Chapter 3

Homotopy perturbation Sawi transform method

3.1 Introduction

There are several well-known integral transforms in the literature, viz. G-transform [94], Sumudu transform [159], Aboodh transform [1], Elzaki transform [48], Pourreza transform [6], Natural transform [89], Mohand transform [124], Sawi transform [101], and Kamal transform [79], etc. These transformations are used to solve various functional equations such as fractional order integral equations, ordinary, and partial fractional differential equations [70, 121, 123, 96, 140, 124]. However, these transformations alone are not enough capable to deal with nonlinear equations because of the difficulties due to the involvement of nonlinear terms.

In recent years, many hybrid methods have been introduced that combine the integral transforms with semi-analytic techniques such as the Sumudu Adomian decomposition method [128], Laplace variational iteration method [26], Residual power series method (RPSM) [91, 169], Homotopy perturbation General transform method [92] and Homotopy analysis Sumudu transform method [117] to solve the Fractional Differential Equations (FDEs). In continuation of the study, the authors here in the present work introduce another powerful method as a combination of the Homotopy perturbation method (HPM) [62, 63] and the Sawi transform [101], and call it: the Homotopy perturbation Sawi transform method (HPSTM), which is capable of dealing with general FDE in an efficient manner, and can be applied not only on various nonlinear wave equations, oscillatory equations with discontinuities and boundary value problems, but it can also deal with different kinds of nonlinear equations.

Certain Well-known definitions and results used in this chapter are as follows:

3.2 Preliminaries

Definition 3.1. The Sawi transform (ST) [101] for the function $u(t)$ is

$$S\{u(t)\} = \frac{1}{v^2}\int_0^\infty u(t)e^{-\frac{t}{v}}dt, t \geq 0, \kappa_1 \leq v \leq \kappa_2, \tag{3.1}$$

for a given function $u(t) \in A$; where $A = \{u(t) : \exists M, \kappa_1, \kappa_2 > 0, |u(t)| < Me^{\frac{|t|}{\kappa_j}}$, if $t \in (-1)^j \times [0, \infty)\}$ and κ_1, κ_2 may be any finite or infinite values, and M must be a finite value.

Theorem 3.1. The Sawi transform of integer ordered derivative as given by Mohand [101] is

$$S\left\{u^{(n)}(x,t)\right\} = \frac{1}{v^n}S\{u(x,t)\} - \sum_{k=0}^{n-1}\frac{1}{v^{n+1-k}}u^{(k)}(x,0). \tag{3.2}$$

where $u^{(n)}(x,t)$ are such that, $\left|u^{(n)}(x,t)\right| < Me^{\frac{|t|}{k_j}}$ with M as a finite and positive value and k_1, k_2 are suitable positive numbers, making $u^{(n)}$'s an exponential order.

Remark 3.1. Working of the Homotopy perturbation method (HPM) [62, 63], is described as below:

We consider a general form of fractional differential equation as

$$D_t^{n\alpha}u(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), 0 < \alpha \leq 1, t > 0, x \in R, \tag{3.3}$$

where $f(x,t)$ is a continuous function and $D_t \left(= \frac{\partial}{\partial t}\right)$ is the differential operator, $R(u)$ are linear terms, and $N(u)$ are nonlinear terms of continuous function $u(x,t)$, subject to initial conditions

$$u(x,0) = \phi_0(x), \frac{\partial u(x,0)}{\partial t} = \phi_1(x), ..., \frac{\partial^{n-1}u(x,0)}{\partial t^{n-1}} = \phi_{n-1}(x). \tag{3.4}$$

Now, applying homotopy technique [62] with perturbation parameter leads to the solution of (3.3), that is

$$u\left(x,t\right)=\lim_{p\to1}\sum_{n=0}^{\infty}p^{n}u_{n}\left(x,t\right)=\lim_{p\to1}[u_{0}\left(x,t\right)+p^{1}u_{1}\left(x,t\right)+p^{2}u_{2}\left(x,t\right)+...],$$
$$=u_{0}(x,t)+u_{1}\left(x,t\right)+u_{2}\left(x,t\right)+.... \tag{3.5}$$

3.3 Main Results

Lemma 3.1. The Sawi transform of Riemann-Liouville fractional integral $D_t^{-(n-\beta)}\left(h(x,t)\right)$ is

$$S\left\{D_t^{-(n-\beta)}\left(h(x,t)\right)\right\}=\frac{1}{\upsilon^{-(n-\beta)}}S\left\{h(x,t)\right\}.$$

Proof. First, we will use the definition of R-L integral as given in (1.1),

$$\begin{aligned}S\left\{D_t^{-(n-\beta)}\left(h(x,t)\right)\right\}&=S\left\{\frac{1}{\Gamma\left(n-\beta\right)}\int_0^t\left(t-\tau\right)^{n-\beta-1}h(x,\tau)d\tau\right\},\\&=\frac{1}{\Gamma\left(n-\beta\right)}\left[S\left\{t^{n-\beta-1}\right\}.S\left\{h(x,t)\right\}\right],\\&=\frac{1}{\Gamma\left(n-\beta\right)}\left[\Gamma\left(n-\beta\right)\upsilon^{n-\beta}.S\left\{h(x,t)\right\}\right],\\&=\frac{1}{\upsilon^{-(n-\beta)}}S\left\{h(x,t)\right\}.\end{aligned}$$

Theorem 3.2. the Sawi transform of Caputo fractional derivative of $u\left(x,t\right)$ is given by

$$S\{D_t^{n\alpha}u\left(x,t\right)\}=\frac{1}{\upsilon^{n\alpha}}S\{u\left(x,t\right)\}-\sum_{k=0}^{n-1}\frac{1}{\upsilon^{n\alpha+1-k}}D_t^{k}u\left(x,0\right). \tag{3.6}$$

