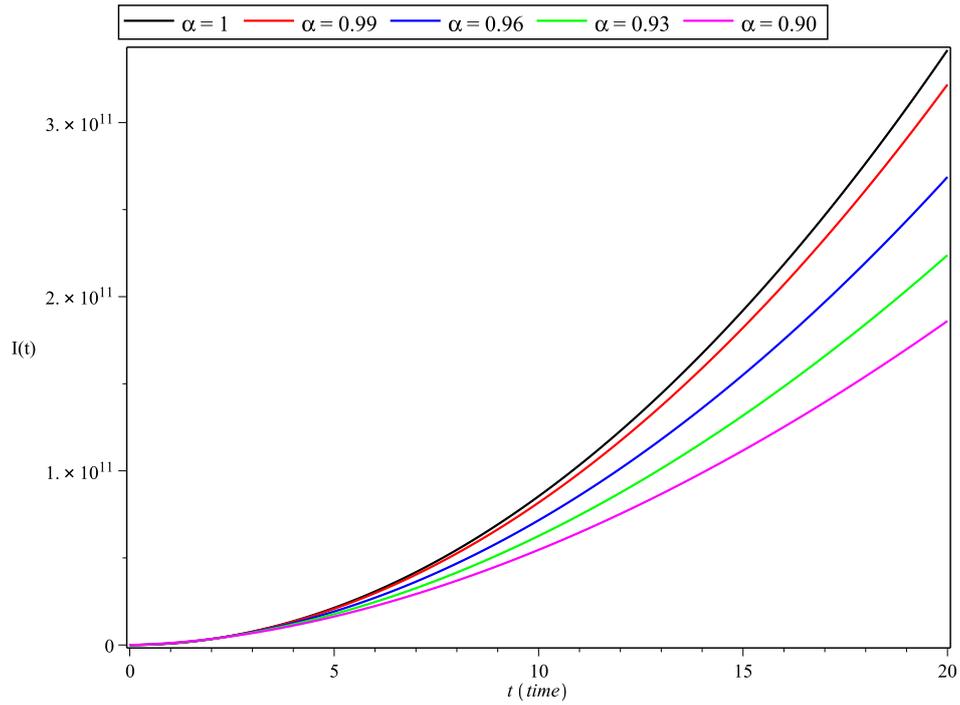


Figure 6.9: The behavior of Infected Population $I(t)$ at various order α .

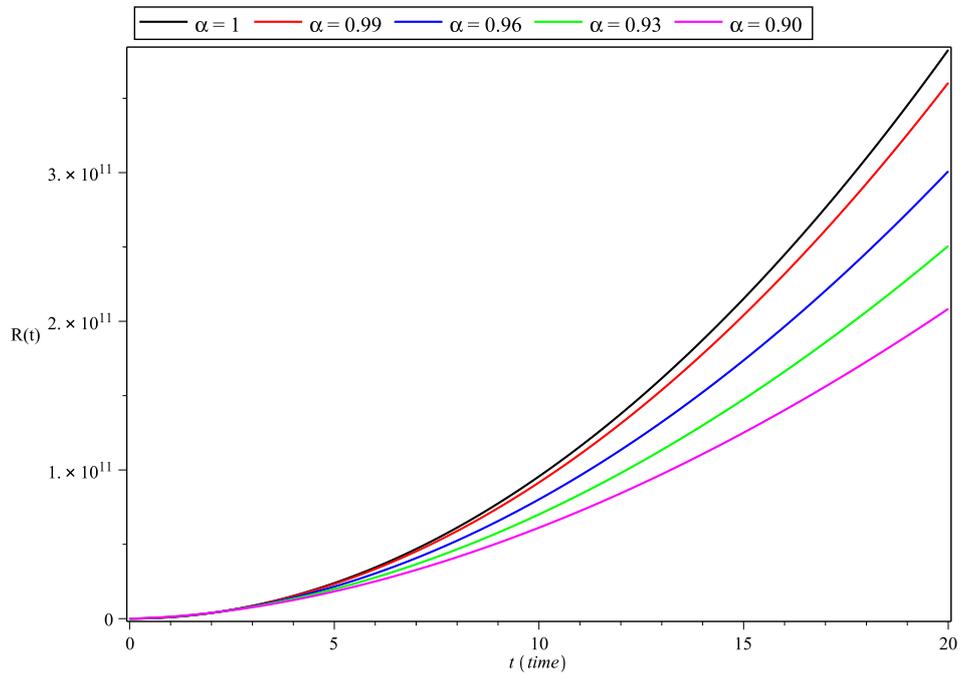


Figure 6.10: The behavior of Recovered Population $R(t)$ at various order α .

