

# Chapter 5

## Adomian Decomposition Shehu Transform Method with Caputo and Atangana-Baleanu Derivatives

### 5.1 Introduction

Recently, several fractional derivatives (viz. Caputo-Hadamard (2012), Caputo-Fabrizio (2015), Atangana-Baleanu (2016), etc.) have been studied by various authors and fruitful results were obtained by implementing these derivatives by number of researchers. As the kernel of the Caputo derivative [33] is singular, Caputo and Fabrizio [34] introduced a non-singular kernel and eventually defined Caputo-Fabrizio (CF) fractional derivative. Because of its advantages, several mathematicians [116, 29, 49, 132] employed this operator to analyse various fractional partial differential equations (FPDEs). However, the CF derivative is a non-local one, but the integrand involved does not have a fractional operator; this issue was addressed by Atangana and Baleanu in 2016. By modifying the definition of Caputo-Fabrizio and introducing Atangana-Baleanu (ABC) derivative, using newly defined non-local and non-singular kernel. This novel fractional operator combines the Reimann-Liouville and Caputo derivatives [24]. This new ABC derivative has a great memory due to the existence of Mittag-Leffler function as its non-local kernel, eventually it results in a better comparative performance as compared to other existing fractional derivative operators.

Validation of the above claim is justified by applying ABC operator, instead of other operators, and solving various scientific models, viz. The general sequential hybrid class of

FDEs [82, 83], controllability of neutral impulsive FDEs with ABC derivatives [30], Covid-19 mathematical model [86, 85], fractional typhoid model [57], wireless sensor network as an application of the fuzzy fractional SIQR model [44], plasma particle model with circular LASER light polarization [171], Hepatitis B model [42], SEIR and Blood Coagulation technologies [118], a fractal-fractional tuberculosis [87] and tobacco [84] mathematical model, a class of population growth model [61], and the fractional nonlinear logistic system [61].

In this chapter, the non-linear time-fractional Sawada-Kotera-Ito equation (TFSKIE) [162, 11] (representing sub-diffusion dynamics for example), namely

$$D_t^\alpha u + 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} + 21uu_{xxxxx} + u_{xxxxxx} = 0, \text{ where } t > 0, 0 < \alpha \leq 1; \text{ is discussed.} \quad (5.1)$$

where  $D_t^\alpha$  is fractional derivative,  $\alpha$  is fractional order, and  $u$  is function of time variable  $t$  and space variable  $x$ . In mathematical physics and engineering, TFSKIE is examined in shallow water in motions of long waves under gravity [75], sub-diffusion dynamics of nano-precipitate growth and destruction [142], interaction phenomena and lump-type solutions [105], acoustic magnetic dispersion in plasma [23], etc. It has also been used in engineering to model the dynamic of mechanical and electrical systems, study of chaotic and nonlinear systems, and the propagation of waves in complex media such as composite materials and porous media [164].

We successfully fabricated a hybrid technique for obtaining semi-analytic solutions of fractional differential equations with different constraints, and name it as Adomian Decomposition Shehu Transform Method (ADShTM), which combines Adomian's [3, 4] well-known Adomian decomposition method (ADM) and Maitama's Shehu transform (ShT) [103]. The suggested hybrid method is a fast converging approach for solving linear and non-linear stiff processes of FPDEs that can be easily adopted by researchers in several areas of science and technology.

## 5.2 Preliminaries

**Definition 5.1.** Shehu transform (ShT) [103] of function  $u(t)$  of exponential order is defined as

$$S_h\{u(t)\} = U(s, r) = \int_0^\infty e^{-\left(\frac{st}{r}\right)} u(t) dt, \text{ where } r > 0, s > 0, \quad (5.2)$$

where function  $u(t)$  valid over the set

$$A = \left\{ u(t) : \exists M, \delta_1, \delta_2 > 0, |u(t)| < M \exp\left(\frac{|t|}{\delta_j}\right), \text{ if } t \in (-1)^j \times [0, \infty) \right\}.$$

Table 5.1: Shehu transform of some basic function  $u(t)$ .

Function $u(t)$	Shehu transform $S_h\{u(t)\}$
$S_h\{k\}$	$\frac{kr}{s}$
$S_h\{t\}$	$\frac{r^2}{s^2}$
$S_h\{t^n\}$	$\frac{\Gamma(n+1)r^{n+1}}{s^{n+1}}, n > 0$
$S_h\{e^{at}\}$	$\frac{u}{s-ar}$
$S_h\{\sin(at)\}$	$\frac{ar^2}{s^2+a^2r^2}$
$S_h\{\cos(at)\}$	$\frac{rs}{s^2+a^2r^2}$

**Definition 5.2.** Inverse Shehu transform (IShT) [103] of function  $u(t)$  is defined as

$$u(t) = S_h^{-1}[U(s, r)] = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{1}{r} \exp\left(\frac{st}{r}\right) S_h\{u(t)\} ds, \quad (5.3)$$

**Theorem 5.1. (Existence of ShT)** If  $u(t)$  is piecewise continuous function in  $0 \leq t \leq \beta$  and with exponential order  $\alpha$  for  $t > \beta$ . Then its ShT  $S_h\{u(t)\}$  exists.

**Proof.** Given in [103].

**Theorem 5.2. (Convolution Th.)** If the Shehu transform of  $u_1(t)$  and  $u_2(t)$  are

$U_1(s, r)$  and  $U_2(s, r)$  respectively, then Convolution of  $u_1$  and  $u_2$  is

$$S_h\{u_1(t) * u_2(t)\} = U_1(s, r) \cdot U_2(s, r), \quad (5.4)$$

or,

$$u_1(t) * u_2(t) = \int_0^t u_1(t)u_2(t - \tau)d\tau. \quad (5.5)$$

**Proof.** Given in [103].

**Theorem 5.3.** The Shehu transform of integer ordered derivative is defined as

$$\begin{aligned} S_h\{u'(t)\} &= \frac{s}{r}S_h\{u(t)\} - u(0), \\ S_h\{u''(t)\} &= \frac{s^2}{r^2}S_h\{u(t)\} - \frac{s}{r}u(0) - u'(0), \end{aligned}$$

in general for  $n^{th}$  order can be written as

$$S_h\{u^{(n)}(t)\} = \left(\frac{s}{r}\right)^n S_h\{u(t)\} - \sum_{k=0}^{n-1} \left(\frac{s}{r}\right)^{n-k-1} u^{(k)}(0). \quad (5.6)$$

**Proof.** Given in [103].

**Lemma 5.1.** Shehu transform of Riemann-Liouville fractional integral  $D_t^{-(n-\alpha)}h(x, t)$  is

$$S_h\{D_t^{-(n-\alpha)}h(x, t)\} = \left(\frac{s}{r}\right)^{-(n-\alpha)} S_h\{h(x, t)\}. \quad (5.7)$$

**Proof.** First, we will use the definition of R-L integral as given in equation (1.1), we get

$$\begin{aligned} S_h\left\{D_t^{-(n-\alpha)}h(x, t)\right\} &= S_h\left\{\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1}h(x, \tau)d\tau\right\}, \\ &= \frac{1}{\Gamma(n-\alpha)} [S_h\{t^{n-\alpha-1}\} \cdot S_h\{h(x, t)\}], \\ &= \frac{1}{\Gamma(n-\alpha)} \left[\Gamma(n-\alpha)\left(\frac{r}{s}\right)^{n-\alpha} S_h\{h(x, t)\}\right], \\ &= \left(\frac{s}{r}\right)^{-(n-\alpha)} S_h\{h(x, t)\}. \end{aligned}$$

**Theorem 5.4.** The Shehu transform of fractional derivative in Caputo sense for continuous function  $u(x, t)$  on  $\Omega$  is

$$S_h\{ {}_0^C D_t^\alpha u(x, t) \} = \left(\frac{s}{r}\right)^\alpha S_h\{u(x, t)\} - \sum_{k=0}^{n-1} \left(\frac{s}{r}\right)^{\alpha-k-1} D_t^k u(x, 0), \quad (5.8)$$

where  $D_t \equiv \frac{\partial}{\partial t}$ ,  $n < \alpha \leq n - 1$  and  $\Omega = (-\infty, \infty) \times (0, \infty)$ .