Proof. We can write the Caputo derivative (1.2) as

$$D_t^{\beta}u\left(x,t\right)=D_t^{-(n-\beta)}h\left(x,t\right),\ \text{where } h\left(x,t\right)=u_t^{(n)}\left(x,t\right),n-1<\beta\leq n. \tag{3.7}$$

Now, according to Lemma – 3.1, we have

$$S\left\{D_t^{-(n-\beta)}h(x,t)\right\} = \frac{1}{v^{-(n-\beta)}}S\{h(x,t)\}. \tag{3.8}$$

Therefore, from both of the above equations (3.7) and (3.8), we get

$$S\left\{D_t^{\beta}u(x,t)\right\} = S\left\{D_t^{-(n-\beta)}h(x,t)\right\} = \frac{1}{v^{-(n-\beta)}}S\{h(x,t)\}. \tag{3.9}$$

According to (3.2), above equation (3.9) leads to

$$S\{h(x,t)\} = S\{u^{(n)}(x,t)\} = \frac{1}{v^n}S\{u(x,t)\} - \sum_{k=0}^{n-1}\frac{1}{v^{n+1-k}}u^{(k)}(x,0). \tag{3.10}$$

Substituting equation (3.10) into (3.9), we have

$$S\left\{D_t^{\beta}u(x,t)\right\} = \frac{1}{v^{-(n-\beta)}}\left[\frac{1}{v^n}S\{u(x,t)\} - \sum_{k=0}^{n-1}\frac{1}{v^{n+1-k}}u^k(x,0)\right], \tag{3.11}$$

$$= \frac{1}{v^{\beta}}S\{u(x,t)\} - \sum_{k=0}^{n-1}\frac{1}{v^{\beta+1-k}}u^k(x,0). \tag{3.12}$$

For $0 < \alpha \leq 1$ considering $\beta = n\alpha$, leads to the desired result (3.6).

3.3.1 Homotopy Perturbation using Sawi Transform Method (HPSTM)

We consider a general form of fractional-order nonlinear differential equation as (3.3) with initial conditions (3.4). First, by operating Sawi transform on (3.3), we have

$$S\{D_t^{n\alpha}u(x,t)\} = -S\{Ru(x,t)\} - S\{Nu(x,t)\} + S\{f(x,t)\}, \tag{3.13}$$

then using (3.6), we get

$$\frac{1}{v^{n\alpha}}S\{u(x,t)\} - \sum_{k=0}^{n-1}\frac{1}{v^{n\alpha+1-k}}D_t^k u(x,0) = -S\{Ru(x,t)\} - S\{Nu(x,t)\} + S\{f(x,t)\}, \tag{3.14}$$

i.e.

$$S\{u(x,t)\} = \frac{1}{v^2}\left[v\phi_0(x) + v^2\phi_1(x) + ... + v^n\phi_{n-1}(x)\right] - v^{n\alpha}S\{Ru(x,t)\}$$
$$- v^{n\alpha}S\{Nu(x,t)\} + v^{n\alpha}S\{f(x,t)\}, \tag{3.15}$$

now taking the inverse Sawi transform of (3.15) gives

$$u(x,t) = G(x,t) - S^{-1}\left[v^{n\alpha}S\{Ru(x,t)\} + v^{n\alpha}S\{Nu(x,t)\}\right], \tag{3.16}$$

where $G(x,t)$ is inverse Sawi transform of initial conditions and last term of (3.15). Applying the Homotopy perturbation method [63] to (3.16), we get

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = G(x,t) - p\left[S^{-1}\left[v^{n\alpha}S\{R\sum_{i=0}^{\infty} p^i u_i(x,t) + N\sum_{i=0}^{\infty} p^i u_i(x,t)\}\right]\right]. \tag{3.17}$$

In (3.17), nonlinear terms are decomposed using He's polynomial [63],

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{3.18}$$

where $H_n(u_0, u_1, u_2, ..., u_n) = \frac{1}{n!}\left[\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, n = 0, 1, 2, ...$
Applying (3.18) into (3.17), we find that

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = G(x,t) - p\left[S^{-1}\left[v^{n\alpha}S\{R\sum_{i=0}^{\infty} p^i u_i(x,t) + \sum_{i=0}^{\infty} p^i H_i(u)\}\right]\right]. \tag{3.19}$$

From the above equation (3.19), we get

$$\begin{aligned}
&p^0 : u_0(x,t) = G(x,t),\\
&p^1 : u_1(x,t) = -S^{-1}\left[v^{n\alpha}S\{Ru_0(x,t)\} + v^{n\alpha}S\{H_0(u)\}\right],\\
&p^2 : u_2(x,t) = -S^{-1}\left[v^{n\alpha}S\{Ru_1(x,t)\} + v^{n\alpha}S\{H_1(u)\}\right],\\
&\vdots\\
&p^n : u_n(x,t) = -S^{-1}\left[v^{n\alpha}S\{Ru_{n-1}(x,t)\} + v^{n\alpha}S\{H_{n-1}(u)\}\right].
\end{aligned} \tag{3.20}$$

Therefore, the solution of (3.3) leads to (3.5).

3.3.2 Convergence of HPSTM

Theorem 3.3. Let the Banach space $B \equiv C([a,b] \times [0,T])$ be defined on rectangular interval $[a,b] \times [0,T]$. Then equation (3.5) defined as $u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$ is convergent series, if $u_0 \in B$ is Bounded and $||u_{k+1}|| \leq \delta ||u_k||, \forall u_k \in B$, and for $0 < \delta < 1$.