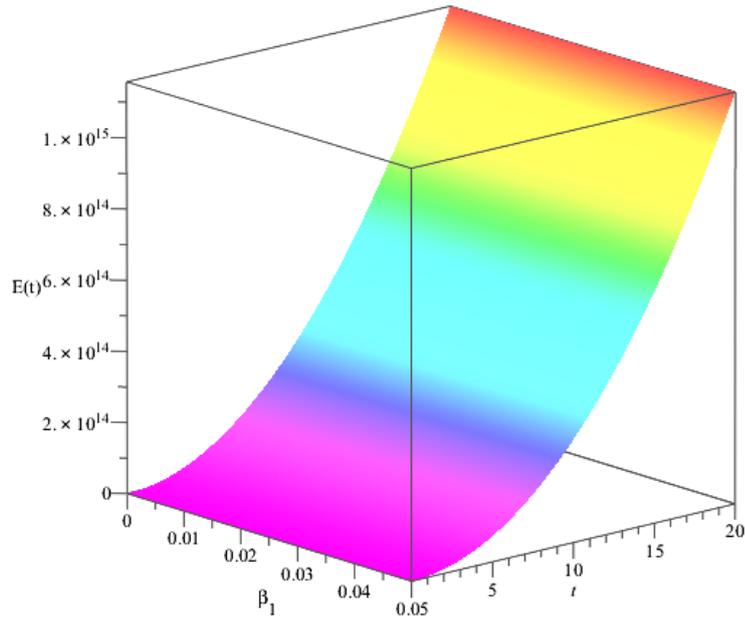


Figure 6.11: Rate of Transmission from $\mathbb{S}(t)$ class to $\mathbb{E}(t)$ class when β_1 is 0 to 0.05.

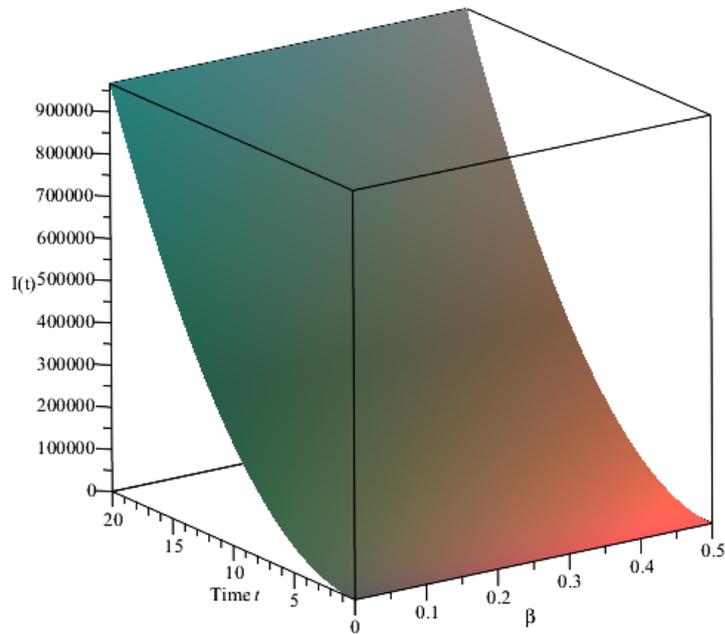


Figure 6.12: Rate of Transmission from $\mathbb{S}(t)$ class to $\mathbb{I}(t)$ class when β_2 is 0 to 0.5.