**Proof.** We have the Caputo fractional derivative of  $u(x, t)$  as in (1.2) given below:

$$D_t^\alpha [u(x, t)] = D_t^{-(n-\alpha)} [h(x, t)], \text{ where } h(x, t) = u_t^{(n)}(x, t), \text{ } n - 1 < \alpha \leq n. \quad (5.9)$$

Applying the Shehu transform to  ${}_0^C D_t^\alpha u(x, t)$ , using (5.9) it is

$$S_h\{ {}_0^C D_t^\alpha u(x, t) \} = S_h\{ D_t^{-(n-\alpha)} h(x, t) \}, \quad (5.10)$$

Lemma – 5.1 leads to

$$S_h\{ D_t^\alpha u(x, t) \} = S_h\{ D_t^{-(n-\alpha)} h(x, t) \} = \left(\frac{s}{r}\right)^{-(n-\alpha)} S_h\{h(x, t)\}, \quad (5.11)$$

where now using (5.6), we get

$$S_h\{h(x, t)\} = S_h\{u_t^{(n)}(x, t)\} = \left(\frac{s}{r}\right)^n S_h\{u(x, t)\} - \sum_{k=0}^{n-1} \left(\frac{s}{r}\right)^{n-k-1} D_t^k u(x, 0). \quad (5.12)$$

Using identities (5.11) and (5.12) into (5.10), we have

$$S_h\{ {}_0^C D_t^\alpha u(x, t) \} = \left(\frac{s}{r}\right)^\alpha S_h\{u(x, t)\} - \sum_{k=0}^{n-1} \left(\frac{s}{r}\right)^{\alpha-k-1} D_t^k u(x, 0). \quad (5.13)$$

**Lemma 5.2.** Shehu transform of Mittag-Leffler function is given as

$$S_h \left\{ E_\alpha \left( \frac{-\alpha}{n-\alpha} t^\alpha \right) \right\} = \frac{1}{\left(\frac{s}{r}\right)^\alpha + \frac{\alpha}{n-\alpha}}$$

**Proof.** First, we will use the definition of Mittag-Leffler function of order one as given

in (1.3), we have

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

now replacing  $z$  with  $\frac{-\alpha}{n-\alpha}t^\alpha$ , it leads to

$$E_\alpha\left(\frac{-\alpha}{n-\alpha}t^\alpha\right) = \sum_{k=0}^{\infty} \left(\frac{-\alpha}{n-\alpha}\right)^k \frac{t^{k\alpha}}{\Gamma(\alpha k + 1)}.$$

Applying Shehu transform to above equation, we get

$$\begin{aligned} S_h \left\{ E_\alpha \left( \frac{-\alpha}{n-\alpha} t^\alpha \right) \right\} &= S_h \left\{ \sum_{k=0}^{\infty} \left( \frac{-\alpha}{n-\alpha} \right)^k \frac{t^{k\alpha}}{\Gamma(\alpha k + 1)} \right\}, \\ &= \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{(n-\alpha)^k} \left( \frac{r}{s} \right)^{\alpha(k+1)}, \\ &= \left( \frac{r}{s} \right)^\alpha \left( 1 - \frac{\alpha}{n-\alpha} \left( \frac{r}{s} \right)^\alpha + \left( \frac{\alpha}{n-\alpha} \left( \frac{r}{s} \right)^\alpha \right)^2 - \left( \frac{\alpha}{n-\alpha} \left( \frac{r}{s} \right)^\alpha \right)^3 + \dots \right), \\ &= \left( \frac{r}{s} \right)^\alpha \left( \frac{1}{1 + \frac{\alpha}{n-\alpha} \left( \frac{r}{s} \right)^\alpha} \right), \\ &= \frac{1}{\left( \frac{s}{r} \right)^\alpha + \frac{\alpha}{n-\alpha}}. \end{aligned}$$

**Theorem 5.5.** The Shehu transform of fractional derivative in the ABC sense for continuous function  $u(x, t) \in H^1(a, b)$ ,  $b > a$  is

$$S_h \{ {}_0^{ABC} D_t^\alpha [u(x, t)] \} = \frac{M(\alpha)}{n-\alpha} \left[ \frac{\left( \frac{s}{r} \right)^\alpha S_h \{ u(x, t) \} - \sum_{k=0}^{n-1} \left( \frac{s}{r} \right)^{\alpha-k-1} D_t^k u(x, 0)}{\left( \frac{s}{r} \right)^\alpha + \left( \frac{\alpha}{n-\alpha} \right)} \right], \quad (5.14)$$

where  $n-1 < \alpha \leq n$ .

**Proof.** Let  $S_h \{ u(x, t) \}$  be the ShT of the function  $u(x, t)$ .

For  $n-1 < \alpha \leq n$ , the time fractional derivative of  $u(x, t) \in H^1(a, b)$ , ( $b > a$ ) with order  $\alpha$  in Atangana-Baleanu sense is defined in (1.5) as below:

$${}_0^{ABC} D_t^\alpha [u(x, t)] = \frac{M(\alpha)}{n-\alpha} \int_0^t E_\alpha \left( \frac{-\alpha(t-\tau)^\alpha}{(n-\alpha)} \right) \frac{\partial^n u(x, \tau)}{\partial \tau^n} d\tau. \quad (5.15)$$

As per the property of the Convolution theorem, using (5.5), we get

$$\int_0^t \frac{\partial^n u(x, t)}{\partial t^n} E_\alpha \left( \frac{-\alpha}{n-\alpha} (t-\tau)^\alpha \right) d\tau = D_t^\alpha u(x, t) * E_\alpha \left( \frac{-\alpha}{n-\alpha} t^\alpha \right). \quad (5.16)$$

Applying the Shehu transform to the equation (5.15), gives

$$\begin{aligned} S_h \{ {}_0^{ABC} D_t^\alpha [u(x, t)] \} &= S_h \left\{ \frac{M(\alpha)}{n-\alpha} \int_0^t E_\alpha \left( \frac{-\alpha(t-\tau)^\alpha}{(n-\alpha)} \right) \frac{\partial^n u(x, \tau)}{\partial \tau^n} d\tau \right\}, \\ &= \frac{M(\alpha)}{n-\alpha} S_h \left\{ D_t^\alpha u(x, t) * E_\alpha \left( \frac{-\alpha}{n-\alpha} t^\alpha \right) \right\}, \\ &= \frac{M(\alpha)}{n-\alpha} S_h \{ D_t^\alpha u(x, t) \} \cdot S_h \left\{ E_\alpha \left( \frac{-\alpha}{n-\alpha} t^\alpha \right) \right\}, \end{aligned} \quad (5.17)$$

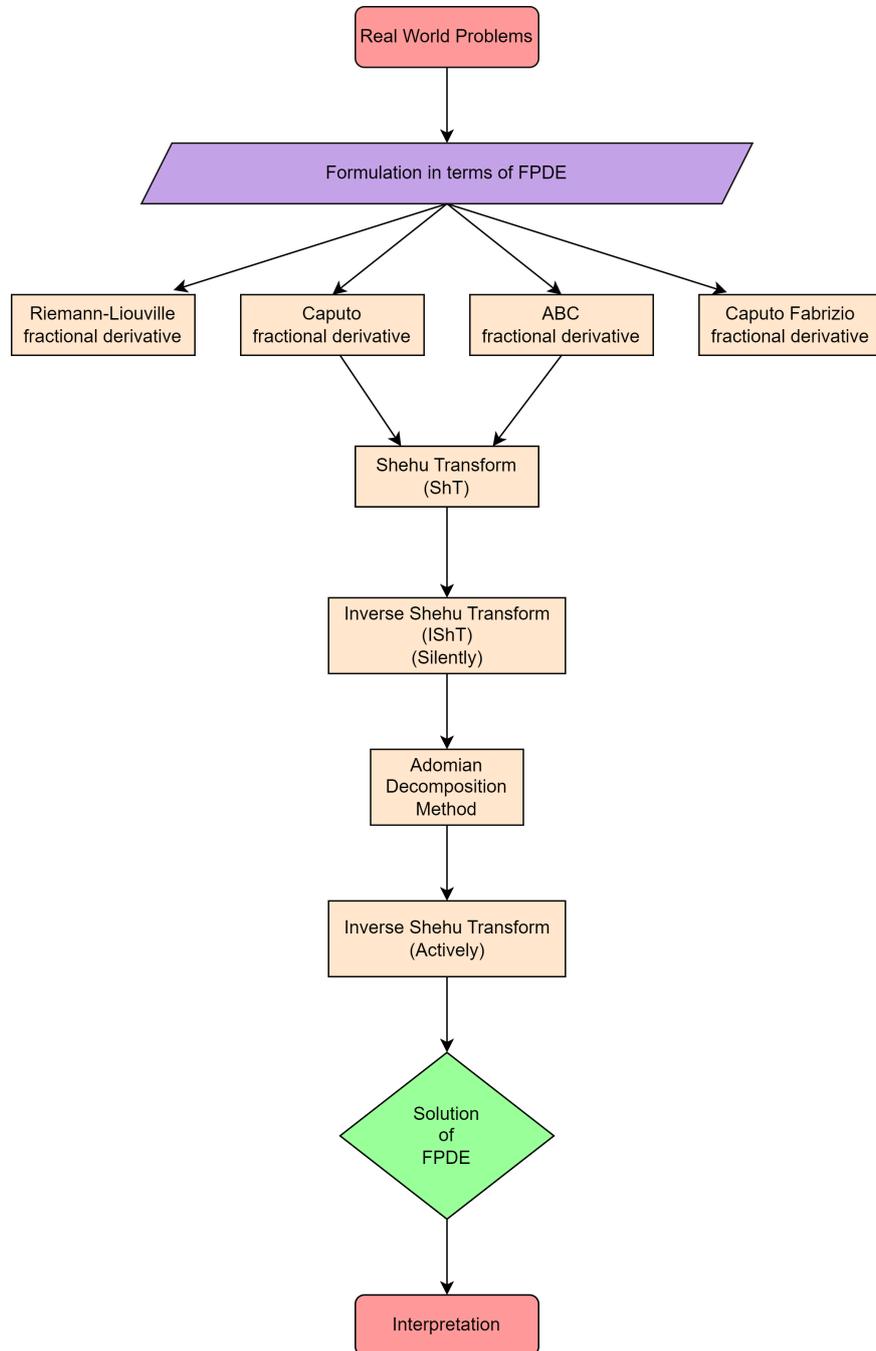
employing the result of Theorem – 5.4 as equation (5.8) and Lemma – 5.2, gives us

$$\begin{aligned} S_h \{ {}_0^{ABC} D_t^\alpha [u(x, t)] \} &= \frac{M(\alpha)}{n-\alpha} \left( \left( \frac{s}{r} \right)^\alpha S_h \{ u(x, t) \} \right. \\ &\quad \left. - \sum_{k=0}^{n-1} \left( \frac{s}{r} \right)^{\alpha-k-1} D_t^k u(x, 0) \right) \cdot \left( \frac{1}{\left( \frac{s}{r} \right)^\alpha + \frac{\alpha}{n-\alpha}} \right), \\ &= \frac{M(\alpha)}{n-\alpha} \left[ \frac{\left( \frac{s}{r} \right)^\alpha S_h \{ u(x, t) \} - \sum_{k=0}^{n-1} \left( \frac{s}{r} \right)^{\alpha-k-1} D_t^k u(x, 0)}{\left( \frac{s}{r} \right)^\alpha + \frac{\alpha}{n-\alpha}} \right]. \end{aligned} \quad (5.18)$$

Hence, the proof is complete.