Proof. Considering the sequence $\{A_q\}$ as partial sums of equation (3.5), we have

$$\begin{aligned}
A_0 &= u_0(x,t), \\
A_1 &= u_0(x,t) + u_1(x,t), \\
A_2 &= u_0(x,t) + u_1(x,t) + u_2(x,t), \\
&\vdots \\
A_q &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ... + u_q(x,t).
\end{aligned} \tag{3.21}$$

To prove this theorem, we next prove that $\{A_q\}_{q=0}^{\infty}$ is a Cauchy sequence [21] in B. Now, we take

$$\begin{aligned}
\|A_{q+1} - A_q\| &= \|u_{q+1}(x,t)\| \\
&\leq \delta \|u_q(x,t)\| \\
&\leq \delta^2 \|u_{q-1}(x,t)\| \\
&\vdots \\
&\leq \delta^{q+1} \|u_0(x,t)\|.
\end{aligned} \tag{3.22}$$

Therefore, for any $q, n \in N$, such that $q > n$, we get

$$\begin{aligned}
\|A_q - A_n\| &= \|(A_q - A_{q-1}) + (A_{q-1} - A_{q-2}) + (A_{q-2} - A_{q-3}) + ... + (A_{n+1} - A_n)\| \\
&\leq \|A_q - A_{q-1}\| + \|A_{q-1} - A_{q-2}\| + \|A_{q-2} - A_{q-3}\| + ... + \|A_{n+1} - A_n\| \\
&\leq \delta^q \|u_0(x,t)\| + \delta^{q-1} \|u_0(x,t)\| + ... + \delta^{n+1} \|u_0(x,t)\| \\
&\leq \beta \|u_0(x,t)\|,
\end{aligned} \tag{3.23}$$

where $\beta = \frac{(1-\delta^{q-n})}{(1-\delta)} \delta^{n+1}$.

Since $u_0(x,t)$ is bounded, therefore $\|u_0(x,t)\| < \infty$.

For $0 < \delta < 1$, as the value of n increases and $n \to \infty$ leads to $\beta \to 0$, therefore

$$\lim_{\substack{n\to\infty \\ q\to\infty}} \|A_q - A_n\| = 0. \tag{3.24}$$

Hence, $\{A_q\}_{q=0}^{\infty}$ is a Cauchy sequence in B.

It concludes that the solution of equation (3.3) as a series is convergent.

Theorem 3.4. If the approximate series solution of equation (3.3) is $\sum\limits_{k=0}^{n} u_k(x,t)$, then the maximum Absolute error is estimated by

$$\left\| u(x,t) - \sum_{k=0}^{n} u_k(x,t) \right\| \le \frac{\delta^{n+1}}{1-\delta} \|u_0(x,t)\|, \tag{3.25}$$

where δ is a number such that $\frac{\|u_{k+1}\|}{\|u_k\|} \le \delta$.

Proof. From equation (3.23) in Theorem – 3.3, we have

$$\|A_q - A_n\| \le \beta \|u_0(x,t)\|, \text{ where } \beta = \frac{(1-\delta^{q-n})}{(1-\delta)} \delta^{n+1}. \tag{3.26}$$

Here, $\{A_q\}_{q=0}^{\infty} \to u(x,t)$ as $q \to \infty$ and from (3.21), we get $A_n = \sum\limits_{k=0}^{n} u_k(x,t)$,

$$\left\| u(x,t) - \sum_{k=0}^{n} u_k(x,t) \right\| \le \beta \|u_0(x,t)\|, \tag{3.27}$$

Now, $(1-\delta^{q-n}) < 1$ since $0 < \delta < 1$, then

$$\left\| u(x,t) - \sum_{k=0}^{n} u_k(x,t) \right\| \le \frac{\delta^{n+1}}{1-\delta} \|u_0(x,t)\|. \tag{3.28}$$

Hence, the proof.

3.4 Application of the HPSTM and Numerical Discussions

3.4.1 Solution of Time-fractional logistic equation

In many applications, there arise a logistic equation like viz. biology, medicine, economy, Nieto [113] studied the time-fractional logistic equation as a nonlinear ODE defined as below:

$$D_t^{\alpha} u\left(t\right) = u\left(t\right)\left[1 - u\left(t\right)\right],\ 0 < \alpha \le 1, t > 0; \tag{3.29}$$

where the initial condition is

$$u(0) = u_0. \tag{3.30}$$

For $u_0 = \frac{1}{2}$ and $\alpha = 1$, the exact solution [113] is

$$u\left(t\right) = \frac{1}{1 + e^{-t}}. \tag{3.31}$$

On applying the Sawi transform to equation (3.29), this gives

$$S\left\{u_t^{\alpha}(t)\right\} = S\left\{u\left(t\right)\right\} - S\left\{u^2(t)\right\}; \tag{3.32}$$

further by using (3.6) and (3.32), we get

$$\frac{1}{v^{\alpha}} S\left\{u\left(t\right)\right\} - \frac{1}{v^{\alpha+1}} u(0) = S\left\{u\left(t\right)\right\} - S\left\{u^2(t)\right\}. \tag{3.33}$$

On considering (3.30), this leads to

$$S\left\{u\left(t\right)\right\} = \frac{u_0}{v} + v^{\alpha} S\left\{u\left(t\right) - u^2(t)\right\}. \tag{3.34}$$

And the inverse Sawi transform of (3.34) holds

$$u\,(t) = u_0 + S^{-1}\left[v^{\alpha} S\left\{u\,(t) - u^2(t)\right\}\right]. \tag{3.35}$$

Applying the Homotopy perturbation method to (3.35), leads to

$$\sum_{i=0}^{\infty} p^i u_i(t) = u_0 + pS^{-1}\left[v^{\alpha} S\left\{R\left(\sum_{i=0}^{\infty} p^i u_i\right) - N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right\}\right],$$

in above equation, nonlinear terms are decomposed by He's polynomial as

$$\sum_{i=0}^{\infty} p^i u_i(t) = u_0 + pS^{-1}\left[v^{\alpha} S\left\{R\left(\sum_{i=0}^{\infty} p^i u_i\right) - \sum_{i=0}^{\infty} p^i H_i(u)\right\}\right], \tag{3.36}$$

where He's polynomial $H_i(u)$ [63] is

$$\begin{aligned} H_0(u) =& {u_0}^2, \\ H_1(u) =& 2u_0 u_1, \\ H_2(u) =& 2u_0 u_2 + {u_1}^2, \\ & \vdots \end{aligned} \tag{3.37}$$

