# Chapter 4

# Homotopy perturbation General transform method

## 4.1 Introduction

Fractional calculus is the branch of mathematics that deals with arbitrary ordered differentiation and integration. The fractional differential equations (FDEs) are used to obtain the memory properties of many scientific and engineering problems. Thus, numerous researchers are working on developing new techniques to obtain the analytic and numerical solution of fractional order differential equations [120]. Integral transforms are used to obtain analytic solutions for certain types of differential equations, partial differential equations, FDEs, integral equations, integro-differential equations as it can be expressed easily by its inversion formulas. Several new integral transforms in the class of Laplace transform [39] have been established in the previous two decades, including Aboodh [1], Elzaki [48], G-transform [93], Kamal [79], Mohand [108], Pourreza [6], Sawi [101], and Sumudu [73] transforms. The list of some transforms and their definitions are mentioned by Jafari [70]. Other analytical and semi-analytical techniques such as Adomian decomposition method (ADM) [153], homotopy perturbation method (HPM) [5], Homotopy analysis method (HAM) [18], residual power series method (RPSM) [91], are used to obtain the solution of FDE's and FPDE's. Also, to increase the efficacy, different kernels were also considered to solve FDEs [7, 8, 9].

Furthermore, the merger of many analytical methods with these integral transforms have been utilized to numerous problems to minimize calculations and obtain rapid convergence. Laplace residual power series method [13], Laplace Adomian decomposition method [138], Aboodh transform iterative method [25], Natural decomposition method [156], Elzaki transform homotopy perturbation method [168], Sumudu decomposition method [73], Homotopy analysis Shehu transform method [104], modified Laplace Variational iteration method [112], Homotopy perturbation Laplace transform method [77], q - Homotopy analysis transform method [157], Yang transform Homotopy perturbation method [5], Antagana-Baleanu fractional derivative operator and Laplace transform [125], two dimensional fuzzy fractional wave equation using natural transform [20], etc.

The motive of the present work is to establish a new hybrid technique for obtaining a semi-analytic solution of fractional differential equations namely Homotopy Perturbation General Transform Method (HPGTM) as an amalgamation of He's [62, 64] well-known Homotopy Perturbation Method (HPM) and General Integral Transform (GT) [70]. The intention of considering GT over other integral transforms is to overcome some limitations of exponential ordered integral transforms by substitution of $\rho(s)$ and $q(s)$ in (4.1) as discussed by Jafari [70]. Thus the proposed technique is an efficient tool for solving linear and non-linear stiff fractional differential equations (FDEs). The HPGTM is more reliable and provides rapid convergence compared to other conventional method of its class, including approximate, analytic, and semi-analytic methods.

## 4.2 Prerequisites

**Definition 4.1.** General transform [70] of function $u(t)$ is defined as

$$G(s) = \rho(s) \int_0^\infty u(t) e^{-q(s)t} dt; \quad k_1 \leq s \leq k_2, \tag{4.1}$$

where $t \geq 0$, $\rho(s) \neq 0$, and $q(s)$ are positive real functions. The general transform of some basic functions are listed in Table 4.1.
Where the General transform of functions $u(t)$ which are not continuously differentiable contains terms with negative or fractional powers of $q(s)$. The relation between other well-known transforms with GT were explained by Jafari [70].

Table 4.1: General transform $G(s)$ of some basic function $u(t)$.

| Function $u(t) = G^{-1}\{G(s)\}$ | General transform $G(s) = G\{u(t)\}$ |
|---|---|
| $G\{1\}$ | $\frac{\rho(s)}{q(s)}$ |
| $G\{t\}$ | $\frac{\rho(s)}{q^2(s)}$ |
| $G\{t^n\}$ | $\frac{\Gamma(n+1)\rho(s)}{q(s)^{n+1}}, n > 0$ |
| $G\{\sin(at)\}$ | $\frac{a\rho(s)}{q^2(s)+a^2}$, if $q(s) > \lvert\exists(a)\rvert$ |
| $G\{\cos(at)\}$ | $\frac{\rho(s)q(s)}{q^2(s)+a^2}$ |
| $G\{e^{at}\}$ | $\frac{\rho(s)}{q(s)-a}$ |
| $G\{tH(t-a)\}$ | $\frac{e^{-aq(s)}(q(s)+a)\rho(s)}{q(s)^2}$ |

**Theorem 4.1. (Existence theorem for GT)** If $u(t)$ is piecewise continuous function of exponential order $k \geq 0$, and satisfies $|u(t)| \leq Me^{kt}$, then its general transform $G(s)$ exists for all $q(s) > k$.

**Proof.** we have

$$\begin{aligned} |G(s)| &= \left|\rho(s)\int_0^\infty u(t)e^{-q(s)t}dt\right|, \\ &\leq \rho(s)\int_0^\infty |u(t)|e^{-q(s)t}dt, \\ &\leq \rho(s)\int_0^\infty Me^{kt}e^{-q(s)t}dt, \\ &\leq \frac{\rho(s)M}{q(s)-k}, \quad \forall q(s) > k. \end{aligned} \tag{4.2}$$

Thus, the proof is complete.

**Theorem 4.2.** General transform of integer order derivatives is given by Jafari [70] as follows

$$\begin{aligned} G\{u'(t)\} &= q(s)G(s) - \rho(s)u(0), \\ G\{u''(t)\} &= q(s)^2G(s) - \rho(s)q(s)u(0) - \rho(s)u'(0), \end{aligned} \tag{4.3}$$

in general for $n^{\text{th}}$ order

$$G\{u^{(n)}(t)\} = q(s)^nG(s) - \rho(s)\sum_{k=0}^{n-1}(q(s))^{n-1-k}u^{(k)}(0). \tag{4.4}$$

where function $f(t)$ is differentiable and $\rho(s)$ and $q(s)$ are positive real functions.

**Theorem 4.3.(Convolution theorem [70, 106])** If $G\{u_1(t)\} = G_1(s)$ and $G\{u_2(t)\} = G_2(s)$ then

$$u_1(t) * u_2(t) = \int_0^t u_1(\tau)u_2(t-\tau)d\tau = \frac{1}{\rho(s)}[G_1(s) \cdot G_2(s)] \tag{4.5}$$

**Theorem 4.4.** For any $0 < \alpha \leq 1$, and fixed positive integer $n$, GT of Caputo fractional derivative of continuous function $u(x,t)$ is given by

$$G\{D_t^{\alpha}u(x,t)\} = q(s)^{\alpha}G\{u(x,t)\} - \rho(s)q(s)^{\alpha-1}u(x,0),$$
$$G\{D_t^{2\alpha}u(x,t)\} = q(s)^{2\alpha}G\{u(x,t)\} - \rho(s)q(s)^{2\alpha-1}u(x,0) - \rho(s)q(s)^{2\alpha-2}D_t u(x,0) \tag{4.6}$$

In general, it can be defined as

$$G\{D_t^{n\alpha}u(x,t)\} = q(s)^{n\alpha}G\{u(x,t)\} - \rho(s)\sum_{k=0}^{n-1} q(s)^{n\alpha-1-k}D_t^k u(x,0). \tag{4.7}$$

where $D_t \equiv \frac{\partial}{\partial t}$, $D_t^k u(x,0)$ is the value of $D_t^k u(x,t)$ at $t = 0$ for $k = 1, 2, \ldots (n-1)$.

**Proof.** Let us write Caputo fractional derivative (1.2) in the form

$$D_t^{\alpha}u(x,t) = D_t^{-(n-\beta)}h(x,t), \text{ where } h(x,t) = u^n(x,t), n-1 < \beta \leq n, \tag{4.8}$$

now the General transform of Riemann-Liouville fractional integral is

$$G\{D_t^{\alpha}u(x,t)\} = q(s)^{-(n-\beta)}G\{u(x,t)\}, \tag{4.9}$$

from the above both equations (4.8) and (4.9), we get

$$G\left\{D_t^{\beta}u(x,t)\right\} = G\left\{D_t^{-(n-\beta)}h(x,t)\right\} = q(s)^{-(n-\beta)}G\{h(x,t)\}, \tag{4.10}$$

Using the General transform of integer order derivative as defined in (4.4),

$$G\{h(x,t)\} = G\{u^n(x,t)\} = q(s)^n G\{u(x,t)\} - \rho(s)\sum_{k=0}^{n-1}(q(s))^{n-1-k}D_t^k u(x,0) \tag{4.11}$$

Substituting equation (4.11) into (4.10), we achieve

$$G\left\{D_t^{\beta}u(x,t)\right\} = q(s)^{-(n-\beta)}\left[q(s)^n G\{u(x,t)\} - \rho(s)\sum_{k=0}^{n-1} q(s)^{n-1-k} D_t^k u(x,0)\right], \quad (4.12)$$

$$= q(s)^{\beta} G\{u(x,t)\} - \rho(s)\sum_{k=0}^{n-1} q(s)^{\beta-1-k} D_t^k u(x,0). \quad (4.13)$$

For any fractional-order $0 < \alpha \leq 1$ taking $\beta = n\alpha$, we have (4.7).