## 5.3 Adomian Decomposition Shehu Transform Method

### 5.3.1 Flow Diagram of Proposed Scheme



### 5.3.2 Working of the ADShTM

Let us consider a general fractional-ordered non-linear differential equation

$$D_t^\alpha u(x, t) + Ru(x, t) + Nu(x, t) = f(x, t), \text{ where } x \in R, t \geq 0, n - 1 < \alpha \leq n, \quad (5.19)$$

subject to initial conditions

$$u(x, 0) = \phi_0(x), \frac{\partial u(x, 0)}{\partial t} = \phi_1(x), \dots, \frac{\partial^{n-1} u(x, 0)}{\partial t^{n-1}} = \phi_{n-1}(x). \quad (5.20)$$

where  $D_t$  ( $\equiv \frac{\partial}{\partial t}$ ) is the derivative operator, R denotes linear operator and N represents the non-linear differential operator on the continuous function  $u(x, t)$ .

Operating the Shehu transform to (5.19) yields

$$S_h \{D_t^\alpha u(x, t)\} = -S_h \{Ru(x, t)\} - S_h \{Nu(x, t)\} + S_h \{f(x, t)\}, \quad (5.21)$$

There are two different cases:

(I) Shehu transform of the Caputo fractional derivative.

(II) Shehu transform of the Atangana-Baleanu fractional derivative.

#### Case - (I)

let us use derivative property (5.8) of the Shehu transform in the Caputo sense, (5.21) can be written as

$$\begin{aligned} \left(\frac{s}{r}\right)^\alpha S_h \{u(x, t)\} - \sum_{k=0}^{n-1} \left(\frac{s}{r}\right)^{\alpha-k-1} D_t^k u(x, 0) = -S_h \{Ru(x, t)\} - S_h \{Nu(x, t)\} \\ + S_h \{f(x, t)\}, \end{aligned} \quad (5.22)$$

or

$$\begin{aligned} S_h \{u(x, t)\} = \left[ \left(\frac{r}{s}\right) \phi_0(x) + \left(\frac{r}{s}\right)^2 \phi_1(x) + \dots + \left(\frac{r}{s}\right)^{n-1} \phi_{n-1}(x) \right] \\ - \left(\frac{r}{s}\right)^\alpha S_h \{Ru(x, t)\} - \left(\frac{r}{s}\right)^\alpha S_h \{Nu(x, t)\} + \left(\frac{r}{s}\right)^\alpha S_h \{f(x, t)\}. \end{aligned} \quad (5.23)$$

Applying inverse Shehu transform on (5.23) takes the form

$$u(x, t) = \phi(x, t) - S_h^{-1} \left[ \left( \frac{r}{s} \right)^\alpha S_h \{Ru(x, t)\} + \left( \frac{r}{s} \right)^\alpha S_h \{Nu(x, t)\} \right], \quad (5.24)$$

where  $\phi(x, t)$  is inverse Shehu transform of the first and last term of (5.23), considering together.

Now, applying the Adomian decomposition method [4] to equation (5.24) gives

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \phi(x, t) - S_h^{-1} \left[ \left( \frac{r}{s} \right)^\alpha S_h \left\{ R \sum_{n=0}^{\infty} u_n(x, t) \right\} + \left( \frac{r}{s} \right)^\alpha S_h \left\{ N \sum_{n=0}^{\infty} u_n(x, t) \right\} \right], \quad (5.25)$$

an appropriate strategy to determine non-linear terms of (5.25) using the Adomian Polynomial [3] is

$$A_n = \frac{1}{n!} \frac{d^n}{d\theta^n} \left[ N \left( \sum_{j=0}^{\infty} \theta^j u_j(x, t) \right) \right]_{\theta=0}, \quad n = 0, 1, 2, \dots \quad (5.26)$$

Substituting (5.26) in (5.25), leads to

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \phi(x, t) - S_h^{-1} \left[ \left( \frac{r}{s} \right)^\alpha S_h \left\{ R \sum_{n=0}^{\infty} u_n(x, t) \right\} + \left( \frac{r}{s} \right)^\alpha S_h \left\{ \sum_{n=0}^{\infty} A_n \right\} \right], \quad (5.27)$$

the recurrence relation in (5.25) yields

$$\begin{aligned}
u_0(x, t) &= \phi(x, t) \\
u_1(x, t) &= -S_h^{-1} \left[ \left( \frac{r}{s} \right)^\alpha S_h \{ Ru_0(x, t) \} + \left( \frac{r}{s} \right)^\alpha S_h \{ A_0 \} \right], \\
u_2(x, t) &= -S_h^{-1} \left[ \left( \frac{r}{s} \right)^\alpha S_h \{ Ru_1(x, t) \} + \left( \frac{r}{s} \right)^\alpha S_h \{ A_1 \} \right], \\
&\vdots \\
u_n(x, t) &= -S_h^{-1} \left[ \left( \frac{r}{s} \right)^\alpha S_h \{ Ru_{n-1}(x, t) \} + \left( \frac{r}{s} \right)^\alpha S_h \{ A_{n-1} \} \right].
\end{aligned} \tag{5.28}$$

Thus, the solution  $u(x, t)$  can be approximated as

$$\begin{aligned}
u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\
&= u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots
\end{aligned} \tag{5.29}$$

### Case - (II)

Using differential property (5.14) of the Shehu transform in the ABC sense, (5.21) can be written as

$$\begin{aligned}
&\frac{1}{(n - \alpha) \left( \frac{s}{r} \right)^\alpha + \alpha} \left[ \left( \frac{s}{r} \right)^\alpha S_h \{ u(x, t) \} - \sum_{k=0}^{n-1} \left( \frac{s}{r} \right)^{\alpha-k-1} D_t^k u(x, 0) \right] \\
&= -S_h \{ Ru(x, t) \} - S_h \{ Nu(x, t) \} + S_h \{ f(x, t) \},
\end{aligned} \tag{5.30}$$

or

$$\begin{aligned}
S_h \{ u(x, t) \} &= \left[ \left( \frac{r}{s} \right) \phi_0(x) + \left( \frac{r}{s} \right)^2 \phi_1(x) + \dots + \left( \frac{r}{s} \right)^{n-1} \phi_{n-1}(x) \right] \\
&\quad - \left( \frac{(n - \alpha) \left( \frac{s}{r} \right)^\alpha + \alpha}{\left( \frac{s}{r} \right)^\alpha} \right) S_h \{ Ru(x, t) \} \\
&\quad - \left( \frac{(n - \alpha) \left( \frac{s}{r} \right)^\alpha + \alpha}{\left( \frac{s}{r} \right)^\alpha} \right) S_h \{ Nu(x, t) \} + \left( \frac{(n - \alpha) \left( \frac{s}{r} \right)^\alpha + \alpha}{\left( \frac{s}{r} \right)^\alpha} \right) S_h \{ f(x, t) \}.
\end{aligned} \tag{5.31}$$

Taking inverse Shehu transform of (5.31), leads to

$$u(x, t) = \phi(x, t) - S_h^{-1} \left[ \left( \frac{(n - \alpha) \left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \{Ru(x, t)\} + \left( \frac{(n - \alpha) \left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \{Nu(x, t)\} \right], \quad (5.32)$$

Now, applying the Adomian decomposition method [4] to equation (5.32) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x, t) = \phi(x, t) - S_h^{-1} & \left[ \left( \frac{(n - \alpha) \left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \left\{ R \sum_{n=0}^{\infty} u_n(x, t) \right\} \right. \\ & \left. + \left( \frac{(n - \alpha) \left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \left\{ N \sum_{n=0}^{\infty} u_n(x, t) \right\} \right], \end{aligned} \quad (5.33)$$

an appropriate strategy to determine non-linear terms of (5.33) using the Adomian Polynomial [3], we get

$$A_n = \frac{1}{n} \frac{d^n}{d\theta^n} \left[ N \left( \sum_{j=0}^n \theta^j u_j(x, t) \right) \right]_{\theta=0, n=0,1,2,\dots} \quad (5.34)$$

Substituting (5.34) into (5.33), it reach to

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x, t) = \phi(x, t) - S_h^{-1} & \left[ \left( \frac{(n - \alpha) \left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \left\{ R \sum_{n=0}^{\infty} u_n(x, t) \right\} \right. \\ & \left. + \left( \frac{(n - \alpha) \left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \left\{ \sum_{n=0}^{\infty} A_n \right\} \right], \end{aligned} \quad (5.35)$$

the recurrence relation of (5.33) yields

$$\begin{aligned} u_0(x, t) &= \phi(x, t) \\ u_1(x, t) &= -S_h^{-1} \left[ \left( \frac{(n - \alpha) \left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \{Ru_0(x, t)\} + \left( \frac{(n - \alpha) \left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \{A_0\} \right], \\ u_2(x, t) &= -S_h^{-1} \left[ \left( \frac{(n - \alpha) \left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \{Ru_1(x, t)\} + \left( \frac{(n - \alpha) \left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \{A_1\} \right], \\ &\vdots \\ u_n(x, t) &= -S_h^{-1} \left[ \left( \frac{(n - \alpha) \left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \{Ru_{n-1}(x, t)\} + \left( \frac{(n - \alpha) \left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \{A_{n-1}\} \right]. \end{aligned} \quad (5.36)$$

The solution  $u(x, t)$  is approximated as

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\ &= u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots \end{aligned} \quad (5.37)$$

### 5.3.3 Convergence of ADShTM

**Theorem 5.6.** [21] Let the Banach space  $B \equiv C([a, b] \times [0, T])$  is defined on rectangular interval  $[a, b] \times [0, T]$ . Then equation (5.37) as  $u(x, t) = \sum_{k=0}^{\infty} u_k(x, t)$  is convergent series, if  $u_0 \in B$  is Bounded and  $\|u_{k+1}\| \leq \sigma \|u_k\|, \forall u_k \in B$ , where  $0 < \sigma < 1$  and  $\|u\| = \sup_{\substack{x \in [a, b] \\ t \in [0, T]}} |u(x, t)|$ .