From (3.36) and (3.37), we find that

$$p^0 : u_0(t) = u_0, \tag{3.38}$$

$$\begin{aligned} p^1 : u_1(t) &= S^{-1}\left[v^{\alpha} S\left\{u_0(t)\right\} - v^{\alpha} S\left\{H_0(u)\right\}\right], \\ &= \left(u_0 - u_0^2\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned} \tag{3.39}$$

$$\begin{aligned} p^2 : u_2(t) &= S^{-1}\left[v^{\alpha} S\left\{u_1(t)\right\} - v^{\alpha} S\left\{H_1(u)\right\}\right], \\ &= S^{-1}\left[v^{\alpha} S\left\{\left(u_0 - u_0^2\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right\} - v^{\alpha} S\left\{2u_0 u_1\right\}\right], \\ &= \left(u_0 - 3u_0^2 + 2u_0^3\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{3.40}$$

$$\vdots$$

Therefore, we can write the solution of (3.29) with consideration of (3.30) as

$$
\begin{aligned}
u\left(t\right) &= \sum_{n=0}^{\infty} u_n(t) = u_0(t) + u_1(t) + u_2(t) + u_3(t)...,\\
u\left(t\right) &= u_0 + \left(u_0 - u_0^2\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left(u_0 - 3u_0^2 + 2u_0^3\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad + \left(u_0 - 3u_0^2 + 4u_0^3 - 6u_0^4\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ... \qquad (3.41)
\end{aligned}
$$

For the sake of simplicity, $u_0 = \frac{1}{2}$ and $\alpha = 1$ in equation (3.41), becomes

$$
\begin{aligned}
u\left(t\right) &= \frac{1}{2} + \frac{t}{4} - \frac{t^3}{48} + \frac{t^5}{480} - ...\\
&= \frac{1}{1+e^{-t}}\\
&= \frac{1}{1+E_1(-t)}. \qquad (3.42)
\end{aligned}
$$

Thus, from (3.31) and (3.42), we see that the HPSTM gives an exact solution of the time-fractional logistic equation in the form of the Mittag-Leffler function [55, 139].

3.4.2 Solution of Time-fractional Fornberg-Whitham equation

Considering this nonlinear time-fractional Fornberg-Whitham equation reported by Gupta et al. [58] as follows:

$$u_t^{\alpha} - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx},\ 0 < \alpha \leq 1, t > 0, x \in R, \qquad (3.43)$$

with initial condition

$$u\left(x,0\right) = \frac{4}{3}e^{\frac{x}{2}}. \qquad (3.44)$$

When $u = 1$, the exact solution of (3.43), given by Zhang et al. [169], is

$$u\left(x,t\right) = \frac{4}{3}e^{\left(\frac{x}{2}-\frac{2t}{3}\right)}. \qquad (3.45)$$

Applying, the Sawi trasform to (3.43), we get

$$S\{u_t^\alpha\} - S\{u_{xxt}\} + S\{u_x\} = S\{uu_{xxx}\} - S\{uu_x\} + 3S\{u_x u_{xx}\}, \tag{3.46}$$

using (3.3), we have

$$\left[\frac{1}{v^\alpha} S\{u\,(x,t)\} - \frac{1}{v^{\alpha+1}} u\,(x,0)\right] - S\{u_{xxt}\} + S\{u_x\} = S\{uu_{xxx}\} - S\{uu_x\} + 3S\{u_x u_{xx}\}, \tag{3.47}$$

on considering the initial condition (3.44) gives

$$S\{u\,(x,t)\} = \frac{4}{3v} e^{\frac{x}{2}} + v^\alpha S\left\{u_{xxt} - u_x + uu_{xxx} - uu_x + 3u_x u_{xx}\right\}, \tag{3.48}$$

and hence the inverse Sawi transform of (3.48) holds

$$u\,(x,t) = \frac{4}{3} e^{\frac{x}{2}} + S^{-1}\left[v^\alpha S\left\{u_{xxt} - u_x + uu_{xxx} - uu_x + 3u_x u_{xx}\right\}\right]. \tag{3.49}$$

Employing the homotopy perturbation method to (3.49), gives

$$\sum_{i=0}^{\infty} p^i u_i(t) = \frac{4}{3} e^{\frac{x}{2}} + pS^{-1}\left[v^\alpha S\left\{R\left(\sum_{i=0}^{\infty} p^i u_i\right) - N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right\}\right],$$

in above equation, nonlinear terms are decomposed by He's polynomial as

$$\sum_{i=0}^{\infty} p^i u_i\,(x,t) = \frac{4}{3} e^{\frac{x}{2}} + pS^{-1}\left[v^\alpha S\left\{R\left(\sum_{i=0}^{\infty} p^i u_i\right) + \sum_{i=0}^{\infty} p^i H_i(u)\right\}\right], \tag{3.50}$$

where nonlinear terms of equation (3.49) are solved by He's polynomial $H_i(u)$ [63], as

$$\begin{aligned}
H_0(u) =& u_0 u_{0xxx} - u_0 u_{0x} + 3u_{0x}u_{0xx},\\
H_1(u) =& u_1 u_{0xxx} + u_0 u_{1xxx} - u_1 u_{0x} - u_0 u_{1x} + 3u_{1x}u_{0xx} + 3u_{0x}u_{1xx},\\
H_2(u) =& u_2 u_{0xxx} + 2u_1 u_{1xxx} + u_0 u_{2xxx} - u_2 u_{0x} - 2u_1 u_{1x} - u_0 u_{2x}\\
& + 3u_{2x}u_{0xx} + 6u_{1x}u_{1xx} + 3u_{0x}u_{2xx},\\
& \vdots
\end{aligned} \tag{3.51}$$