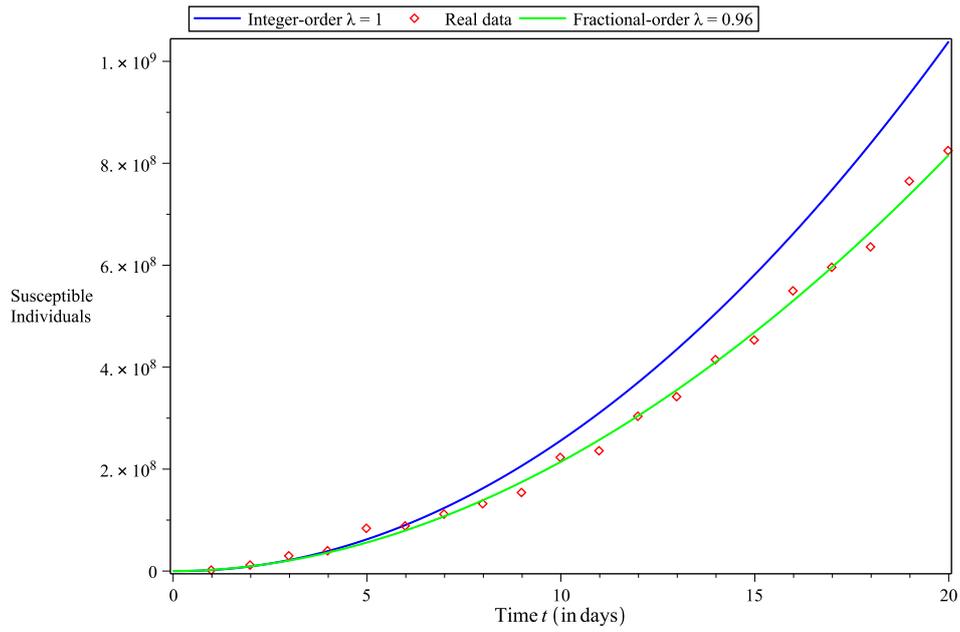


Figure 6.13: Model fitting of $\mathbb{S}(t)$ class Real data with Integer-order and best fitting fractional-order.

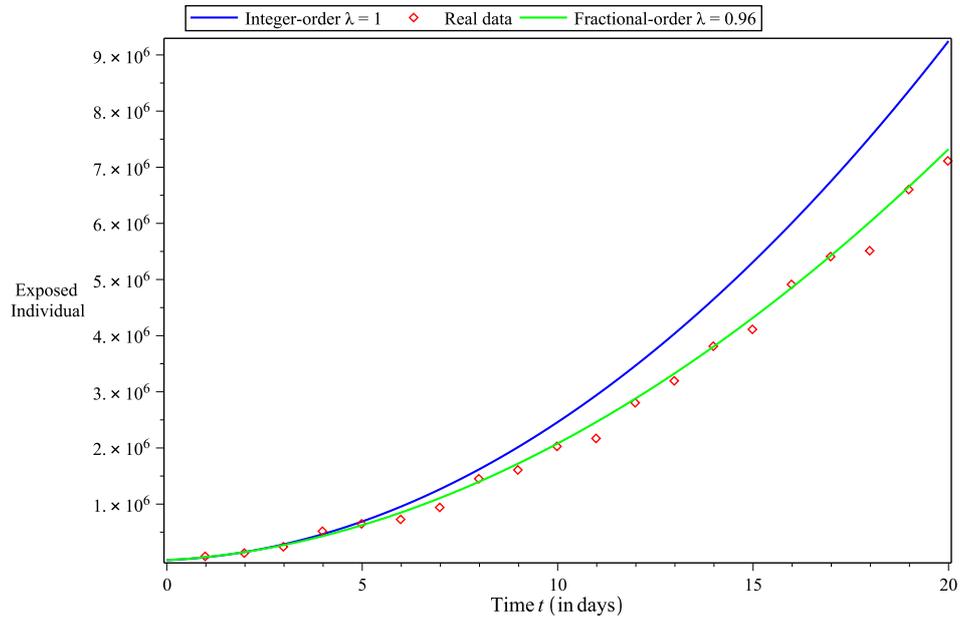


Figure 6.14: Model fitting of $\mathbb{E}(t)$ class Real data with Integer-order and best fitting fractional-order.