**Proof.** Considering the sequence  $\{A_q\}$  as partial sum of (5.37), we have

$$\begin{aligned} A_0 &= u_0(x, t), \\ A_1 &= u_0(x, t) + u_1(x, t), \\ A_2 &= u_0(x, t) + u_1(x, t) + u_2(x, t), \\ &\vdots \\ A_q &= u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots + u_q(x, t). \end{aligned} \quad (5.38)$$

To prove this theorem, we will try to prove  $\{A_q\}_{q=0}^{\infty}$  be a Cauchy sequence in  $B$ . Now, we take

$$\begin{aligned} \|A_{q+1} - A_q\| &= \|u_{q+1}(x, t)\|, \\ &\leq \sigma \|u_q(x, t)\|, \\ &\leq \sigma^2 \|u_{q-1}(x, t)\|, \\ &\leq \sigma^3 \|u_{q-2}(x, t)\|, \\ &\vdots \\ &\leq \sigma^{q+1} \|u_0(x, t)\|. \end{aligned} \quad (5.39)$$

Therefore, any  $q, n \in N$ , and  $q \geq n$ , we achieve

$$\begin{aligned}
\|A_q - A_n\| &= \|(A_q - A_{q-1}) + (A_{q-1} - A_{q-2}) + (A_{q-2} - A_{q-3}) + \dots + (A_{n+1} - A_n)\|, \\
&\leq \|A_q - A_{q-1}\| + \|A_{q-1} - A_{q-2}\| + \|A_{q-2} - A_{q-3}\| + \dots + \|A_{n+1} - A_n\|, \\
&\leq \sigma^q \|u_0(x, t)\| + \sigma^{q-1} \|u_0(x, t)\| + \sigma^{q-2} \|u_0(x, t)\| + \dots + \sigma^{n+1} \|u_0(x, t)\|, \\
&\leq (\sigma^q + \sigma^{q-1} + \sigma^{q-2} + \dots + \sigma^{n+1}) \|u_0(x, t)\|, \\
&\leq \frac{(1 - \sigma^{q-n})}{(1 - \sigma)} \sigma^{n+1} \|u_0(x, t)\|, \\
&\leq \beta \|u_0(x, t)\|.
\end{aligned} \tag{5.40}$$

where  $\beta = \frac{(1 - \sigma^{q-n})}{(1 - \sigma)} \sigma^{n+1}$ . Since  $u_0(x, t)$  is bounded then  $\|u_0(x, t)\| < \infty$ .

As the value of  $n$  increases with  $0 < \sigma < 1$ , and  $n \rightarrow \infty$  leads to

$$\lim_{\substack{n \rightarrow \infty \\ q \rightarrow \infty}} \|A_q - A_n\| \leq \lim_{\substack{n \rightarrow \infty \\ q \rightarrow \infty}} \beta \|u_0(\zeta, t)\|, \tag{5.41}$$

$$= \|u_0(\zeta, t)\| \lim_{\substack{n \rightarrow \infty \\ q \rightarrow \infty}} \left( \frac{1 - \sigma^{q-n}}{1 - \sigma} \right) \sigma^{n+1}, \tag{5.42}$$

$$= \|u_0(\zeta, t)\| \left( \frac{1}{1 - \sigma} \right) \lim_{n \rightarrow \infty} \sigma^{n+1}, \tag{5.43}$$

$$= 0. \tag{5.44}$$

Thus, for sufficiently large values of  $q$  and  $n$ , we have  $\|A_q - A_n\| \rightarrow 0$ .

Hence,  $\{A_q\}_{q=0}^{\infty}$  is Cauchy sequence in  $B$ .

It concludes that the solution of equation (5.19) as series is convergent.

**Theorem 5.7.** If the approximate series solution of (5.19) is  $\sum_{k=0}^n u_k(x, t)$ , then the maximum Absolute error is estimated by

$$\left\| u(x, t) - \sum_{k=0}^n u_k(x, t) \right\| \leq \frac{\sigma^{n+1}}{1 - \sigma} \|u_0(x, t)\|. \tag{5.45}$$

**Proof.** From equation (5.40) in Theorem – 5.6, we have

$$\|A_q - A_n\| \leq \beta \|u_0(x, t)\|, \text{ where } \beta = \frac{(1 - \sigma^{q-n})}{(1 - \sigma)} \sigma^{n+1}. \tag{5.46}$$

Here,  $\{A_q\}_{q=0}^{\infty} \rightarrow u(x, t)$  as  $q \rightarrow \infty$  and from equation (5.38), we have  $A_n = \sum_{k=0}^n u_k(x, t)$ ,

$$\left\| u(x, t) - \sum_{k=0}^n u_k(x, t) \right\| \leq \beta \|u_0(x, t)\|, \quad (5.47)$$

Now,  $(1 - \sigma^{q-n}) < 1$  since  $0 < \sigma < 1$ , then

$$\left\| u(x, t) - \sum_{k=0}^n u_k(x, t) \right\| \leq \frac{\sigma^{n+1}}{1 - \sigma} \|u_0(x, t)\|. \quad (5.48)$$

Thus, Theorem is proved.

## 5.4 Numerical Application

Consider a non-linear time-fractional Sawada-Kotera-Ito equation (TFSKIE) [162, 11] as

$$\begin{aligned} D_t^\alpha u + 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} \\ + 21uu_{xxxxx} + u_{xxxxxxx} = 0, \end{aligned} \quad (5.49)$$

where  $t > 0$ ,  $0 < \alpha \leq 1$  and  $x \in R$ ,

with initial condition

$$u(x, 0) = \frac{4}{3}b^2 (2 - 3\tanh^2(bx)), \quad (5.50)$$

where  $b$  is an arbitrary constant.

The exact solution [10] of equation (5.49) for standard motion i.e.  $\alpha = 1$ , is given by

$$u(x, t) = \frac{4}{3}b^2 \left[ 2 - 3\tanh^2 \left( b \left( \frac{256b^6t}{3} + x \right) \right) \right]. \quad (5.51)$$

In this section, we are dealing with TFSKIE having two different fractional derivative operators known as the Caputo and the ABC respectively.

### Case - (I).

Let us consider the time-fractional derivative in the Caputo sense and operating the ShT

on equation (5.49) gives

$$\begin{aligned} S_h \{ {}_0^C D_t^\alpha u \} = & - S_h \{ 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{xx}u_{xxx} \\ & + 42u_xu_{xxxx} + 21uu_{xxxxx} + u_{xxxxxxx} \}, \end{aligned} \quad (5.52)$$

using (5.7) into (5.52), gives

$$\begin{aligned} \left[ \left( \frac{s}{r} \right)^\alpha S_h \{ u(x, t) \} - \left( \frac{s}{r} \right)^{\alpha-1} u(x, 0) \right] = & - S_h \{ 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} \\ & + 126u^2u_{xxx} + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} \\ & + 21uu_{xxxxx} + u_{xxxxxxx} \}, \end{aligned} \quad (5.53)$$

applying initial condition (5.50), we have

$$\begin{aligned} S_h \{ u(x, t) \} = & \frac{4b^2r}{3s} (2 - 3\tanh^2(bx)) - \left( \frac{r}{s} \right)^\alpha S_h \{ 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} \\ & + 126u^2u_{xxx} + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} + 21uu_{xxxxx} + u_{xxxxxxx} \}, \end{aligned} \quad (5.54)$$

operating inverse Shehu transform (5.3) to above equation

$$\begin{aligned} u(x, t) = & \frac{4b^2}{3} (2 - 3\tanh^2(bx)) - S_h^{-1} \left[ \left( \frac{r}{s} \right)^\alpha S_h \{ 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} \right. \\ & \left. + 126u^2u_{xxx} + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} + 21uu_{xxxxx} + u_{xxxxxxx} \} \right]. \end{aligned} \quad (5.55)$$

Now applying the Adomian decomposition method [4] to (5.55) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x, t) = & \frac{4b^2}{3} (2 - 3\tanh^2(bx)) - S_h^{-1} \left[ \left( \frac{r}{s} \right)^\alpha S_h \left\{ R \left( \sum_{n=0}^{\infty} u_n(x, t) \right) \right\} \right. \\ & \left. + \left( \frac{r}{s} \right)^\alpha S_h \left\{ N \left( \sum_{n=0}^{\infty} u_n(x, t) \right) \right\} \right]. \end{aligned} \quad (5.56)$$