From (3.50) and (3.51), we find that

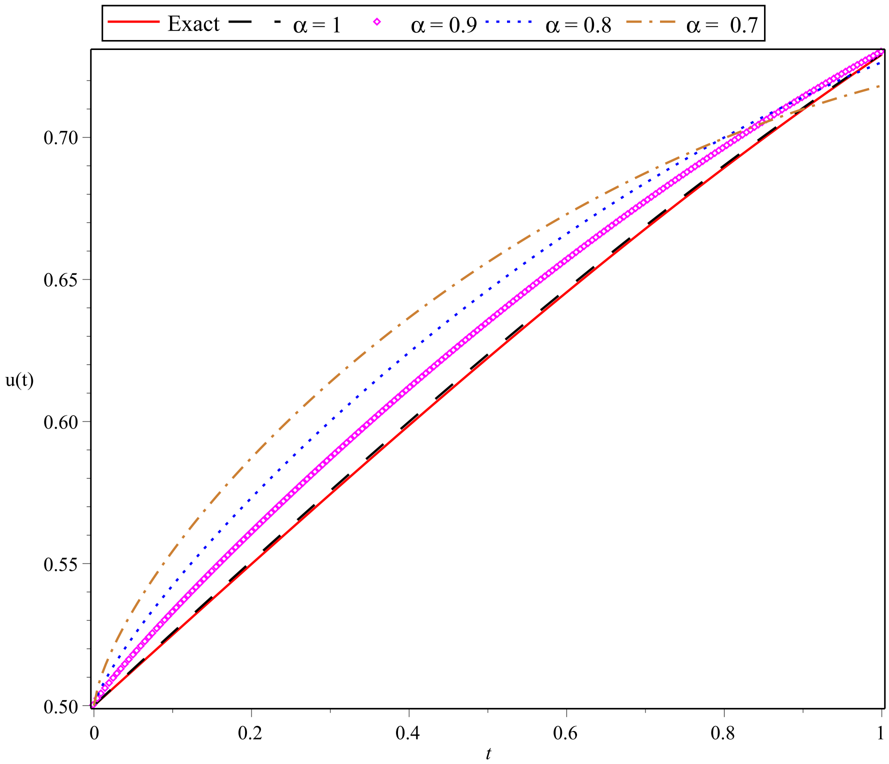
$$\begin{aligned}
p^0 : u_0(x,t) =& \frac{4}{3}e^{\frac{x}{2}},\\
p^1 : u_1(x,t) =& S^{-1}\left[v^\alpha S\left\{Ru_0(x,t)\right\} + v^\alpha S\left\{H_0(u)\right\}\right],\\
=& -\frac{2}{3}e^{\frac{x}{2}}\frac{t^\alpha}{\Gamma(\alpha+1)},\\
p^2 : u_2(x,t) =& S^{-1}\left[v^\alpha S\left\{Ru_1(x,t)\right\} + v^\alpha S\left\{H_1(u)\right\}\right],\\
=& -\frac{1}{6}e^{\frac{x}{2}}\left[\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right],\\
p^3 : u_3(x,t) =& S^{-1}\left[v^\alpha S\left\{Ru_2(x,t)\right\} + v^\alpha\left\{H_2(u)\right\}\right],\\
=& -\frac{1}{24}e^{\frac{x}{2}}\left[\frac{t^\alpha}{\Gamma(\alpha+1)} - 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2}{3}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right],\\
& \vdots
\end{aligned} \tag{3.52}$$

As per (3.5), the approximate solution of (3.44) can be given by

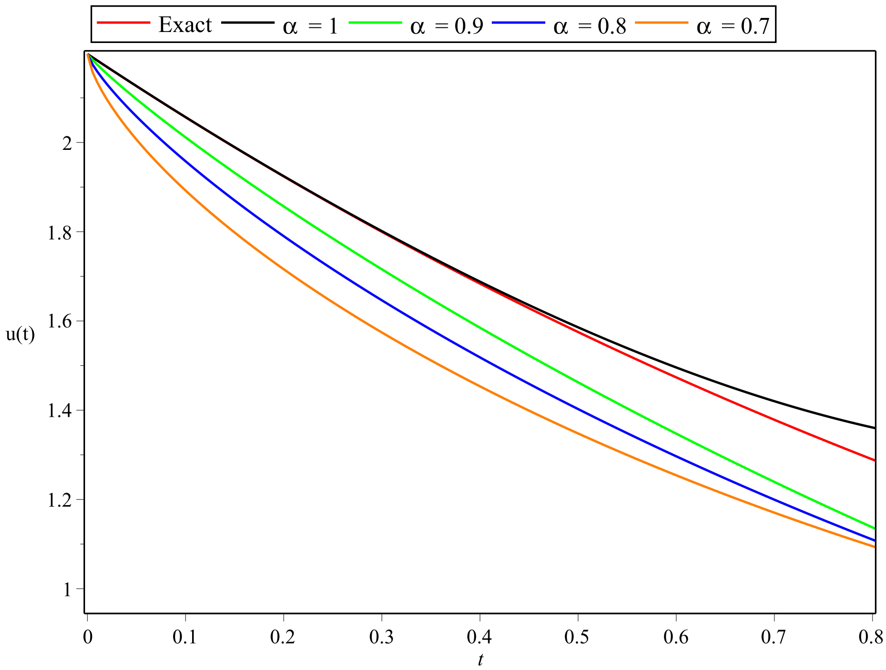
$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t)..., \tag{3.53}$$

which eventually takes the form

$$u(x,t) = \frac{1}{3}e^{\frac{x}{2}}\left[4 - \frac{85}{32}\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{27}{32}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{7}{48}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{1}{3}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - ...\right]. \tag{3.54}$$