## 4.3 Homotopy Perturbation General Transform Method

Researchers should be prepared to deal with functional differential equations while modeling dynamical systems which may be highly non-linear and stochastic in parameters, or with specified conditions. Some of the well-known non-linear differential equations occurring in nature are Riccati's equation (arise in the study of the nature of the universe), KdV equation (solitary waves on shallow water), backward Kolmogorov equation (describe the time evolution of the velocity of a particle in Brownian motion), Klein-Gordon equation (arise in special relativity particularly in Quantum field theory), etc. The general functional equation representing wide class of physical phenomenon is given by

$$Fu(x,t) = f(x,t)$$

where $F$ represents a general non-linear differential operator involving linear and non-linear terms. Considering a general time-fractional non-linear differential equation

$$D_t^{n\alpha}u(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), \text{ for } 0 < \alpha \leq 1,\ x \in R,\ t \leq 0, \quad (4.14)$$

subject to initial conditions

$$u(x,0) = \phi_0(x), \frac{\partial u(x,0)}{\partial t} = \phi_1(x), ..., \frac{\partial^{n-1}u(x,0)}{\partial t^{n-1}} = \phi_{n-1}(x). \quad (4.15)$$

where $D_t \left(\equiv \frac{\partial}{\partial t}\right)$, $R$ and $N$ represents the differential operators, having linear and non-linear terms of the continuous function $u(x,t)$ respectively and $f(x,t)$ is a continuous function of $x$ and $t$.

Operating General transform to (4.14) yields

$$G\{D_t^{n\alpha}u(x,t)\} = -G\{Ru(x,t)\} - G\{Nu(x,t)\} + G\{f(x,t)\}. \tag{4.16}$$

Using differentiation properties (4.7) of General transform, (4.16) can be written as

$$q(s)^{n\alpha}G\{u(x,t)\} - \rho(s)\sum_{k=0}^{n-1} q(s)^{n\alpha-1-k}D_t^k u(x,0) = -G\{Ru(x,t)\} - G\{Nu(x,t)\} + G\{f(x,t)\}, \tag{4.17}$$

or

$$\begin{aligned} G\{u(x,t)\} &= \frac{\rho(s)}{q(s)^{n\alpha}}\sum_{k=0}^{n-1} q(s)^{n\alpha-1-k}D_t^k u(x,0) - \frac{1}{q(s)^{n\alpha}}\left[G\left\{Ru(x,t)\right\} + G\left\{Nu(x,t)\right\}\right. \\ &\quad \left. - G\left\{f(x,t)\right\}\right], \\ &= \rho(s)\left[\frac{1}{q(s)}\phi_0(x) + \frac{1}{q(s)^2}\phi_1(x) + \cdots + \frac{1}{q(s)^n}\phi_{n-1}(x)\right] - \frac{1}{q(s)^{n\alpha}}G\left\{Ru(x,t)\right\} \\ &\quad - \frac{1}{q(s)^{n\alpha}}G\left\{Nu(x,t)\right\} + \frac{1}{q(s)^{n\alpha}}G\left\{f(x,t)\right\}. \end{aligned} \tag{4.18}$$

Taking Inverse General transform (IGT),

$$u(x,t) = \phi(x,t) - G^{-1}\left[\frac{1}{q(s)^{n\alpha}}G\{Ru(x,t)\} + \frac{1}{q(s)^{n\alpha}}G\{Nu(x,t)\}\right], \tag{4.19}$$

where $\phi(x,t)$ is IGT of first and last term of equation (4.18).

Now, applying HPM [62, 64] to equation (4.19) gives

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = \phi(x,t) - pG^{-1}\left[\frac{1}{q(s)^{n\alpha}}G\left\{R\sum_{i=0}^{\infty} p^i u_i(x,t)\right\} + \frac{1}{q(s)^{n\alpha}}G\left\{N\sum_{i=0}^{\infty} p^i u_i(x,t)\right\}\right]. \tag{4.20}$$

For any $x \in R$ and $t < 1$. In equation (4.20), nonlinear terms are decomposed using He's Polynomial [64], which is

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{4.21}$$

where $H_n(u_0, u_1, u_2, ..., u_n) = \frac{1}{n!}\left[\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \quad n = 0, 1, 2, ...$ .

Substituting equation (4.21) into (4.20),

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = \phi(x,t) - p\left[G^{-1}\left[\frac{1}{q(s)^{n\alpha}} G\left\{R\sum_{i=0}^{\infty} p^i u_i(x,t)\right\} + \frac{1}{q(s)^{n\alpha}} G\left\{\sum_{i=0}^{\infty} p^i H_i(u)\right\}\right]\right]. \tag{4.22}$$

Comparing both sides of the '$p$' term's of (4.22),

$$\begin{aligned}
&p^0 : u_0(x,t) = \phi(x,t),\\
&p^1 : u_1(x,t) = -G^{-1}\left[\tfrac{1}{q(s)^{n\alpha}} G\{Ru_0(x,t)\} + \tfrac{1}{q(s)^{n\alpha}} G\{H_0(u)\}\right],\\
&p^2 : u_2(x,t) = -G^{-1}\left[\tfrac{1}{q(s)^{n\alpha}} G\{Ru_1(x,t)\} + \tfrac{1}{q(s)^{n\alpha}} G\{H_1(u)\}\right],\\
&\vdots\\
&p^n : u_n(x,t) = -G^{-1}\left[\tfrac{1}{q(s)^{n\alpha}} G\{Ru_{n-1}(x,t)\} + \tfrac{1}{q(s)^{n\alpha}} G\{H_{n-1}(u)\}\right].
\end{aligned} \tag{4.23}$$

Eventually, the solution of (4.14) obtained as

$$u(x,t) = \lim_{p\to 1} u_n(x,t) = u_0(x,t) + p^1 u_1(x,t) + p^2 u_2(x,t) + ..., \tag{4.24}$$

or

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + .... \tag{4.25}$$

## 4.4 Convergence of HPGTM

**Theorem 4.5.** Let $B \equiv C([a,b] \times [0,T])$ be the Banach space of all continuous real-valued functions defined on rectangular region $[a,b] \times [0,T]$ and for $u \in B$, define $||u|| = \sup_{x\in[a,b],\, t\in[0,\,T]} |u(x,t)|$, then $\sum_{k=0}^{\infty} u_k(x,t)$ converges, if $u_0 \in B$ is bounded then $\forall u_k \in B$, $||u_{k+1}|| \leq \sigma||u_k||$, where $0 < \sigma < 1$.

**Proof.** Let $\{A_q\}$ be the sequence as partial sum of equation (4.25) as

$$\begin{aligned} A_0 &= u_0(x,t), \\ A_1 &= u_0(x,t) + u_1(x,t), \\ A_2 &= u_0(x,t) + u_1(x,t) + u_2(x,t), \\ &\vdots \\ A_q &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ... + u_q(x,t). \end{aligned} \tag{4.26}$$

To get the desired result, we will prove that $\{A_q\}_{q=0}^{\infty}$ forms a Cauchy sequence [21] in Banach space $B$. Further, let us take

$$\begin{aligned} \|A_{q+1} - A_q\| &= \|u_{q+1}(x,t)\|, \\ &\leq \sigma \|u_q(x,t)\|, \\ &\leq \sigma^2 \|u_{q-1}(x,t)\|, \\ &\leq \sigma^3 \|u_{q-2}(x,t)\|, \\ &\vdots \\ &\leq \sigma^{q+1} \|u_0(x,t)\|. \end{aligned} \tag{4.27}$$

For any $q, n \in N$, with $q \geq n$, we acquire

$$\begin{aligned} \|A_q - A_n\| &= \|(A_q - A_{q-1}) + (A_{q-1} - A_{q-2}) + (A_{q-2} - A_{q-3}) + ... + (A_{n+1} - A_n)\|, \\ &\leq \|A_q - A_{q-1}\| + \|A_{q-1} - A_{q-2}\| + \|A_{q-2} - A_{q-3}\| + ... + \|A_{n+1} - A_n\|, \\ &\leq \sigma^q \|u_0(x,t)\| + \sigma^{q-1} \|u_0(x,t)\| + \sigma^{q-2} \|u_0(x,t)\| + ... + \sigma^{n+1} \|u_0(x,t)\|, \\ &\leq \left(\sigma^q + \sigma^{q-1} + \sigma^{q-2} + ... + \sigma^{n+1}\right) \|u_0(x,t)\|, \\ &\leq \frac{(1-\sigma^{q-n})}{(1-\sigma)} \sigma^{n+1} \|u_0(x,t)\|, \\ &\leq \beta \|u_0(x,t)\|. \end{aligned} \tag{4.28}$$

Where $\beta = \frac{(1-\sigma^{q-n})}{(1-\sigma)} \sigma^{n+1}$. Since $u_0(x,t)$ is bounded, then $\|u_0(x,t)\| < \infty$. Also, for any finite value of $n$ and considering $q \to \infty$ then $\beta \to 0$, that means

$$\lim_{q\to\infty} \|A_q - A_n\| = 0. \tag{4.29}$$

Hence, $\{A_q\}_{q=0}^{\infty}$ is Cauchy sequence in $B$. It follows that the series solution of equation (4.14) as equation (4.25) is convergent.