To determine the nonlinear terms of (5.56), using the Adomian Polynomial [4] is

$$\begin{aligned}
A_0 &= 252u_0^3u_{0x} + 63u_{0x}^3 + 378u_0u_{0x}u_{0xx} + 126u_0^2u_{0xxx} + 63u_{0xx}u_{0xxx} \\
&\quad + 42u_{0x}u_{0xxxx} + 21u_0u_{0xxxxx}, \\
A_1 &= 756u_0^2u_1u_{0x} + 252u_0^3u_{1x} + 189u_{0x}^2u_1 + 378u_1u_{0x}u_{0xx} + 378u_0u_{1x}u_{0xx} \\
&\quad + 378u_0u_{0x}u_{1xx} + 252u_0u_1u_{0xxx} + 126u_0^2u_{1xxx} + 63u_{1xx}u_{0xxx} \\
&\quad + 63u_{0xx}u_{1xxx} + 42u_{1x}u_{0xxxx} + 42u_{0x}u_{1xxxx} + 21u_1u_{0xxxxx} \\
&\quad + 21u_0u_{1xxxxx}, \\
A_2 &= 1512u_0u_1^2u_{0x} + 756u_0^2u_2u_{0x} + 1521u_0^2u_1u_{1x} + 256u_0^3u_{2x} + 378u_{0x}u_{1x}u_1 \\
&\quad + 378u_2u_{0x}u_{0xx} + 756u_1u_{1x}u_{0xx} + 756u_1u_{0x}u_{1xx} + 756u_0u_{1x}u_{1xx} \\
&\quad + 189u_{0x}^2u_2 + 378u_0u_{2x}u_{0xx} + 378u_0u_{0x}u_{2xx} + 252u_1^2u_{0xxx} \\
&\quad + 252u_0u_2u_{0xxx} + 504u_0u_1u_{1xxx} + 126u_0^2u_{2xxx} + 63u_{2xx}u_{0xxx} \\
&\quad + 126u_{1xx}u_{1xxx} + 63u_{0xx}u_{2xxx} + 42u_{2x}u_{0xxxx} + 84u_{1x}u_{1xxx} \\
&\quad + 42u_{0x}u_{2xxxx} + 21u_2u_{0xxxxx} + 42u_1u_{1xxxxx} + 21u_0u_{2xxxxx}, \\
&\quad \vdots
\end{aligned} \tag{5.57}$$

Comparing the both sides of series terms in (5.56), yields

$$\begin{aligned}
u_0(x, t) &= \frac{4b^2}{3} (2 - 3\tanh^2(bx)), \\
u_1(x, t) &= -S_h^{-1} \left[ \left(\frac{r}{s}\right)^\alpha S_h\{u_{0xxxxxx}\} + \left(\frac{r}{s}\right)^\alpha S_h\{A_0\} \right], \\
&= \frac{2048b^9}{3} \tanh(bx) \operatorname{sech}^2(bx) \frac{t^\alpha}{\Gamma(1+\alpha)}, \\
u_2(x, t) &= -S_h^{-1} \left[ \left(\frac{r}{s}\right)^\alpha S_h\{u_{1xxxxxx}\} + \left(\frac{r}{s}\right)^\alpha S_h\{A_1\} \right], \\
&= \frac{524288b^{16}}{9} \operatorname{sech}^4(bx) (\cosh(2bx) - 2) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\
u_3(x, t) &= -S_h^{-1} \left[ \left(\frac{r}{s}\right)^\alpha S_h\{u_{2xxxxxx}\} + \left(\frac{r}{s}\right)^\alpha S_h\{A_2\} \right], \\
&= \frac{268435456b^{23}}{27} \tanh(bx) \operatorname{sech}^4(bx) (\cosh(2bx) - 5) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \\
&\quad \vdots
\end{aligned} \tag{5.58}$$

Thus, the solution of (5.49) can be obtained as

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \dots \quad (5.59)$$

It holds

$$\begin{aligned} u(x, t) = & \frac{4b^2}{3} (2 - 3\tanh^2(bx)) + \frac{2048b^9}{3} \tanh(bx)\operatorname{sech}^2(bx) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & + \frac{524288b^{16}}{9} \operatorname{sech}^4(bx) (\cosh(2bx) - 2) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \frac{268435456b^{23}}{27} \tanh(bx) \operatorname{sech}^4(bx) (\cosh(2bx) - 5) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \dots \end{aligned} \quad (5.60)$$

Here, equation (5.60) is the approximate solution of TFSKIE using the ADShTM in the Caputo sense.

### Case - (II).

Let us consider the time-fractional derivative in the ABC sense and operating the Shehu transform on (5.49) gives

$$\begin{aligned} S_h \{ {}_0^{ABC} D_t^\alpha u \} = & - S_h \{ 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} \\ & + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} + 21uu_{xxxxx} + u_{xxxxxx} \}, \end{aligned} \quad (5.61)$$

using derivative property (5.14) with  $M(\alpha) = 1$  and  $n = 1$ , we get

$$\begin{aligned} & \frac{1}{(1-\alpha) \left( \left( \frac{s}{r} \right)^\alpha + \frac{\alpha}{1-\alpha} \right)} \left[ \left( \frac{s}{r} \right)^\alpha Sh\{u(x, t)\} - \left( \frac{s}{r} \right)^{\alpha-1} u(x, 0) \right] \\ & = - S_h \{ 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} \\ & \quad + 126u^2u_{xxx} + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} \\ & \quad + 21uu_{xxxxx} + u_{xxxxxx} \}, \end{aligned} \quad (5.62)$$

applying initial condition (5.50), we have

$$\begin{aligned}
S_h\{u(x, t)\} &= \frac{4b^2r}{3s} (2 - 3\tanh^2(bx)) \\
&\quad - \left( \frac{(1 - \alpha)\left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \{252u^3u_x + 63u_x^3 + 378uu_xu_{xx} \\
&\quad + 126u^2u_{xxx} + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} + 21uu_{xxxxx} + u_{xxxxxx}\}, \quad (5.63)
\end{aligned}$$

operating inverse Shehu transform (5.3) to above equation

$$\begin{aligned}
u(x, t) &= \frac{4b^2}{3} (2 - 3\tanh^2(bx)) - S_h^{-1} \left[ \left( \frac{(1 - \alpha)\left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \{252u^3u_x + 63u_x^3 \right. \\
&\quad + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} + 21uu_{xxxxx} \\
&\quad \left. + u_{xxxxxx}\} \right]. \quad (5.64)
\end{aligned}$$

Now applying the Adomian decomposition method [4] to equation (5.64) gives

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x, t) &= \frac{4b^2}{3} (2 - 3\tanh^2(bx)) \\
&\quad - S_h^{-1} \left[ \left( \frac{(1 - \alpha)\left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \left\{ R \left( \sum_{n=0}^{\infty} u_n(x, t) \right) \right\} \right. \\
&\quad \left. + \left( \frac{(1 - \alpha)\left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h \left\{ N \left( \sum_{n=0}^{\infty} u_n(x, t) \right) \right\} \right]. \quad (5.65)
\end{aligned}$$

To determine the non-linear terms of equation (5.65), using the Adomian Polynomial [3] is

$$\begin{aligned}
A_0 &= 252u_0^3u_{0x} + 63u_0^3u_x + 378u_0u_{0x}u_{0xx} + 126u_0^2u_{0xxx} + 63u_{0xx}u_{0xxx} \\
&\quad + 42u_{0x}u_{0xxxx} + 21u_0u_{0xxxxx}, \\
A_1 &= 756u_0^2u_1u_{0x} + 252u_0^3u_{1x} + 189u_0^2u_1u_x + 378u_1u_{0x}u_{0xx} + 378u_0u_{1x}u_{0xx} \\
&\quad + 378u_0u_{0x}u_{1xx} + 252u_0u_1u_{0xxx} + 126u_0^2u_{1xxx} + 63u_{1xx}u_{0xxx} \\
&\quad + 63u_{0xx}u_{1xxx} + 42u_{1x}u_{0xxxx} + 42u_{0x}u_{1xxx} + 21u_1u_{0xxxxx} \\
&\quad + 21u_0u_{1xxxxx},
\end{aligned}$$

$$\begin{aligned}
A_2 = & 1512u_0u_1^2u_{0x} + 756u_0^2u_2u_{0x} + 1521u_0^2u_1u_{1x} + 256u_0^3u_{2x} + 378u_{0x}u_{1x}u_1 \\
& + 378u_2u_{0x}u_{0xx} + 756u_1u_{1x}u_{0xx} + 756u_1u_{0x}u_{1xx} + 756u_0u_{1x}u_{1xx} \\
& + 189u_{0x}^2u_2 + 378u_0u_{2x}u_{0xx} + 378u_0u_{0x}u_{2xx} + 252u_1^2u_{0xxx} \\
& + 252u_0u_2u_{0xxx} + 504u_0u_1u_{1xxx} + 126u_0^2u_{2xxx} + 63u_{2xx}u_{0xxx} \\
& + 126u_{1xx}u_{1xxx} + 63u_{0xx}u_{2xxx} + 42u_{2x}u_{0xxx} + 84u_{1x}u_{1xxx} \\
& + 42u_{0x}u_{2xxx} + 21u_2u_{0xxxx} + 42u_1u_{1xxxx} + 21u_0u_{2xxxx}, \tag{5.66} \\
& \vdots
\end{aligned}$$