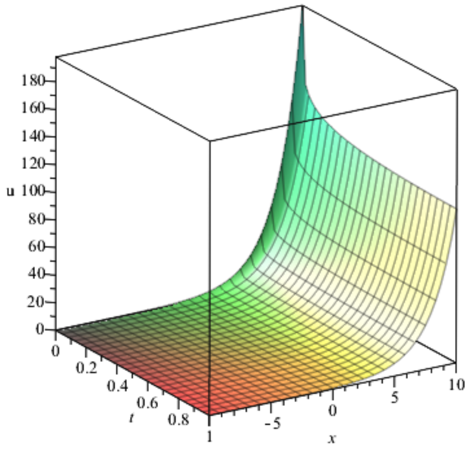


(a) Time-fractional logistic equation

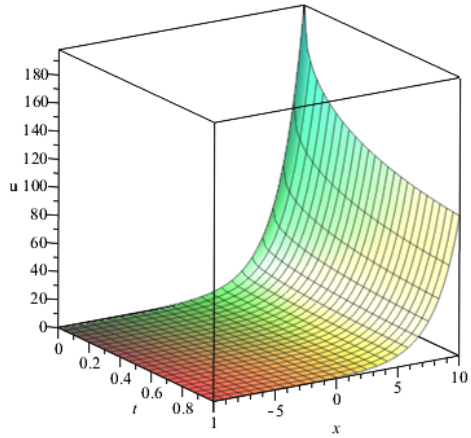


(b) Time-fractional F-W equation at $x = 1$

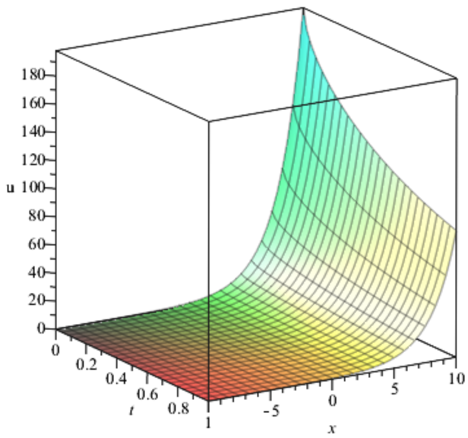
Figure 3.1: Comparision of the solution of the fractional logistic and F-W equation by using the HPSTM method with different fractional values α.



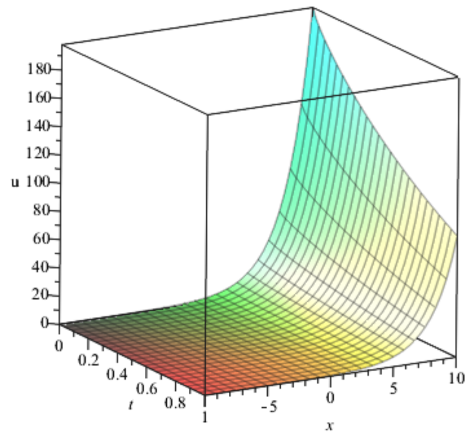
(a) $\alpha = 0.2$



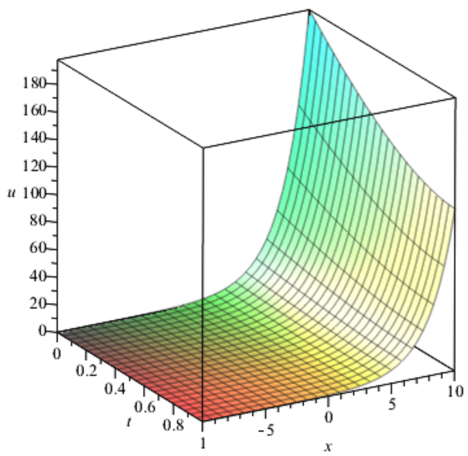
(b) $\alpha = 0.4$



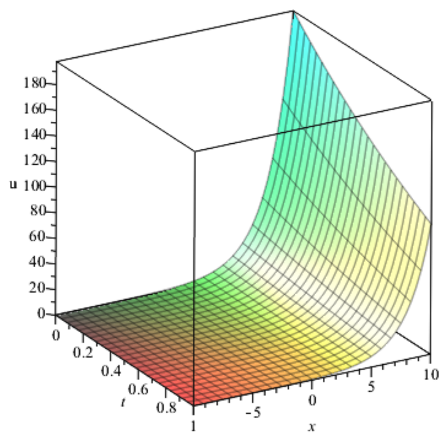
(c) $\alpha = 0.6$



(d) $\alpha = 0.8$



(e) $\alpha = 1$



(f) Exact solution

Figure 3.2: The behaviour of the F-W equation by the HPSTM with different order α.

Table 3.1: The solution of the logistic equation at different fractional order α.

t	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$	$\alpha = 0.04$	$\alpha = 0.05$
0.1	0.7279291460	0.7246049271	0.7212790875	0.7179533744	0.7146295346
0.2	0.7292915794	0.7273104082	0.7253074187	0.7232835548	0.7212397720
0.3	0.7300911275	0.7289034084	0.7276874339	0.7264438047	0.7251731306
0.4	0.7306595666	0.7300382954	0.7293865849	0.7287048401	0.7279934728
0.5	0.7311011391	0.7309212251	0.7307105298	0.7304693304	0.7301979088

Table 3.2: The absolute error in the solution of the F-W equation by the HPSTM and the RPSM [169].