**Theorem 4.6.** If $\sum\limits_{k=0}^{n} u_k(x,t)$ is obtained as approximate solution of equation (4.14), then the maximum absolute error is estimated as

$$\left\| u(x,t) - \sum_{k=0}^{n} u_k(x,t) \right\| \leq \frac{\sigma^{n+1}}{1-\sigma} \|u_0(x,t)\|. \tag{4.30}$$

**Proof.** From equation (4.28) of Theorem – 4.5, we have

$$\|A_q - A_n\| \leq \beta \|u_0(x,t)\|, \text{ where } \beta = \frac{(1-\sigma^{\mathrm{q-n}})}{(1-\sigma)}\sigma^{\mathrm{n+1}}. \tag{4.31}$$

Here, $\{A_q\}_{q=0}^{\infty} \to u(x,t)$ as $q \to \infty$ and from equation (4.26), we can get $A_n = \sum\limits_{k=0}^{n} u_k(x,t)$,

$$\left\| u(x,t) - \sum_{k=0}^{n} u_k(x,t) \right\| \leq \beta \|u_0(x,t)\|, \tag{4.32}$$

now $(1-\sigma^{q-n}) < 1$ since $0 < \sigma < 1$, then

$$\left\| u(x,t) - \sum_{k=0}^{n} u_k(x,t) \right\| \leq \frac{\sigma^{n+1}}{1-\sigma} \|u_0(x,t)\|. \tag{4.33}$$

Thus, Theorem is proved.

**Theorem 4.7.** If the series solution $\sum\limits_{n=0}^{\infty} u_n$ given in (4.25) is convergent, then it is an exact solution of the general form of FDE (4.14), by considering $n-1 < \alpha \leq n$.

**Proof.** Let consider the sequence $A_n = u_0 + u_1 + u_2 + ... + u_n$.

By using the iterative scheme

$$\begin{aligned}
&A_0 = u_{0,}\\
&A_1 = u_0 + u_{1,}\\
&\vdots\\
&A_n = u(x,0) - G^{-1}\left[\tfrac{1}{q(s)^\alpha} G\left\{u_1 + u_2 + ... + u_n\right\}\right] + G^{-1}\left[\tfrac{1}{q(s)^\alpha} G\left\{f(x,t)\right\}\right].
\end{aligned} \tag{4.34}$$

Suppose that the series solution (4.25) converges, and $u = \sum\limits_{n=0}^{\infty} u_n$, then we have

$$\begin{aligned}
u(x,t) = \lim_{n\to\infty} A_n = u(x,0) - G^{-1}\left[\frac{1}{q(s)^\alpha} G\left\{R \lim_{n\to\infty} u_n + \lim_{n\to\infty} H_n(u)\right\}\right]\\
+ G^{-1}\left[\frac{1}{q(s)^\alpha} G\left\{f(x,t)\right\}\right],
\end{aligned} \tag{4.35}$$

where $H_n(u_0, u_1, u_2, ..., u_n) = \frac{1}{n!} N\left(\sum\limits_{i=0}^{\infty} u_i\right), n = 0, 1, 2, ...$ then by HPM we obtain

$$\begin{aligned}
u(x,t) = \sum_{n=0}^{\infty} p^n u_n = u(x,0) - pG^{-1}\left[\frac{1}{q(s)^\alpha} G\left\{R\left(\sum_{n=0}^{\infty} p^n u_n\right) + N\left(\sum_{n=0}^{\infty} p^n u_n\right)\right\}\right]\\
+ G^{-1}\left[\frac{1}{q(s)^\alpha} G\left\{f(x,t)\right\}\right],
\end{aligned} \tag{4.36}$$

for homotopy parameter $p = 1$, it leads to

$$\begin{aligned}
u(x,t) = \sum_{n=0}^{\infty} u_n = u(x,0) - G^{-1}\left[\frac{1}{q(s)^\alpha} G\left\{R\left(\sum_{n=0}^{\infty} u_n\right) + N\left(\sum_{n=0}^{\infty} u_n\right)\right\}\right]\\
+ G^{-1}\left[\frac{1}{q(s)^\alpha} G\left\{f(x,t)\right\}\right],
\end{aligned} \tag{4.37}$$

or

$$u(x,t) = u(x,0) - G^{-1}\left[\frac{1}{q(s)^\alpha} G\left\{R(u(x,t)) + N(u(x,t))\right\}\right] + G^{-1}\left[\frac{1}{q(s)^\alpha} G\left\{f(x,t)\right\}\right]. \tag{4.38}$$

Applying GT,

$$q(s)^{\alpha}G\{u(x,t)\} - \rho(s)q(s)^{\alpha}u(x,0) = -G\{Ru(x,t)\} - G\{Nu(x,t)\} + G\{f(x,t)\}, \tag{4.39}$$

$$G\{D_t^{\alpha}u(x,t)\} = -G\{Ru(x,t)\} - G\{Nu(x,t)\} + G\{f(x,t)\}, \tag{4.40}$$

Hence, IGT of (4.40) gives (4.14), which proves the desired result.

## 4.5 Applications

### 4.5.1 Radioactive decay model

Time-fractional radioactive decay model [68] is given by

$$D_t^{\alpha}u(t) = -\lambda u(t), \text{ where } 0 < \alpha \leq 1,\ t > 0,\ \lambda \in R, \tag{4.41}$$

with an initial condition

$$u(0) = u_0. \tag{4.42}$$

operating General transform to equation (4.41),

$$G\{D_t^{\alpha}u(t)\} = -\lambda G\{u(t)\}, \tag{4.43}$$

using derivative properties (4.6), we get

$$q(s)^{\alpha}G\{u(t)\} - \rho(s)q(s)^{\alpha-1}u(0) = -\lambda G\{u(t)\}, \tag{4.44}$$

substituting equation (4.42),

$$G\{u(t)\} = \frac{\rho(s)}{q(s)}u_0 - \lambda\frac{1}{q(s)^{\alpha}}G\{u(t)\}, \tag{4.45}$$

Taking inverse General transform

$$u(t) = G^{-1}\left[\frac{\rho(s)}{q(s)}u_0\right] - G^{-1}\left[\lambda\frac{1}{q(s)^{\alpha}}G\left\{u(t)\right\}\right], \tag{4.46}$$

$$= u_0 - \lambda G^{-1}\left[\frac{1}{q(s)^{\alpha}}G\left\{u(t)\right\}\right]. \tag{4.47}$$

By applying homotopy perturbation method on (4.47), yields

$$\sum_{i=0}^{\infty} p^i u_i(t) = u_0 - p\lambda G^{-1}\left[\frac{1}{q(s)^{\alpha}}G\left\{\sum_{i=0}^{\infty} p^i u_i(t)\right\}\right]. \tag{4.48}$$

Now comparing the power of 'p' in terms of (4.48),

$$\begin{aligned}
&p^0 : u_0(t) = u_0,\\
&p^1 : u_1(t) = -\lambda G^{-1}\left[\frac{1}{q(s)^{\alpha}}G\left\{u_0\right\}\right] = -\lambda u_0\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
&p^2 : u_2(t) = -\lambda G^{-1}\left[\frac{1}{q(s)^{\alpha}}G\left\{u_1\right\}\right] = \lambda u_0\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
&p^3 : u_3(t) = -\lambda G^{-1}\left[\frac{1}{q(s)^{\alpha}}G\left\{u_2\right\}\right] = \lambda u_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&\qquad\vdots
\end{aligned} \tag{4.49}$$

Thus, the exact solution of (4.41) can be obtained from (4.25) as

$$u(t) = u_0\left[1 - \lambda\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \lambda\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \lambda\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right],$$

$$= u_0 E_{\alpha}(-\lambda t^{\alpha}). \tag{4.50}$$

Where $E_{\alpha}(-\lambda t^{\alpha})$ is Mittag-Leffler function [120] and hence the obtained solution in convergent for $t < 1$ in light of Theorem – 4.5.