Comparing the both sides of (5.65), we achieve

$$\begin{aligned}
u_0(x, t) &= \frac{4b^2}{3} (2 - 3\tanh^2(bx)), \\
u_1(x, t) &= -S_h^{-1} \left[ \left( \frac{(1-\alpha)\left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h\{u_{0xxxxxx}\} + \left( \frac{(1-\alpha)\left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h\{A_0\} \right], \\
&= \frac{2048b^9}{3} \tanh(bx) \operatorname{sech}^2(bx) \left( 1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(1+\alpha)} \right), \\
u_2(x, t) &= -S_h^{-1} \left[ \left( \frac{(1-\alpha)\left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h\{u_{1xxxxxx}\} + \left( \frac{(1-\alpha)\left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h\{A_1\} \right], \\
&= \frac{524288b^{16}}{9} \operatorname{sech}^4(bx) (\cosh(2bx) - 2) \left[ (1-\alpha)^2 + \frac{2\alpha(1-\alpha)t^\alpha}{\Gamma(1+\alpha)} \frac{\alpha^2 t^{2\alpha}}{\Gamma(1+2\alpha)} \right], \\
u_3(x, t) &= -S_h^{-1} \left[ \left( \frac{(1-\alpha)\left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h\{u_{2xxxxxx}\} + \left( \frac{(1-\alpha)\left(\frac{s}{r}\right)^\alpha + \alpha}{\left(\frac{s}{r}\right)^\alpha} \right) S_h\{A_2\} \right], \\
&= \frac{268435456b^{23}}{27} \tanh(bx) \operatorname{sech}^4(bx) (\cosh(2bx) - 5) \left[ (1-\alpha)^3 + \frac{3\alpha(1-\alpha)^2 t^\alpha}{\Gamma(1+\alpha)} \right. \\
&\quad \left. + \frac{3\alpha^2(1-\alpha)t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{\alpha^3 t^{3\alpha}}{\Gamma(1+3\alpha)} \right], \tag{5.67} \\
& \vdots
\end{aligned}$$

Thus, the solution of (5.49) can be obtained as

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \dots \tag{5.68}$$

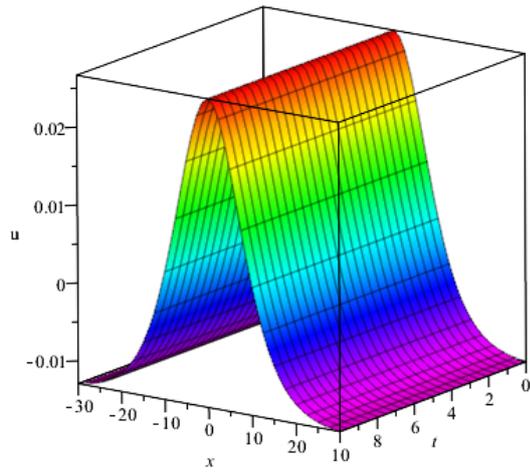
It holds

$$\begin{aligned}
 u(x, t) = & \frac{4b^2}{3} (2 - 3 \tanh^2(bx)) + \frac{2048b^9}{3} \tanh(bx) \operatorname{sech}^2(bx) \left( 1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(1 + \alpha)} \right) \\
 & + \frac{524288b^{16}}{9} \operatorname{sech}^4(bx) (\cosh(2bx) - 2) \left[ (1 - \alpha)^2 + \frac{2\alpha(1 - \alpha)t^\alpha}{\Gamma(1 + \alpha)} \right. \\
 & \left. + \frac{\alpha^2 t^{2\alpha}}{\Gamma(1 + 2\alpha)} \right] + \frac{268435456b^{23}}{27} \tanh(bx) \operatorname{sech}^4(bx) (\cosh(2bx) \\
 & - 5) \left[ (1 - \alpha)^3 + \frac{3\alpha(1 - \alpha)^2 t^\alpha}{\Gamma(1 + \alpha)} + \frac{3\alpha^2(1 - \alpha)t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \frac{\alpha^3 t^{3\alpha}}{\Gamma(1 + 3\alpha)} \right] + \dots \quad (5.69)
 \end{aligned}$$

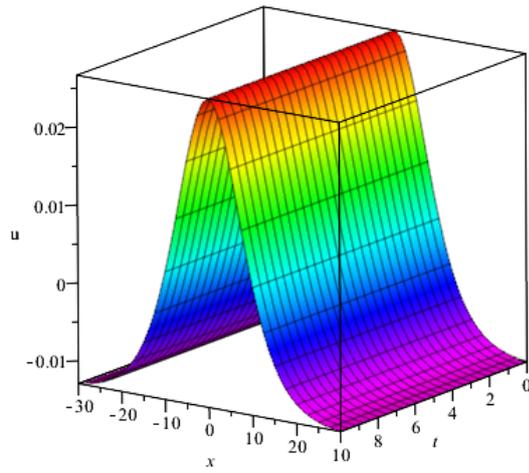
Here, equation (5.69) is the approximate solution of TFSKIE using the ADShTM in the ABC sense.

Table 5.2: The absolute error in the solution of TFSKIE by the ADShTM in the Caputo sense and the ABC sense for  $\alpha = 1$ .

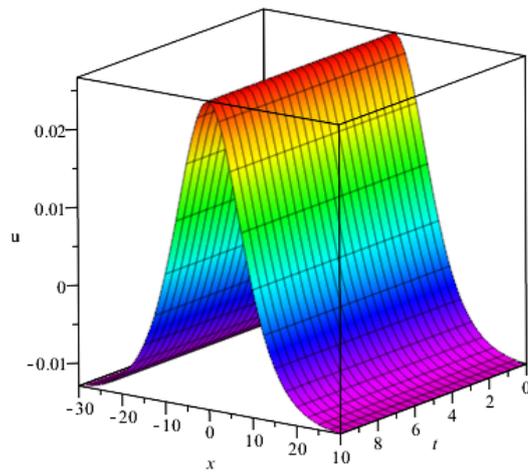
$x$	$t$	Exact	ADShTM (Caputo sense)	ADShTM (ABC sense)	Exact-ADShTM  (Caputo sense)	Exact-ADShTM  (ABC sense)
-10	0.1	0.0034656622	0.0034656185	0.0034656685	4.36750E-08	6.32500E-09
	0.2	0.0034656840	0.0034655967	0.0034656967	8.73490E-08	1.26510E-08
	0.3	0.0034657058	0.0034655748	0.0034657748	1.31024E-07	6.89760E-08
	0.4	0.0034657277	0.0034655530	0.0034658030	1.74685E-07	7.53150E-08
	0.5	0.0034657495	0.0034655311	0.0034655313	2.18360E-07	2.18177E-07
-5	0.1	0.0181246008	0.0181245512	0.0181246012	4.96200E-08	3.80000E-10
	0.2	0.0181246256	0.0181245264	0.0181246264	9.92300E-08	7.70000E-10
	0.3	0.0181246504	0.0181245016	0.0181246552	1.48850E-07	4.74600E-09
	0.4	0.0181246752	0.0181244768	0.0181246791	1.98480E-07	3.82000E-09
	0.5	0.0181247000	0.0181244519	0.0181247519	2.48100E-07	5.19000E-08
5	0.1	0.0181245512	0.0181246008	0.0181245501	4.96300E-08	1.09000E-09
	0.2	0.0181245264	0.0181246256	0.0181245286	9.92500E-08	2.20100E-09
	0.3	0.0181245016	0.0181246504	0.0181245042	1.48860E-07	2.64000E-09
	0.4	0.0181244768	0.0181246752	0.0181245752	1.98480E-07	9.84800E-08
	0.5	0.0181244519	0.0181247000	0.0181245500	2.48110E-07	9.80740E-08
10	0.1	0.0034656185	0.0034656622	0.0034656122	4.36480E-08	6.35200E-09
	0.2	0.0034655967	0.0034656840	0.0034655940	8.73380E-08	2.66200E-09
	0.3	0.0034655748	0.0034657058	0.0034655106	1.30999E-07	6.42444E-08
	0.4	0.0034655530	0.0034657277	0.0034656069	1.74662E-07	5.39000E-08
	0.5	0.0034655311	0.0034657495	0.0034655749	2.18350E-07	4.38027E-08



(a)  $\alpha = 1$  by Caputo



(b)  $\alpha = 1$  by ABC



(c) Exact solution

Figure 5.1: Behavior of TFSKIE by the ADShTM with order  $\alpha = 1$  and  $b = 0.1$  in the Caputo sense, the ABC sense and the exact solution.

Table 5.3: The solution  $u(x, t)$  via ADShTM in the ABC sense with the various fractional values of  $\alpha$  and  $x = 10$ .