x	t	Exact	HPSTM	RPSM	|Exact-HPSTM|	|Exact-RPSM|
-10	0.1	0.00840452864	0.008406038080	0.008402202315	1.50944E-06	2.32632E-06
	0.2	0.00786249525	0.007865130666	0.007820475300	2.63541E-06	4.20200E-05
	0.3	0.00735541923	0.007361937038	0.007238748282	6.51781E-06	1.16671E-04
	0.4	0.00688104606	0.006898983932	0.006657021266	1.79379E-05	2.24025E-04
	0.5	0.00643726666	0.006480594855	0.006075294249	4.33282E-05	3.61972E-04
-5	0.1	0.10238811940	0.102406508144	0.102359778960	1.83887E-05	2.83404E-05
	0.2	0.09578480094	0.095816906847	0.095272893116	3.21059E-05	5.11908E-04
	0.3	0.08960735032	0.089686753511	0.088186007271	7.94032E-05	1.42134E-03
	0.4	0.08382830210	0.084046830091	0.081099121369	2.18528E-04	2.72918E-03
	0.5	0.07842196221	0.078949807683	0.074012235501	5.27845E-04	4.40973E-03
1	0.1	2.0565203530	2.05688970049	2.05595111975	3.69347E-04	5.69233E-04
	0.2	1.9238891560	1.92453402034	1.91360721245	6.44864E-04	1.02819E-02
	0.3	1.7998117440	1.80140659916	1.77126330515	1.59486E-03	2.85484E-02
	0.4	1.6837364570	1.68812570906	1.62891939670	4.38925E-03	5.48171E-02
	0.5	1.5751472170	1.58574927730	1.48657548890	1.06021E-02	8.85717E-02
5	0.1	15.1957442500	15.19847338630	15.19153816050	2.72914E-03	4.20609E-03
	0.2	14.2157249100	14.22048984063	14.13975104413	4.76493E-03	7.59739E-02
	0.3	13.2989099400	13.31069441818	13.08796392776	1.17845E-02	2.10946E-01
	0.4	12.4412231400	12.47365556636	12.03617680286	3.24324E-02	4.05046E-01
	0.5	11.6388511600	11.71719036881	10.98438968283	7.83392E-02	6.54461E-01

3.5 Results and Conclusion

The semi-analytical solutions of the time-fractional logistic equation and F-W equation are discussed in this chapter using the HPSTM and found that the obtained results are in a good match for the given values of parameters. The comparison of the solution of the time-fractional logistic equation and F-W equation with different fractional order α are given in Figures − 3.1(a) and 3.1(b). Table − 3.1 shows the logistic equation result at some fractional order α. Looking to the above graph, the proposed method gives us the exact solution to the logistic equation. The dynamics of the F-W equation at different fractional orders $\alpha = 0.2$, 0.4, 0.6, and 0.8 along with $\alpha = 1$ and exact solution are shown in Fig.− 3.2(a) to 3.2(f). Further, the comparative output of the F-W equation using present HPSTM, RPSM, and analytic solutions are given in Table − 3.2. The HPSTM can be viewed as a good refinement of the existing HPM method blended with the Sawi transform and can become a popular one with its widespread applicability, reliability and computational ease. Thus, satisfactory results are achieved through the HPSTM, which are more suitable as compared to the RPSM technique in its class. Clearly, the main advantage of the HPSTM is to obtain solutions to complicated problems conveniently.

3.6 A Maple implementation and graphs for Logistic and Fornberg-Whitham Equations

3.6.1 A Maple code with 2D plot for the exact and approximate solution of Logistic equation by HPSTM at some fractional order α

```
pde := diff(u(x, t), t)-u(x, t)+u(x, t)^2 = 0;
```

$$\frac{\partial}{\partial t}u(x,t) - u(x,t) + u(x,t)^2 = 0$$

```
pdsolve(pde);
```

$$u(x,t) = \frac{1}{1 + e^{-t}\_F1(x)}$$

```
ivp := (u(x, 0) =1/2);
```

$$u(x,0) = \frac{1}{2}$$

```
pdsolve([pde, ivp]);
```

$$u(x,t) = \frac{1}{1 + e^{-t}}$$

\# By using HPSTM manually calculated

u := u0+(-u0^2+u0)*t^a/GAMMA(1+a)+(2*u0^3-3*u0^2+u0)*t$^{(2*a)}$/GAMMA(1+2*a) +(-6*u0^4+4*u0^3-3*u0^2+u0)*t$^{(3*a)}$/GAMMA(1+3*a);

$$u0 + \frac{(-u0^2 + u0)\, t^a}{\Gamma(1+a)} + \frac{(2u0^3 - 3u0^2 + u0)\, t^{2a}}{\Gamma(1+2a)} + \frac{(-6u0^4 + 4u0^3 - 3u0^2 + u0)\, t^{3a}}{\Gamma(1+3a)}$$

```
v := eval(u, [u0 = 1/2, a = 1]);
```

$$\frac{1}{2} + \frac{1}{4}t - \frac{1}{48}t^3$$

```
v1 := eval(u, [u0 = 1/2, a = 0.9]);
```

$$0.5 + 0.2599385336t^{0.9} - 0.02997133457t^{2.7}$$

```
v2 := eval(u, [u0 = 1/2, a = 0.8]);
```

$$0.5 + 0.2684178185t^{0.8} - 0.04192933400t^{2.4}$$

```
v3 := eval(u, [u0 = 1/2, a = 0.7]);
```

$$0.5 + 0.2751368514t^{0.7} - 0.05687970813t^{2.1}$$

```
v0 := (1/(1+exp(-t)))
```

$$\left(\frac{1}{(1+e^{-t})}\right)$$

```
plot([v, v0, v1, v2, v3], t = 0..1, color = ["red", "black", "magenta", "blue",
"gold"]);
```

See figure – (3.1a)