### 4.5.2 Riccati equation

Consider a non-linear time-fractional Riccati equation [69, 133] (arise in the study of the nature of the universe) as

$$D_t^{\alpha} u(t) = -u^2(t) + 1, \text{ where } 0 < \alpha \leq 1, \ t > 0, \tag{4.51}$$

with the initial condition

$$u(0) = 0. \tag{4.52}$$

And the exact solution [69] when $\alpha = 1$ is

$$u(t) = \frac{e^{2t} - 1}{e^{2t} + 1}. \tag{4.53}$$

Operating GT on (4.51) gives

$$G\left\{D_t^{\alpha} u(t)\right\} = -G\left\{u^2(t)\right\} + G\left\{1\right\}, \tag{4.54}$$

using derivative properties (4.6), it leads to

$$q(s)^{\alpha} G\left\{u(t)\right\} - \rho(s) q(s)^{\alpha-1} u(0) = \frac{\rho(s)}{q(s)} - G\left\{u^2(t)\right\}, \tag{4.55}$$

applying initial condition (4.52), we have

$$G\left\{u(t)\right\} = \frac{\rho(s)}{q(s)} u(0) + \frac{\rho(s)}{q(s)^{\alpha+1}} - \frac{1}{q(s)^{\alpha}} G\left\{u(t)^2\right\}, \tag{4.56}$$

using inverse General transform, we have

$$\begin{aligned} u(t) &= G^{-1}\left[\frac{\rho(s)}{q(s)^{\alpha+1}}\right] - G^{-1}\left[\frac{1}{q(s)^{\alpha}} G\left\{u(t)^2\right\}\right], \\ &= \frac{t^{\alpha}}{\Gamma(\alpha+1)} - G^{-1}\left[\frac{1}{q(s)^{\alpha}} G\left\{u(t)^2\right\}\right]. \end{aligned} \tag{4.57}$$

Now applying homotopy perturbation method, we get

$$\sum_{i=0}^{\infty} p^i u_i(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} - pG^{-1}\left[\frac{1}{q(s)^\alpha} G\left\{R\left(\sum_{i=0}^{\infty} p^i u_i\right) + \sum_{i=0}^{\infty} p^i H_i(u)\right\}\right], \tag{4.58}$$

where non-linear term of equation (4.58) is decomposed using He's Polynomial $H_i(u)$ as follows

$$\begin{aligned}
H_0(u) &= u_0^2,\\
H_1(u) &= 2u_0u_1,\\
H_2(u) &= 2u_0u_2 + u_1^2,\\
H_3(u) &= 2u_0u_3 + 2u_1u_2,\\
&\vdots
\end{aligned} \tag{4.59}$$

By comparing the coefficient of the power of "p" terms

$$\begin{aligned}
p^0 : u_0(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)},\\
p^1 : u_1(t) &= -G^{-1}\left[\frac{1}{q(s)^\alpha} G\left\{H_0(u)\right\}\right],\\
&= -\frac{\Gamma(2\alpha+1)}{(\Gamma(1+\alpha))^2}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)},\\
p^2 : u_2(t) &= -G^{-1}\left[\frac{1}{q(s)^\alpha} G\{H_1(u)\}\right],\\
&= \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma(1+\alpha))^3\Gamma(3\alpha+1)}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)},\\
p^3 : u_3(t) &= -G^{-1}\left[\frac{1}{q(s)^\alpha} G\{H_2(u)\}\right],\\
&= \frac{-2\Gamma(2\alpha+1)\Gamma(6\alpha+1)}{(\Gamma(1+\alpha))^4\Gamma(3\alpha+1)}\left(\frac{4\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)\frac{t^{7\alpha}}{\Gamma(7\alpha+1)},\\
&\vdots
\end{aligned} \tag{4.60}$$

Thus the solution of (4.51) can be obtained using (4.25) as

$$u(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} - \frac{\Gamma(2\alpha+1)\Gamma(6\alpha+1)}{(\Gamma(\alpha+1))^4\Gamma(3\alpha+1)}\left(\frac{4\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)\frac{t^{7\alpha}}{\Gamma(7\alpha+1)} + \dots \tag{4.61}$$

Here, equation (4.61) is the approximate semi-analytical solution of the time-fractional Riccati equation using HPGTM. For any positive $\epsilon$ and $t < 1$, we have $|p^m| < \epsilon$ for $m = 1, 2, \dots$. Further for $0 < \alpha \leq 1$, (4.61) is decreasing series.

Thus, the obtained solution is convergent from Theorem – 4.5 and Theorem – 4.6. The absolute error in the solution of Riccati equation by HPGTM (up to $4^{th}$ term), RPSM [17] (up to $4^{th}$ term) and ADM [110] (up to $4^{th}$ term) for $\alpha = 1$ is shown in Table 4.2.

Table 4.2: The Absolute error in the solution of Riccati equation by HPGTM, RPSM [17], and ADM [110] for $\alpha = 1$.

| $t$ | Exact Solution (ES) | Absolute error (for HPGTM) \|ES-HPGTM\| (first 4 terms of HPGTM) | Absolute error (for RPSM) \|ES-RPSM\| (first 4 terms of RPSM) | Absolute error (for ADM) \|ES-ADM\| (first 4 terms of ADM) |
|---|---|---|---|---|
| 0.1 | 0.09966799456 | 4.00000E-11 | 6.43563E-04 | 2.34600E-10 |
| 0.2 | 0.19737532030 | 1.11000E-08 | 1.54744E-03 | 8.34500E-08 |
| 0.3 | 0.29131261240 | 4.15300E-07 | 4.36363E-03 | 4.34567E-06 |
| 0.4 | 0.37994896220 | 5.38380E-06 | 7.23523E-03 | 1.56745E-05 |
| 0.5 | 0.46211715720 | 3.87842E-05 | 1.03463E-02 | 7.12537E-04 |

Table 4.3: The approximate solution $u(t)$ by HPGTM for different fractional orders $\alpha$.

| t | $\alpha = 0.2$ | $\alpha = 0.4$ | $\alpha = 0.6$ | $\alpha = 0.8$ | $\alpha = 1$ |
|---|---|---|---|---|---|
| 0.1 | 0.39748255730 | 0.39315488330 | 0.26892225050 | 0.16800233020 | 0.09966799460 |
| 0.2 | 0.20447907300 | 0.47018399580 | 0.38669861860 | 0.28523920740 | 0.19737530920 |
| 0.3 | 0.08284363800 | 0.48857715310 | 0.46679428780 | 0.38183598960 | 0.29131219710 |
| 0.4 | 0.43845154800 | 0.45375737300 | 0.52237448420 | 0.46270091400 | 0.37994357840 |
| 0.5 | 0.85003066400 | 0.35851209870 | 0.55547999640 | 0.53011939030 | 0.46207837300 |

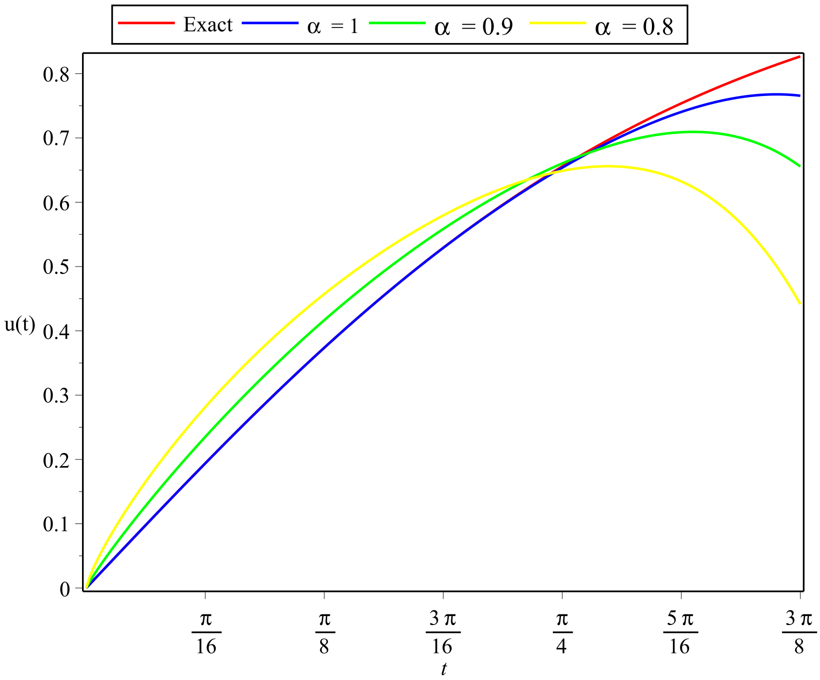



Figure 4.1: Solution of time-fractional Riccati equation for different values of $\alpha$ and exact solution.