$t$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0.1	0.00346586058	0.00346582717	0.00346577490	0.00346571693	0.00346566216
0.2	0.00346586504	0.00346583969	0.00346579389	0.00346573896	0.00346568399
0.3	0.00346586796	0.00346584879	0.00346580927	0.00346575879	0.00346570583
0.4	0.00346587017	0.00346585621	0.00346582269	0.00346577732	0.00346572766
0.5	0.00346587198	0.00346586258	0.00346583481	0.00346579493	0.00346574950

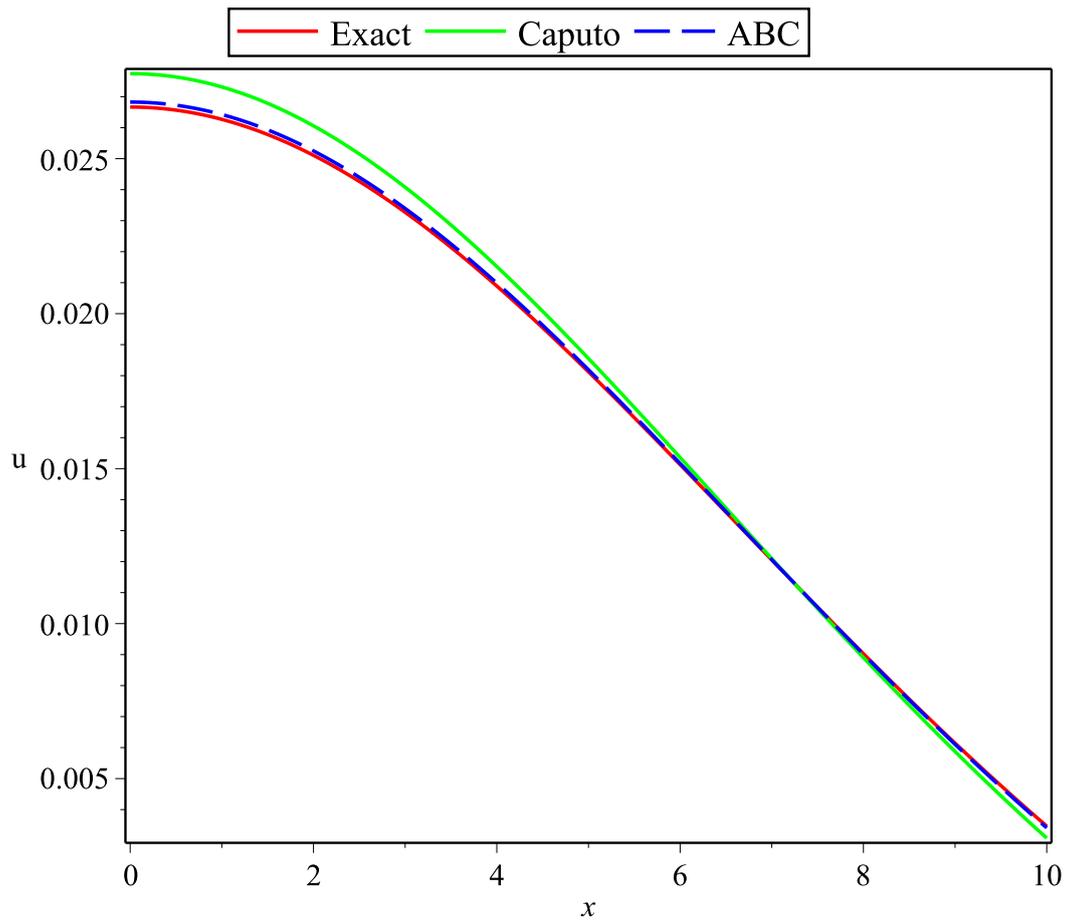
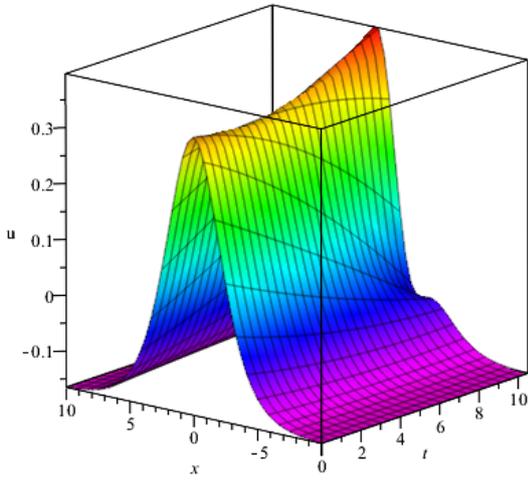
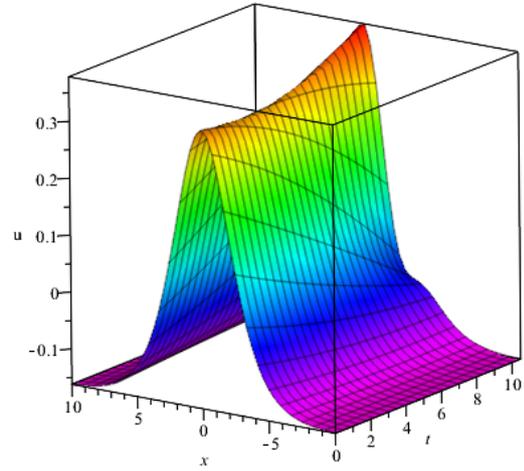


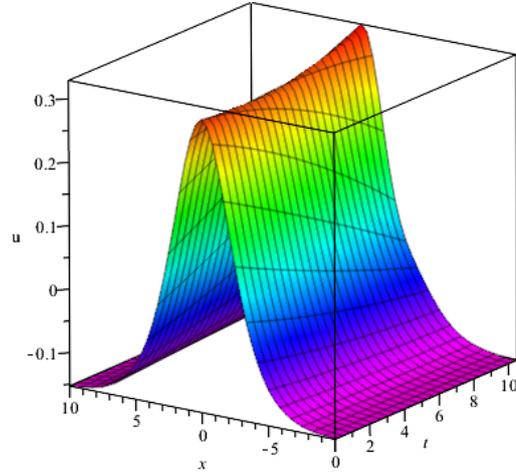
Figure 5.2: Comparison of the solution of TFSKIE by the Caputo sense, the ABC sense, and the exact solution at the values of  $\alpha = 1, b = 0.1$  and  $t = 4$ .



(a)  $\alpha = 0.4$  by ABC



(b)  $\alpha = 0.6$  by ABC



(c)  $\alpha = 0.8$  by ABC

Figure 5.3: Behavior of TFSKIE by the ADShTM with the different fractional order  $\alpha$  and  $b = 0.1$  in the ABC sense.

Table 5.4: The absolute error in the solution of TFSKIE by ADShTM, q-HAM [10], RPSM [12], and ADTM [164] for  $\alpha = 1$ ,  $b = 0.1$  and  $t = 1$  with the exact solution.

$x$	Exact	ADShTM (ABC)	q-HAM (Caputo)	RPSM (Caputo)	ADTM (ABC)	Exact-ADShTM (ABC)	Exact-q-HAM (Caputo)	Exact-RPSM (Caputo)	Exact-ADTM (ABC)
1	0.026269251	0.026269252	0.026269286	0.026269386	0.026269203	7.30000E-10	3.47400E-08	1.34730E-07	4.82300E-08
2	0.025108256	0.025108256	0.025108515	0.025108515	0.025108377	9.43000E-10	2.58990E-07	2.58980E-07	1.20946E-07
3	0.023271963	0.023271967	0.023272327	0.023272327	0.023271491	3.96000E-09	3.63970E-07	3.63960E-07	4.72130E-07
4	0.020891996	0.020891990	0.020892440	0.020892440	0.020891184	5.72997E-09	4.43870E-07	4.43860E-07	8.12320E-07
5	0.018124328	0.018124382	0.018124824	0.018124824	0.018125320	5.45290E-08	4.96210E-07	4.96210E-07	9.92380E-07
6	0.015129516	0.015129638	0.015130038	0.015130038	0.015130560	1.21760E-07	5.21760E-07	5.21760E-07	1.04352E-06
7	0.012055988	0.012056512	0.012065122	0.012056512	0.012058036	5.23760E-07	9.13311E-06	5.23760E-07	2.04754E-06
8	0.009028620	0.009029127	0.009021268	0.009029127	0.009029634	5.06859E-07	7.35192E-06	5.06859E-07	5.01374E-06
9	0.006143123	0.006143599	0.006145992	0.006145992	0.006144075	4.76197E-07	2.86906E-06	2.86909E-06	7.52425E-06
10	0.003465422	0.003465859	0.003468587	0.003468587	0.003466295	4.36689E-07	3.16473E-06	3.16477E-06	8.73427E-06

## 5.5 Results and Conclusion

ADShTM is used in this paper to solve the time-fractional Sawada-Kotera-Ito equation (TFSKIE) using the Caputo and ABC senses. We investigated the convergence of the method using Banach's fixed point theory. In view of Figure – 5.2 and Table – 5.2, we can state that the solution of TFSKIE by the ABC derivative is much better as compared to the Caputo derivative. Additionally, the efficiency of the proposed method is verified in Table – 5.4. The results show that the approximate solution of the TFSKIE by ADShTM is more accurate as compared to other established methods such as q-HAM, RPSM, and ADTM. Figures – 5.3(a), 5.3(b), and 5.3(c) show that the behavior of TFSKIE by ADShTM with different fractional-order  $\alpha = 0.4, 0.6,$  and  $0.8$  are compatible in the ABC sense. The ADShTM using the ABC operator demonstrates effectiveness, accuracy, important characteristics, and ease of computation. We believe that our work will pave the way for many more fractional-order problems in the nature of the non-linear fractional partial differential equations.