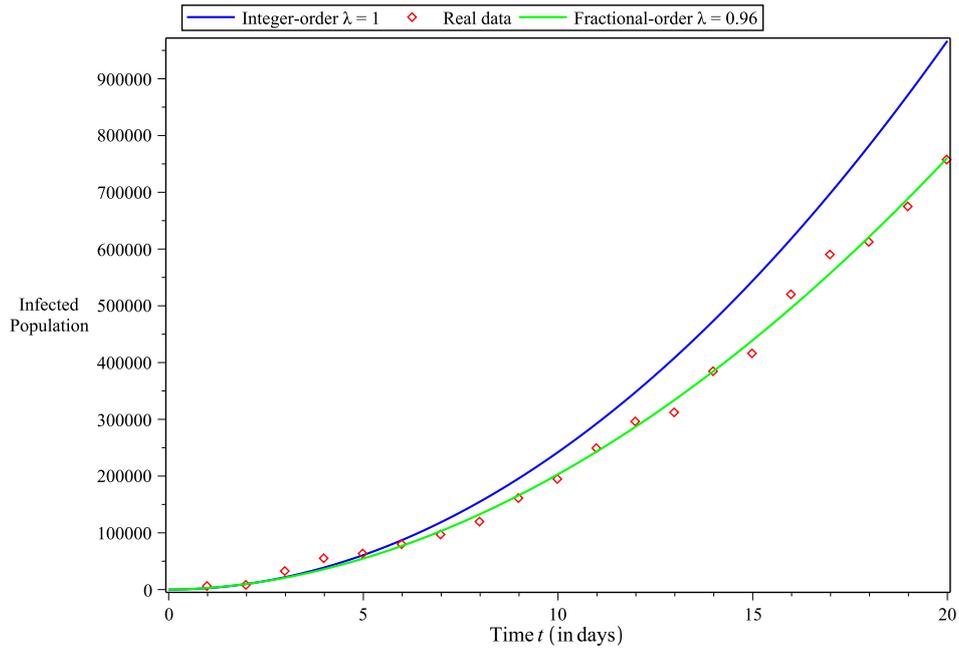


Figure 6.15: Model fitting of $\mathbb{I}(t)$ class Real data with Integer-order and best fitting fractional-order.

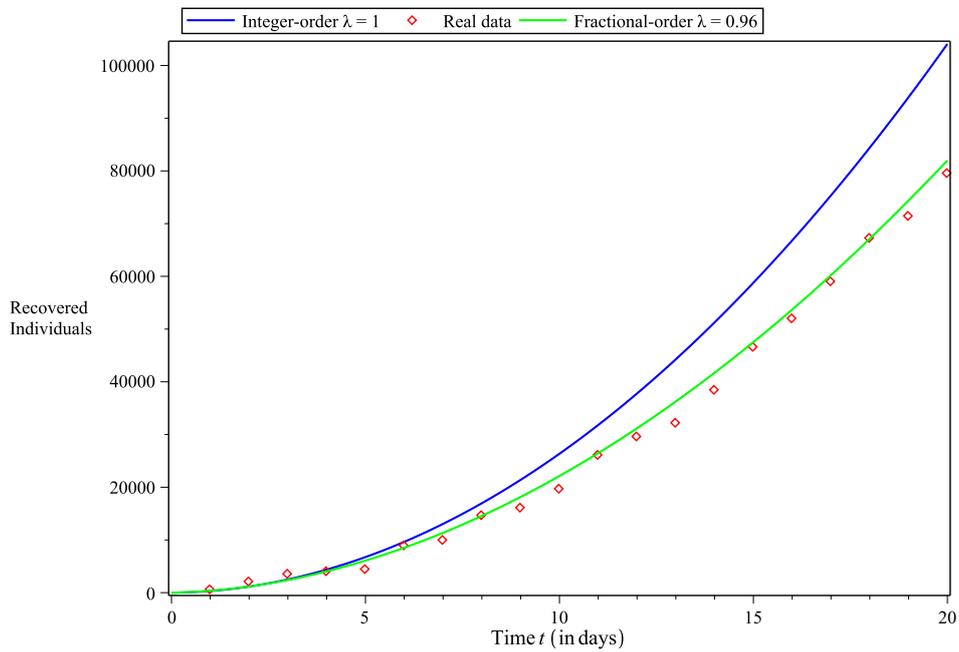


Figure 6.16: Model fitting of $\mathbb{R}(t)$ class Real data with Integer-order and best fitting Fractional-order.

6.7 Results and Conclusion

In this current study, the suggested model is solved by the HPLTM technique as a non-linear FDEs with the latest ABC derivative. Sections – 6.3 and 6.4 include the discussion of the qualitative analysis of the SEIR model’s existence and uniqueness via Banach’s fixed point theory and stability analysis respectively. Figures - 6.2 to 6.5 demonstrate the graphical expressions of the solution of the compartments $\mathbb{S}(t)$, $\mathbb{E}(t)$, $\mathbb{I}(t)$, and $\mathbb{R}(t)$ of the proposed model (6.1) with distinct fractional orders $\alpha = 0.90, 0.93, 0.96, 0.99$, and 1, respectively. Figures - 6.6 and 6.7 showcases the graphical results of the rate of transmission from $\mathbb{S}(t)$ class to $\mathbb{E}(t)$ class and $\mathbb{I}(t)$ class, respectively. The current study highlights the value of the mathematical model while outlining practical issues and the effectiveness of the fractional operator under consideration. Furthermore, we conclude that the strategy for the estimated result is quite methodical and effective in addressing the system of FDEs. We also conclude with the applicability of the suggested technique for other epidemic models, as well as some connected issues that one can take up as further research towards service of humanity.