### 4.5.3 Backward Kolmogorov equation

Linear time-fractional partial differential equation, backward Kolmogorov equation [153] (describe the time evolution of the velocity of a particle in Brownian motion) in fractional-order

$$D_t^{\alpha} u(x,t) - (x+1)u_x - x^2 e^t u_{xx} = 0, \text{ where } 0 < \alpha \leq 1, \ t > 0, \ x \in R, \tag{4.62}$$

here $u_x = \frac{\partial u}{\partial x}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$
with initial condition

$$u(x,0) = x + 1. \tag{4.63}$$

Now, operating GT to (4.62), yields

$$G\{D_t^\alpha u(x,t)\} - G\{(x+1)u_x\} - G\{x^2 e^t u_{xx}\} = G\{0\}, \tag{4.64}$$

using differentiation properties (4.6), (4.64) can be written as

$$\left[q(s)^\alpha G\left\{u(x,t)\right\} - p(s)q(s)^{\alpha-1}u(x,0)\right] - G\{(x+1)u_x\} - G\{x^2 e^t u_{xx}\} = 0, \tag{4.65}$$

Substituting (4.63) into (4.65), we get

$$G\{u(x,t)\} = \frac{\rho(s)}{q(s)}(x+1) + \frac{1}{q(s)^\alpha}G\{(x+1)u_x\} + \frac{1}{q(s)^\alpha}G\{x^2 e^t u_{xx}\}, \tag{4.66}$$

Taking IGT, it leads to

$$u(x,t) = (x+1) + G^{-1}\left[\frac{1}{q(s)^\alpha}G\{(x+1)u_x\} + \frac{1}{q(s)^\alpha}G\{x^2 e^t u_{xx}\}\right]. \tag{4.67}$$

By applying the HPM, we have

$$\begin{aligned}\sum_{i=0}^{\infty} p^i u_i(x,t) =& (x+1) + pG^{-1}\left[\frac{1}{q(s)^\alpha}G\left\{(x+1)\frac{\partial}{\partial x}\left(\sum_{i=0}^{\infty} p^i u_i(x,t)\right)\right\}\right] \\ &+ pG^{-1}\left[+\frac{1}{q(s)^\alpha}G\left\{x^2 e^t \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{\infty} p^i u_i(x,t)\right)\right\}\right].\end{aligned} \tag{4.68}$$

By comparing the power of '$p$' terms of equation (4.68),

$$
\begin{aligned}
p^0 &: u_0(x,t) = x+1,\\
p^1 &: u_1(x,t) = G^{-1}\left[\frac{1}{q(s)^\alpha}G\left\{(x+1)\frac{\partial u_0}{\partial x} + x^2e^t\frac{\partial^2 u_0}{\partial x^2}\right\}\right],\\
&\qquad\quad = (x+1)\frac{t^\alpha}{\Gamma(\alpha+1)},\\
p^2 &: u_2(x,t) = G^{-1}\left[\frac{1}{q(s)^\alpha}G\left\{(x+1)\frac{\partial u_1}{\partial x} + x^2e^t\frac{\partial^2 u_1}{\partial x^2}\right\}\right],\\
&\qquad\quad = (x+1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
p^3 &: u_3(x,t) = G^{-1}\left[\frac{1}{q(s)^\alpha}G\left\{(x+1)\frac{\partial u_2}{\partial x} + x^2e^t\frac{\partial^2 u_2}{\partial x^2}\right\}\right],\\
&\qquad\quad = (x+1)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&\ \vdots
\end{aligned}
\tag{4.69}
$$

Proceeding in the same way, the closed form of (4.62) is obtained using (4.25) as

$$
\begin{aligned}
u(x,t) &= (x+1)\left[1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...\right],\\
&= (x+1)E_\alpha(t^\alpha).
\end{aligned}
\tag{4.70}
$$

where $E_\alpha(t^\alpha)$ is Mittag-Leffler function [120].

### 4.5.4 Klein-Gordon equation

Consider non-homogeneous linear partial differential equation (PDE) in fractional order, time-fractional Klein-Gordon(K-G) equation [114] (arise in special relativity particularly in Quantum field theory) as

$$
D_t^{2\alpha}u(x,t) - u_{xx} + u = 6x^3t + (x^3-6x)t^3, \text{ where } 0<\alpha\le 1,\ t>0,\ x\in R, \tag{4.71}
$$

with initial condition

$$
u(x,0) = 0 \ \text{ and } \ D_tu(x,0) = 0. \tag{4.72}
$$

and exact solution [114] when $\alpha = 1$ is

$$u(x,t) = x^3 t^3. \tag{4.73}$$

Operating GT to equation (4.71), we have

$$G\{D_t^{2\alpha} u(x,t)\} = G\{u_{xx}\} - G\{u\} + G\{6x^3 t\} + G\{\left(x^3 - 6x\right) t^3\}, \tag{4.74}$$

using differentiation properties (4.6), it leads to

$$\begin{aligned}\left[q(s)^{2\alpha} G\left\{u(x,t)\right\} - \rho(s) q(s)^{2\alpha-1} u(x,0) - \rho(s) q(s)^{2\alpha-2} D_t u(x,0)\right] =& G\{u_{xx}\} - G\{u\} + G\{6x^3 t\} \\ &+ G\{\left(x^3 - 6x\right) t^3\},\end{aligned} \tag{4.75}$$

substituting equation (4.72), we get

$$G\{u(x,t)\} = 6x^3 \frac{\rho(s)}{q(s)^{2\alpha+2}} + 6(x^3 - 6x) \frac{\rho(s)}{q(s)^{2\alpha+4}} + \frac{1}{q(s)^{2\alpha}} G\left\{u_{xx} - u\right\}, \tag{4.76}$$

applying IGT on (4.76) leads to

$$u(x,t) = 6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 6(x^3 - 6x) \frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} + G^{-1}\left[\frac{1}{q(s)^{2\alpha}} G\left\{u_{xx} - u\right\}\right]. \tag{4.77}$$

By applying the HPM, we have

$$\begin{aligned}\sum_{i=0}^{\infty} p^i u_i(x,t) =& 6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 6(x^3 - 6x) \frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} \\ &+ pG^{-1}\left[\frac{1}{q(s)^{2\alpha}} G\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{\infty} p^i u_i(x,t)\right) - \sum_{i=0}^{\infty} p^i u_i(x,t)\right\}\right].\end{aligned} \tag{4.78}$$

which is convergent series by Theorem – 4.5. Comparing the power of '$p$' in terms of equation (4.78),

$$\begin{aligned}
p^0 : u_0(x,t) &= 6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 6(x^3-6x)\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)},\\
p^1 : u_1(x,t) &= G^{-1}\left[\frac{1}{q(s)^{2\alpha}} G\left\{\frac{\partial^2 u_0}{\partial x^2} - u_0\right\}\right],\\
&= (72x-6x^3)\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} + (72x-6x^3)\frac{t^{4\alpha+3}}{\Gamma(4\alpha+4)},\\
p^2 : u_2(x,t) &= G^{-1}\left[\frac{1}{q(s)^{2\alpha}} G\left\{\frac{\partial^2 u_1}{\partial x^2} - u_1\right\}\right],\\
&= (-108x+6x^3)\frac{t^{6\alpha+1}}{\Gamma(6\alpha+2)} + (-108x+6x^3)\frac{t^{6\alpha+3}}{\Gamma(6\alpha+4)},\\
p^3 : u_3(x,t) &= G^{-1}\left[\frac{1}{q(s)^{2\alpha}} G\left\{\frac{\partial^2 u_2}{\partial x^2} - u_2\right\}\right],\\
&= (144x-6x^3)\frac{t^{8\alpha+1}}{\Gamma(8\alpha+2)} + (144x-6x^3)\frac{t^{8\alpha+3}}{\Gamma(8\alpha+4)},\\
&\vdots
\end{aligned} \tag{4.79}$$

Thus the solution of (4.71) can be obtained from (4.25) as

$$\begin{aligned}
u(x,t) =& 6x^3\left[\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} - \frac{t^{4\alpha+3}}{\Gamma(4\alpha+4)} + \frac{t^{6\alpha+1}}{\Gamma(6\alpha+2)} + \frac{t^{6\alpha+3}}{\Gamma(6\alpha+4)} - ...\right]\\
&+ 36x\left[-\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} + 2\left(\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} + \frac{t^{4\alpha+3}}{\Gamma(4\alpha+4)}\right) - 3\left(\frac{t^{6\alpha+1}}{\Gamma(6\alpha+2)} + \frac{t^{6\alpha+3}}{\Gamma(6\alpha+4)}\right) - ...\right].
\end{aligned} \tag{4.80}$$

Here, equation (4.80) is an approximate solution of the time-fractional K-G equation. For finite $x$ and $t < 1$, from (4.79), it is clear that the series is decreasing. Hence, from Theorem – 4.5, (4.80) is convergent. The Absolute Error in the solution of K-G equation by HPGTM, RPSM [14] (up to $2^{nd}-$ term) and ADM [114] (up to $2^{nd}-$ term) when $\alpha = 1$.