## 5.6 A Maple implementation and graphs for Sawada-Kotera-Ito Equation

### 5.6.1 A Maple code with 3D plot for the exact and approximate solution of Swada-Kotera-Ito equation by HPLTM at order $\alpha = 1$ by Caputo and ABC fractional derivative

```
# Solution of Swada-Kotera-Ito equation using Caputo fractional derivative
u0 := (4*b^2*(1/3))*(2-3*tanh(b*x)^2);
```

$$\frac{4}{3}b^2(2 - 3 \tanh(bx)^2)$$

```
u1 := 2048*b^9*tanh(b*x)*sech(b*x)^2*t^a/(3*GAMMA(a+1));
```

$$\frac{2048 b^9 \tanh(bx) \operatorname{sech}(bx)^2 t^a}{3 \Gamma(a+1)}$$

u2 := 524288\*b<sup>16</sup>\*(cosh(2\*b\*x)-2)\*sech(b\*x)<sup>4</sup>\*t<sup>(2\*a)</sup>/(9\*GAMMA(2\*a+1));

$$\frac{524288 b^{16} (\cosh(2bx) - 2) \operatorname{sech}(bx)^4 t^{2a}}{9 \Gamma(2a + 1)}$$

u3 := 268435456\*b<sup>23</sup>\*(cosh(2\*b\*x)-5)\*tanh(b\*x)\*sech(b\*x)<sup>4</sup>\*t<sup>(3\*a)</sup>/(27\*GAMMA(3\*a+1));

$$\frac{268435456 b^{23} (\cosh(2bx) - 5) \tanh(bx) \operatorname{sech}(bx)^4 t^{3a}}{27 \Gamma(3a + 1)}$$

vc := u0+u1+u2;

$$\frac{4}{3}b^2 (2 - 3 \tanh(bx)^2) + \frac{2048 b^9 \tanh(bx) \operatorname{sech}(bx)^2 t^a}{\Gamma(a+1)} + \frac{524288 b^{16} (\cosh(2bx) - 2) \operatorname{sech}(bx)^4 t^{2a}}{9 \Gamma(2a+1)}$$

vc1 := eval(vc, [b = 0.1, a = 1]);

$$0.02666666667 - 0.04 \tanh(0.1x)^2 + 6.82666666710^{-7} \tanh(0.1x) \operatorname{sech}(0.1x)^2 t + 2.91271111110^{-12} (\cosh(0.2x) - 2) \operatorname{sech}(0.1x)^4 t^2$$

plot3d(vc1, t = 0 .. 10.5, x = -30 .. 30);

See figure – (5.1a)

# Solution of Swada-Kotera-Ito equation using ABC fractional derivative

u0 := (4\*b<sup>2</sup>\*(1/3))\*(2-3\*tanh(b\*x)<sup>2</sup>);

$$\frac{4}{3}b^2 (2 - 3 \tanh(bx)^2)$$

u1 := 2048\*b<sup>9</sup>\*tanh(b\*x)\*sech(b\*x)<sup>2</sup>\*(1-a+a\*t<sup>a</sup>)/(3\*GAMMA(a+1));

$$\frac{2048 b^9 \tanh(bx) \operatorname{sech}(bx)^2 (1 - a + at^a)}{3 \Gamma(a + 1)}$$

```
u2 := 524288*b16*(cosh(2*b*x)-2)*sech(b*x)4*((1-a)2+2*a(1-a)*ta/GAMMA(a+1)
+a2*t(2*a)/GAMMA(2*a+1))*(1/9);
```

$$\frac{524288}{9} b^{16} (\cosh(2bx) - 2) \operatorname{sech}(bx)^4 \left( (1-a)^2 + \frac{2a(1-a)t^a}{\Gamma(a+1)} + \frac{a^2 t^{2a}}{\Gamma(2a+1)} \right)$$

```
u3 := 268435456*b23*(cosh(2*b*x)-5)*tanh(b*x)*sech(b*x)4*((1-a)3
+3*a(1-a)2*ta/GAMMA(a+1)+3*a(1-a)2*t(2*a)/GAMMA(2*a+1)+a3*t(3*a)/GAMMA(3*a+1))*(1/27);
```

$$\frac{268435456}{27} b^{23} (\cosh(2bx) - 5) \tanh(bx) \operatorname{sech}(bx)^4 \left( (1-a)^3 + \frac{3a(1-a)^2 t^a}{\Gamma(a+1)} + \frac{3a(1-a)^2 t^{2a}}{\Gamma(2a+1)} + \frac{a^3 t^{3a}}{\Gamma(3a+1)} \right)$$

```
vabc := u0+u1+u2;
```

$$\frac{4}{3} b^2 (2 - 3 \tanh(bx)^2) + \frac{2048}{3} \frac{b^9 \tanh(bx) \operatorname{sech}(bx)^2 (1-a+t^a)}{\Gamma(a+1)} + \frac{524288}{9} b^{16} (\cosh(2bx) - 2) \operatorname{sech}(bx)^4 \left( (1-a)^2 + \frac{2a(1-a)t^a}{\Gamma(a+1)} + \frac{a^2 t^{2a}}{\Gamma(2a+1)} \right)$$

```
vabc1 := eval(vabc, [b = 0.1, a = 1]);
```

$$0.02666666667 - 0.04 \tanh(0.1x)^2 + 6.82666666710^{-7} \tanh(0.1x) \operatorname{sech}(0.1x)^2 t + 5.82542222210^{-12} (\cosh(0.2x) - 2) \operatorname{sech}(0.1x)^4 \left( 2t + \frac{1}{2} t^2 \right)$$

```
plot3d(vabc1, t = 0..10.5, x = -10..10);
```

See figure – (5.1b)

```
# Exact Solution of Swada-Kotera-Ito equation
```

```
u(x,t):=(4*b2)/(3)(2-3 tanh(2)(b*((256 b6t)/(3)+x)));
```

$$\frac{4}{3} b^2 \left( 2 - 3 \tanh \left( b \left( \frac{256}{3} b^6 t + x \right) \right) \right)^2$$

```
v := eval(u(x, t), [b = 0.1]);
```

$$0.02666666667 - 0.04 \tanh(0.00000853333333333t + 0.1x)^2$$

```
plot3d(v, t = 0..10, x = -30..30);
```

See figure – (5.1c)

**5.6.2 A maple code for the comparison of the solution of Swada-Kotera-Ito equation by the Caputo sense, the ABC sense, and the exact solution at the values of  $\alpha = 1, b = 0.1$  and  $t = 4$**

```
v := eval(u(x, t), [b = 0.1, t = 4]);
```

$$0.02666666667 - 0.04 \tanh(0.00003413333333 + 0.1x)^2$$

```
vc1 := eval(vc, [b = 0.1, a = 1, t = 4]);
```

$$\begin{aligned} &0.027744 - 0.041616 \tanh(0.102x)^2 + 0.000003263399442 \tanh(0.102x) \operatorname{sech}(0.102x)^2 \\ &+ 6.39764508210^{-11} (\cosh(0.204x) - 2) \operatorname{sech}(0.102x)^4 \\ &+ 1.67227920210^{-15} (\cosh(0.204x) - 5) \tanh(0.102x) \operatorname{sech}(0.102x)^4 \end{aligned}$$

```
vabc1 := eval(vabc, [b = 0.1, a = 1, t = 4]);
```

$$\begin{aligned} &0.02682690667 - 0.04024036 \tanh(0.1003x)^2 + 0.00000280528 \tanh(0.1003x) \operatorname{sech}(0.1003x)^2 \\ &+ 9.77827662110^{-11} (\cosh(0.2006x) - 2) \operatorname{sech}(0.1003x)^4 \\ &+ 4.97054894510^{-15} (\cosh(0.2006x) - 5) \tanh(0.1003x) \operatorname{sech}(0.1003x)^4 \end{aligned}$$

```
plot([v, vc1, vabc1], x = 0..10, color = ["red", "green", "blue"]);
```

See figure – (5.2)

### 5.6.3 A maple code for the behavior of Swada-Kotera-Ito equation by the ADShTM with the different fractional order $\alpha$ and $b = 0.1$ in the ABC sense

```
vabc1 := eval(vabc, [b = 0.1, a = 0.4]);
```

$$0.02666666667 - 0.04 \tanh(0.1x)^2 + 7.69406633310^{-7} \tanh(0.1x) \operatorname{sech}(0.1x)^2 (0.6 + 0.4t^{0.4}) + 5.82542222210^{-12} (\cosh(0.2x) - 2) \operatorname{sech}(0.1x)^4 (0.36 + 0.9016483984t^{0.4} + 0.1717874038t^{0.8})$$

```
plot3d(vabc1, t = 0..10.5, x = -10..10);
```

See figure – (5.3a)

```
vabc2 := eval(vabc, [b = 0.1, a = 0.6]);
```

$$0.02666666667 - 0.04 \tanh(0.1x)^2 + 7.64023435310^{-7} \tanh(0.1x) \operatorname{sech}(0.1x)^2 (0.4 + 0.6t^{0.6}) + 5.82542222210^{-12} (\cosh(0.2x) - 2) \operatorname{sech}(0.1x)^4 (0.16 + 1.343009945t^{0.6} + 0.3267373263t^{1.2})$$

```
plot3d(vabc2, t = 0..10.5, x = -10..10);
```

See figure – (5.3b)

```
vabc3 := eval(vabc, [b = 0.1, a = 0.8]);
```

$$0.02666666667 - 0.04 \tanh(0.1x)^2 + 7.32959589710^{-7} \tanh(0.1x) \operatorname{sech}(0.1x)^2 (0.2 + 0.8t^{0.8}) + 5.82542222210^{-12} (\cosh(0.2x) - 2) \operatorname{sech}(0.1x)^4 (0.04 + 1.717874038t^{0.8} + 0.4476699816t^{1.6})$$

```
plot3d(vabc3, t = 0..10.5, x = -10..10);
```

See figure – (5.3c)