3.6.2 A Maple code with 2D plot for the exact and approximate solution of F-W equation by HPSTM at some fractional order α

```
pde := diff(u(x,t),t)-(diff(u(x,t),x,x,t))-u(x,t)*(diff(u(x,t),x,x,x)
+ u(x,t)*(diff(u(x,t),x))+diff(u(x,t),x) - 3*(diff(u(x,t),x))*(diff(u(x,t),x,x))
= 0;
```

$$\frac{\partial}{\partial t}u(x,t) - \left(\frac{\partial^3}{\partial x^2 \partial t}u(x,t)\right) + \frac{\partial}{\partial x}u(x,t) - u(x,t)\left(\frac{\partial^3}{\partial x^3}u(x,t)\right) + u(x,t)\left(\frac{\partial}{\partial x}u(x,t)\right) - 3\left(\frac{\partial}{\partial x}u(x,t)\right)\left(\frac{\partial^2}{\partial x^2}u(x,t)\right) = 0$$

ivp := u(x, 0) = (4/3)*e$^{((1/2)*x)}$;

$$u(x,0) = \frac{4}{3}e^{\frac{1}{2}x}$$

```
pdsolve([pde, ivp]);
```

$$\frac{4}{3}e^{\frac{1}{2}x-\frac{2}{3}t}$$

\# By using HPSTM solution of F-W equation solved manually

u := (1/3*(4-(85/32)*t^{a}/GAMMA(1+a)+(27/32)*t$^{(2*a)}$/GAMMA(1+2*a)
-(7/48)*t$^{(3*a)}$/GAMMA(1+3*a)+(1/3)*t$^{(4*a)}$/GAMMA(1+4*a)))*exp((1/2)*x);

$$\frac{1}{3}\left(4 - \frac{85}{32}\frac{t^a}{\Gamma(1+a)} + \frac{27}{32}\frac{t^{2a}}{\Gamma(1+2a)} - \frac{7}{48}\frac{t^{3a}}{\Gamma(1+3a)} + \frac{1}{3}\frac{t^{4a}}{\Gamma(1+4a)}\right)e^{\frac{1}{2}x}$$

```
u1 := eval(u, [x = 1, a = 1]);
```

$$\frac{1}{3}\left(4-\frac{85}{32}t+\frac{27}{64}t^{2}-\frac{7}{288}t^{3}+\frac{1}{72}t^{4}\right)e^{\frac{1}{2}}$$

```
u2 := eval(u, [x = 1, a = 0.9]);
```

$$\frac{1}{3}\left(4-2.761846918t^{0.9}+0.5032834096t^{1.8}-0.03496655700t^{2.7}+0.02491041119t^{3.6}\right)e^{\frac{1}{2}}$$

```
u3 := eval(u, [x = 1, a = 0.8]);
```

$$\frac{1}{3}\left(4-2.851939322t^{0.8}+0.5901899171t^{1.6}-0.04891755633t^{2.4}+0.04297365930t^{3.2}\right)e^{\frac{1}{2}}$$

```
u4 := eval(u, [x = 1, a = 0.7]);
```

$$\frac{1}{3}\left(4-2.923329045t^{0.7}+0.6792552105t^{1.4}-0.06635965949t^{2.1}+0.07101000490t^{2.8}\right)e^{\frac{1}{2}}$$

```
u0 := eval((4/3)*exp((1/2)*x-2*t*(1/3)), [x = 1]);
```

$$\frac{4}{3}e^{\frac{1}{2}-\frac{2}{3}t}$$

```
plot([u0, u1, u2, u3, u4], t = 0..1, color = ["red", "black", "green", "blue",
"coral"]);
```

See figure – (3.1b)

3.6.3 A Maple code for the solution of F-W equation by HP-STM technique which represent the 3D plot at different fractional order α

```
u := (1/3*(4-(85/32)*t^a/GAMMA(1+a)+(27/32)*t^(2*a)/GAMMA(1+2*a)
-(7/48)*t^(3*a)/GAMMA(1+3*a)+(1/3)*t^(4*a)/GAMMA(1+4*a)))*exp((1/2)*x);
```

$$\frac{1}{3}\left(4-\frac{85}{32}\frac{t^{a}}{\Gamma(1+a)}+\frac{27}{32}\frac{t^{2a}}{\Gamma(1+2a)}-\frac{7}{48}\frac{t^{3a}}{\Gamma(1+3a)}+\frac{1}{3}\frac{t^{4a}}{\Gamma(1+4a)}\right)e^{\frac{1}{2}x}$$

```
u4 := eval(u, [a = 0.2]);
```

$$\frac{1}{3}\left(4-2.892986743t^{0.2}+0.9509572952t^{0.4}-0.1632130141t^{0.6}+0.3578904247t^{0.8}\right)\mathrm{e}^{\frac{1}{2}x}$$

```
plot3d(u4, t = 0..1, x = -10..10);
```

See figure – (3.2a)

```
u3 := eval(u, [a = 0.4]);
```

$$\frac{1}{3}\left(4-2.993754448t^{0.4}+0.9059101374t^{0.8}-0.1323588706t^{1.2}+0.2331614487t^{1.6}\right)\mathrm{e}^{\frac{1}{2}x}$$

```
plot3d(u3, t = 0..1, x = -10..10);
```

See figure – (3.2b)

```
u2 := eval(u, [a = 0.6]);
```

$$\frac{1}{3}\left(4-2.972808472t^{0.6}+0.7657906085t^{1.2}-0.08698725598t^{1.8}+0.1118115573t^{2.4}\right)\mathrm{e}^{\frac{1}{2}x}$$

```
plot3d(u2, t = 0..1, x = -10..10);
```

See figure – (3.2c)

```
u1 := eval(u, [a = 0.8]);
```

$$\frac{1}{3}\left(4-2.851939322t^{0.8}+0.5901899171t^{1.6}-0.04891755633t^{2.4}+0.04297365930t^{3.2}\right)\mathrm{e}^{\frac{1}{2}x}$$

```
plot3d(u1, t = 0..1, x = -10..10);
```

See figure – (3.2d)

```
u0 := eval(u, [a = 1]);
```

$$\frac{1}{3}\left(4-\frac{85}{32}t+\frac{27}{64}t^2-\frac{7}{288}t^3+\frac{1}{72}t^4\right)\mathrm{e}^{\frac{1}{2}x}$$

```
plot3d(u0, t = 0..1, x = -10..10);
```

See figure – (3.2e)

```
v := (4/3)*exp((1/2)*x-2*t*(1/3));
```

$$\frac{4}{3}e^{\frac{1}{2}x-\frac{2}{3}t}$$

```
plot3d(v, t = 0..1, x = -10..10);
```

See figure – (3.2f)