Table 4.4: The Absolute Error in the solution of K-G equation by HPGTM, RPSM, and ADM for $\alpha = 1$ with exact solution (ES).

| $x$ | $t$ | Exact Solution (ES) | Absolute error (for HPGTM) \|ES-HPGTM\| | Absolute error (for RPSM) \|ES-RPSM\| | Absolute error (for ADM) \|ES-ADM\| |
|---|---|---|---|---|---|
| -10 | 0.1 | 1 | 1.49290E-05 | 1.23525E-04 | 1.54900E-07 |
| | 0.2 | 8 | 4.70857E-04 | 4.24635E-04 | 1.98100E-05 |
| | 0.3 | 27 | 3.48879E-03 | 1.23536E-03 | 3.38470E-04 |
| | 0.4 | 64 | 1.41897E-02 | 1.04354E-03 | 2.53562E-03 |
| | 0.5 | 125 | 9.31939E-02 | 9.57309E-02 | 1.20908E-02 |
| -5 | 0.1 | 0.125 | 7.46430E-06 | 3.56980E-05 | 3.27000E-08 |
| | 0.2 | 1 | 2.35429E-04 | 2.56340E-04 | 4.19060E-06 |
| | 0.3 | 3.375 | 1.74439E-03 | 1.35636E-03 | 7.15980E-05 |
| | 0.4 | 8 | 7.09486E-03 | 1.45960E-03 | 5.36381E-04 |
| | 0.5 | 15.625 | 5.64732E-03 | 1.44876E-02 | 2.55767E-03 |
| 1 | 0.1 | 0.001 | 1.49286E-06 | 1.43453E-05 | 3.69080E-06 |
| | 0.2 | 0.008 | 4.70857E-05 | 2.53640E-04 | 4.72381E-05 |
| | 0.3 | 0.027 | 3.48879E-04 | 2.45788E-04 | 8.07107E-05 |
| | 0.4 | 0.064 | 1.41897E-03 | 1.36363E-03 | 6.04648E-05 |
| | 0.5 | 0.125 | 4.12946E-03 | 1.45837E-02 | 2.88318E-04 |
| 5 | 0.1 | 0.125 | 7.46430E-06 | 5.64707E-04 | 3.27000E-08 |
| | 0.2 | 1.000 | 2.39543E-04 | 1.36560E-04 | 4.19060E-06 |
| | 0.3 | 3.375 | 1.74439E-03 | 1.53663E-03 | 7.15980E-05 |
| | 0.4 | 8.000 | 7.09486E-03 | 3.46573E-03 | 5.36381E-04 |
| | 0.5 | 15.625 | 2.06473E-02 | 1.93988E-02 | 2.55767E-03 |

Table 4.5: The approximate solution $u(x,t)$ by HPGTM for different fractional value $\alpha$, and $x = 1$.

| t | $\alpha = 0.2$ | $\alpha = 0.4$ | $\alpha = 0.6$ | $\alpha = 0.8$ | $\alpha = 1$ |
|---|---|---|---|---|---|
| 0.1 | 0.14732910000 | 0.05294213000 | 0.01539627300 | 0.00404551700 | 0.00100000000 |
| 0.2 | 0.35903330000 | 0.17589499200 | 0.06954153800 | 0.02442954700 | 0.00800000000 |
| 0.3 | 0.59719811100 | 0.35106926600 | 0.16673647200 | 0.06976452200 | 0.02700000000 |
| 0.4 | 0.85052178500 | 0.57014449700 | 0.30860452600 | 0.14659733100 | 0.06400000000 |
| 0.5 | 1.11295631800 | 0.82822465600 | 0.49589792600 | 0.26037850300 | 0.12500000000 |

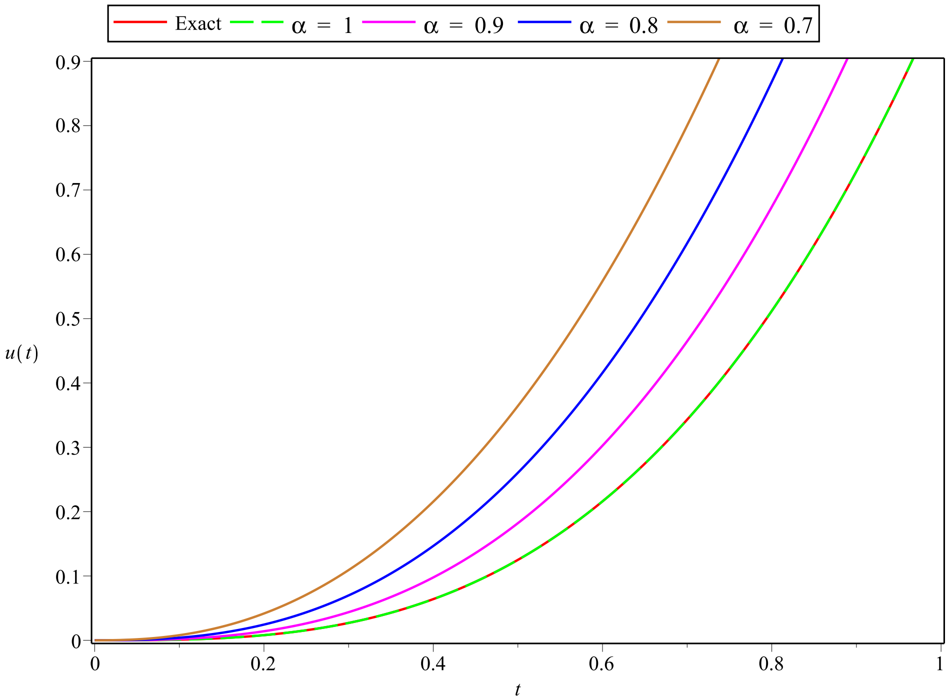
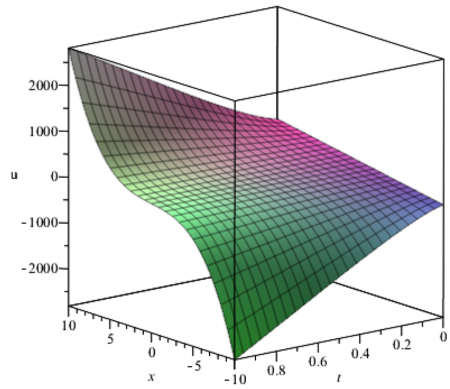
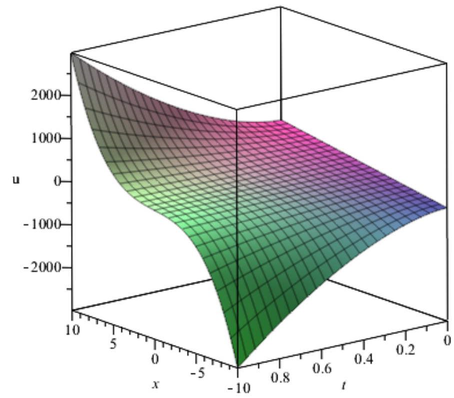


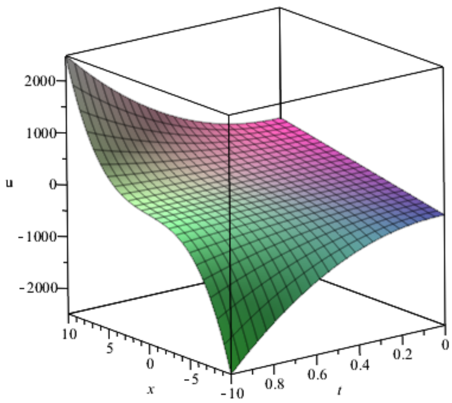

Figure 4.2: Comparison of time-fractional Klein-Gordon equation with different values of $\alpha$ and exact solution at $x = 1$.



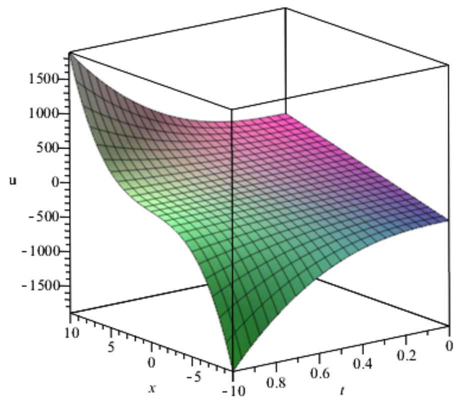

(a) $\alpha = 0.2$



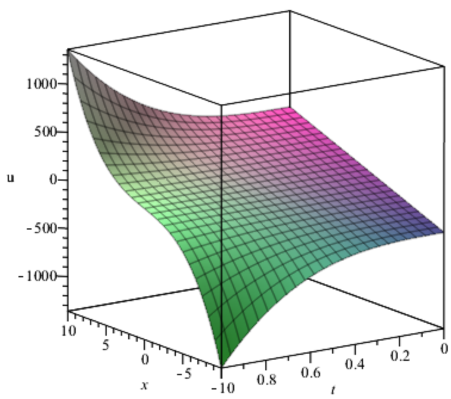

(b) $\alpha = 0.4$



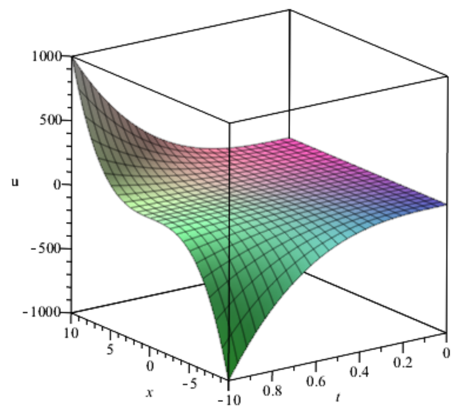

(c) $\alpha = 0.6$




(d) $\alpha = 0.8$




(e) $\alpha = 1$




(f) Exact solution

Figure 4.3: Solution of K-G equation by HPGTM with order $\alpha = 1$ and exact solution.

### 4.5.5 Rosenau-Hyman equation

Consider non-linear time-fractional Rosenau-Hyman(R-H) equation [165, 40] (represent compaction in the use of equipments especially in building work) as

$$D_t^\alpha u(x,t) - uu_{xxx} - uu_x - 3u_x u_{xx} = 0, \text{ where } 0 < \alpha \leq 1, \ t > 0, \ x \in R, \tag{4.81}$$

with initial condition

$$u(x,0) = -\frac{8c}{3}\cos^2\left(\frac{x}{4}\right), \text{ where } c \text{ is arbitrary constant}, \tag{4.82}$$

and exact solution given by Molliq and Noorani [165] as

$$u(x,t) = -\frac{8c}{3}\cos^2\left(\frac{x-ct}{4}\right). \tag{4.83}$$

Operating General transform to equation (4.81), yields

$$G\left\{D_t^\alpha u(x,t)\right\} - G\left\{uu_{xxx}\right\} - G\left\{uu_x\right\} - 3G\left\{u_x u_{xx}\right\} = 0, \tag{4.84}$$

Using differentiation properties (4.6), it leads to

$$\left[q(s)^\alpha G\left\{u\right\} - \rho(s)q(s)^{\alpha-1}u(x,0)\right] = G\left\{uu_{xxx}\right\} + G\left\{uu_x\right\} + 3G\left\{u_x u_{xx}\right\}, \tag{4.85}$$

substituting equation (4.82), we get

$$G\left\{u\right\} = \frac{\rho(s)}{q(s)}\left(-\frac{8c}{3}\cos^2\left(\frac{x}{4}\right)\right) + \frac{1}{q(s)^\alpha}\left[G\left\{uu_{xxx}\right\} + G\left\{uu_x\right\} + 3G\left\{u_x u_{xx}\right\}\right], \tag{4.86}$$

Taking IGT both sides

$$u(x,t) = -\frac{8c}{3}\cos^2\left(\frac{x}{4}\right) + G^{-1}\left[\frac{1}{q(s)^\alpha}G\left\{uu_{xxx} + uu_x + 3u_x u_{xx}\right\}\right]. \tag{4.87}$$

By applying the HPM, we have

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = -\frac{8c}{3}\cos^2\left(\frac{x}{4}\right) + pG^{-1}\left[\frac{1}{q(s)^{\alpha}} G\left\{R\left(\sum_{i=0}^{\infty} p^i u_i\right) + \sum_{i=0}^{\infty} p^i H_i(u)\right\}\right], \quad (4.88)$$

where non-linear terms of equation (4.81) are decomposed using He's Polynomial $H_i(u)$ as follows

$$\begin{aligned}
H_0(u) &= u_0 u_{0xxx} + u_0 u_{0x} + 3u_{0x} u_{0xx}, \\
H_1(u) &= u_0 u_{1xxx} + u_1 u_{0xxx} + u_1 u_{0x} + u_0 u_{1x} + 3u_{1x} u_{0xx} + 3u_{0x} u_{1xx}, \\
H_2(u) &= u_0 u_{2xxx} + u_2 u_{0xxx} + 2u_1 u_{1xxx} + u_2 u_{0x} + u_0 u_{2x} + 2u_1 u_{1x} + 3u_{2x} u_{0xx} + 3u_{0x} u_{2xx} \\
&\quad + 6u_{1x} u_{1xx}, \\
&\vdots
\end{aligned} \quad (4.89)$$

Comparing the power of '$p$' in terms of (4.88), it reach to

$$\begin{aligned}
p^0 : u_0 &= \frac{-8c}{3}\cos^2\left(\frac{x}{4}\right), \\
p^1 : u_1 &= G^{-1}\left[\frac{1}{q(s)^{\alpha}} G\left\{H_0(u)\right\}\right], \\
&= \frac{-2c^2}{3}\sin\left(\frac{x}{2}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\
p^2 : u_2 &= G^{-1}\left[\frac{1}{q(s)^{\alpha}} G\left\{H_1(u)\right\}\right], \\
&= \frac{c^3}{6}\cos\left(\frac{x}{2}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\
p^3 : u_3 &= G^{-1}\left[\frac{1}{q(s)^{\alpha}} G\left\{H_2(u)\right\}\right], \\
&= \frac{c^4}{12}\sin\left(\frac{x}{2}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\
&\vdots
\end{aligned} \quad (4.90)$$

The solution of (4.81) can be obtained from (4.25) as

$$u(x,t) = \frac{-8c}{3}\cos^2\left(\frac{x}{4}\right) - \frac{2c^2}{3}\sin\left(\frac{x}{2}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{c^3}{6}\cos\left(\frac{x}{2}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{c^4}{12}\sin\left(\frac{x}{2}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \dots \tag{4.91}$$

Here, equation (4.91) is an approximate solution of the time-fractional Rosenau-Hyman equation.

Table 4.6: The Absolute Error in the solution of R-H equation by HPGTM, RPSM and ADM for $\alpha = 1$ and $c = 0.25$ with exact solution (ES).

| $x$ | $t$ | Exact Solution (ES) | Absolute error (for HPGTM) \|ES-HPGTM\| | Absolute error (for RPSM) \|ES-RPSM\| | Absolute error (for ADM) \|ES-ADM\| |
|---|---|---|---|---|---|
| -10 | 0.1 | 0.43188 | 3.74560E-06 | 3.79760E-06 | 1.00000E-10 |
| | 0.2 | 0.43585 | 1.51895E-05 | 1.56049E-05 | 1.54662E-05 |
| | 0.3 | 0.43980 | 3.46423E-05 | 3.60430E-05 | 3.55748E-05 |
| | 0.4 | 0.44374 | 6.24127E-05 | 6.57300E-05 | 6.46202E-05 |
| | 0.5 | 0.44767 | 9.88086E-05 | 1.05282E-04 | 1.03114E-04 |
| -5 | 0.1 | 0.06381 | 1.04639E-05 | 1.04962E-05 | 1.04854E-05 |
| | 0.2 | 0.06138 | 4.19838E-05 | 4.22414E-05 | 4.21548E-05 |
| | 0.3 | 0.05899 | 9.47495E-05 | 9.56152E-05 | 9.53230E-05 |
| | 0.4 | 0.05665 | 1.68948E-04 | 1.70991E-04 | 1.70298E-04 |
| | 0.5 | 0.05435 | 2.64761E-04 | 2.68735E-04 | 2.67382E-04 |
| 1 | 0.1 | 0.62784 | 1.14528E-05 | 1.14786E-05 | 1.14699E-05 |
| | 0.2 | 0.62976 | 4.59131E-05 | 4.61188E-05 | 4.60495E-05 |
| | 0.3 | 0.63165 | 1.03532E-04 | 1.04222E-04 | 1.03988E-04 |
| | 0.4 | 0.63348 | 1.84456E-04 | 1.86082E-04 | 1.85528E-04 |
| | 0.5 | 0.63527 | 2.88828E-04 | 2.91987E-04 | 2.90903E-04 |
| 5 | 0.1 | 0.06880 | 1.03990E-05 | 1.03663E-05 | 1.03772E-05 |
| | 0.2 | 0.07136 | 4.14643E-05 | 4.12024E-05 | 4.12890E-05 |
| | 0.3 | 0.07395 | 9.29965E-05 | 9.21088E-05 | 9.24010E-05 |
| | 0.4 | 0.07659 | 1.64793E-04 | 1.62680E-04 | 1.63373E-04 |
| | 0.5 | 0.07927 | 2.56647E-04 | 2.52503E-04 | 2.53856E-04 |

Table 4.7: The approximate solution $u(x,t)$ by HPGTM with different fractional value of $\alpha$, when $x = 4$.

| t | $\alpha = 0.2$ | $\alpha = 0.4$ | $\alpha = 0.6$ | $\alpha = 0.8$ | $\alpha = 1$ |
|---|---|---|---|---|---|
| 0.1 | 0.22105375900 | 0.21178461120 | 0.20532801720 | 0.20108349020 | 0.19841182960 |
| 0.2 | 0.22503556410 | 0.21733022990 | 0.21089441750 | 0.20589843990 | 0.20221647830 |
| 0.3 | 0.22763820740 | 0.22137770280 | 0.21541955110 | 0.21024870860 | 0.20603136860 |
| 0.4 | 0.22961897910 | 0.22468279560 | 0.21938041820 | 0.21432560070 | 0.20985620020 |
| 0.5 | 0.23123685250 | 0.22752676280 | 0.22296936940 | 0.21821245640 | 0.21369067140 |

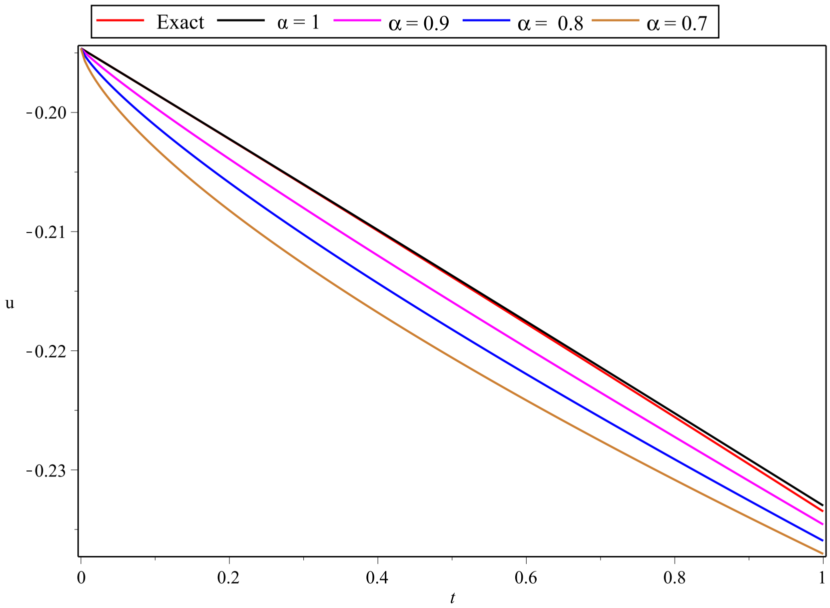
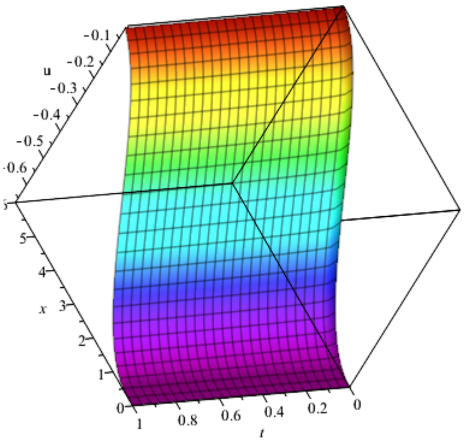
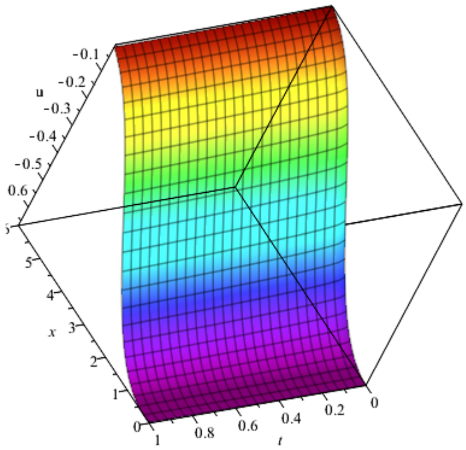


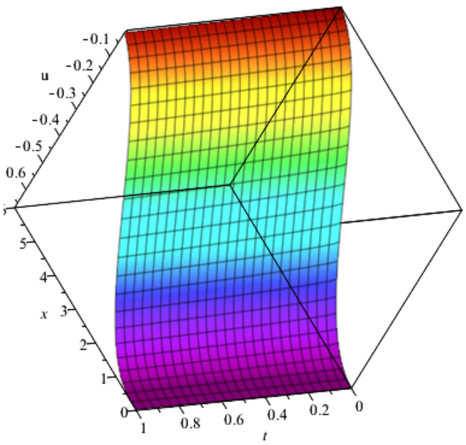

Figure 4.4: Comparison of time-fractional R-H equation with different values of $\alpha$ and exact solution at $x = 4$.



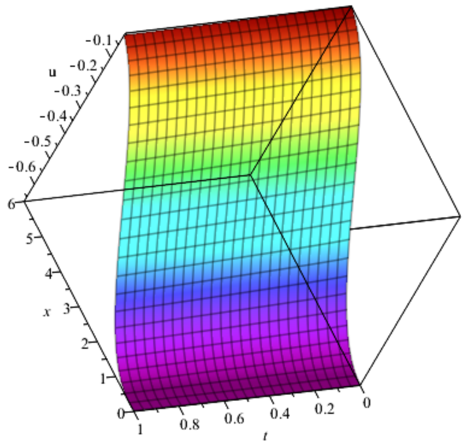

(a) $\alpha = 0.2$



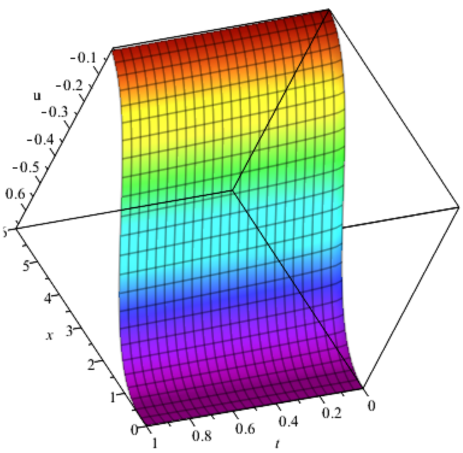

(b) $\alpha = 0.4$



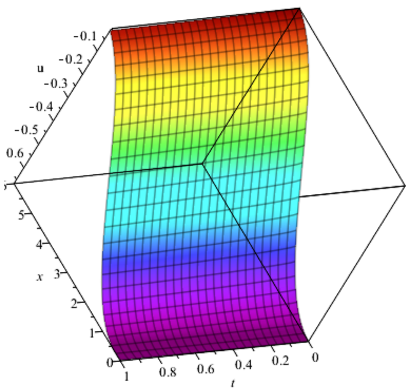

(c) $\alpha = 0.6$




(d) $\alpha = 0.8$




(e) $\alpha = 1$




(f) Exact solution

Figure 4.5: Solution of R-H equation by HPGTM with order $\alpha = 1$ and exact solution.

## 4.6 Results and Conclusion

The present work showcases the analytic solutions of the time-fractional radioactive decay model, and Riccati equation as FODEs and the semi-analytic solutions of the time-fractional backward Kolmogorov, Klein-Gordon and Rosenau-Hyman equations as FPDEs. From Figures – 4.1, 4.2, and 4,4 we can say that we obtained almost the same result by HPGTM at $\alpha = 1$ as compared to the exact solution of Riccati, Klein-Gordon and Rosenau-Hyman equations. The 3D behavior and the comparison of aforesaid equations are discussed in Figures – 4.3 and 4.5 for different fractional orders and the exact solution. Further Tables – 4.2, 4.4, and 4.6 proves that we achieve better results than the RPSM method and almost similar results as compared to the ADM method. From the obtained results we conclude that the proposed method can be applied to a broad range of nonlinear fractional differential equations arising during the analysis of various physical phenomenon, which needs attention in the form of critical scientific investigations.