Chapter 2

Fractional Korteweg-de Vries and Kawahara equations

2.1 Introduction

In the present work, we consider the Kortweg-de Vries (KdV) and Kawahara equations in the form of a fractional partial differential equation (FPDE) and use the Fractional Residual Power Series Method (FRPSM) to obtain the semi-analytic solution. The significance of studying this celebrated equation in the fractional form lies in the fact that using the fractional approach we get more realistic results in real-time situations, as compared to conventional derivatives of integer order.

In 1877, Joseph Valentin Boussinesq [32] initiated theoretical investigations on solitary waves induced on shallow water; later in 1895, Diederik Korteweg and Gustav de Vries (Dutch Mathematicians) has retrieved the weakly non-linear partial differential equation (popularly known as KdV equation) and presented a mathematical model illustrating the wave of shallow water surfaces [95]. In a classical sense, the Korteweg-de Vries (KdV) is a non-linear partial differential equation of order three, given as

$$u_t + 6uu_x + u_{xxx} = 0\,, \tag{2.1}$$

where $u(x,t)$ denotes the elongation of the wave at place x and time t.

In 1972, Kawahara [38] was the first to propose the partial differential equation describing the behavior of solitary waves as follows,

$$u_t + au^m u_x + bu_{xxx} - \lambda u_{xxxxx} = 0, \tag{2.2}$$

where a, b and λ are some specific arbitrary constants. Equation (2.2) arises while modeling wave theory, and scales down to Kortewag-de Vries (KdV) equation [146, 91] for $b = 1$ and $\lambda = 0$. Moreover, the second term of (2.2) is the convective part and the third term is the dispersive part. Karpman and Vanden-Brock [80] proposed that the fifth order term of equation (2.2) shows critical significance for solitary stability.

Easiest way to obtain solutions of non-linear differential equations using the power series method was first proposed by Jordan mathematician Arqub [2, 22] in 2013. Several attempts have been made to solve fractional differential equations (FDEs) and Fractional Partial Differential Equations (FPDEs). FRPSM intends to determine an exact and estimated solution in a semi-analytical way for the fractional physical equations. FRPSM leads to a closed-form solution of several well-known functions as it is working in a well-organized and competent way to solve ordinary and partial differential equations. To solve non-linear time-dependent FPDEs, the FRPSM is one of the dominant techniques, which is established by the generalized formula of the Taylor series. The different types and orders of non-linear FDEs can be solved by the FRPSM effectively. It constructs Residual power series expansion without doing discretization, linearization, or perturbation.

In many fields, the FRPSM introduced by some authors, viz. Tariq et al. [152] obtained an analytic approximate solution to non-linear temporal conformable fractional foam drainage equation, Kumar et al. [97] investigated the approximate analytical solution of fractional Bi-Hamiltonian Boussinesq system, Jena et al. [74] investigated time-fractional fuzzy vibration equation of large membranes. Whereas in 2020, Hasan et al. [60] introduced a solution for linear time fractional Swift-Hohenberg equation, Khalauta et al. [81] presented fractional Bratu-type equation, Dunnimit et al. [45] gave analytic approach to deal with fractional logistic equations.

In perspective on the above literature, the next section comprises of preliminaries followed by an overview of FRPSM and its application to solve the time-fractional KdV and Kawahara equations. Further, the comparison of results with ADM, VIM, and HPM are given. The last section gives details about the results and discussion related to the stability of the solutions obtained.

2.2 Preliminaries

Definition 2.1. [46] A fractional residual power series (FRPS) expansion about $t = t_0$ is expressed in the form

$$\sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + ..., 0 \leq n-1 < \alpha \leq n,\ t \geq t_0. \quad (2.3)$$

Theorem 2.1. [46] Let the FRPS representation at $t = t_0$ for the function u of the form

$$u(t) = \sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha}, t_0 \leq t < t_0 + R \quad (2.4)$$

where R is the radius of convergence.

If $D^{m\alpha}u(t)$, $m = 0, 1, 2, 3, ...$ are continuous on (t_0, t_0+R) then the coefficient c_m are given by the formula $c_m = \frac{D^{m\alpha}u(t_0)}{\Gamma(1+m\alpha)}$, $m = 0, 1, 2, ...$ where $D^{m\alpha} = D^{\alpha}. D^{\alpha}. D^{\alpha} ... D^{\alpha} (m\,\text{times})$.

Theorem 2.2. [46] Let the FRPS representation at $t = t_0$ for the function u is expressed by the form

$$u(x,t) = \sum_{m=0}^{\infty} f_m(x)(t-t_0)^{m\alpha}, \text{where } x \in \mathbb{R}\,, t_0 \leq t < t_0 + R, 0 \leq n-1 < \alpha \leq n. \quad (2.5)$$

If $D^{m\alpha}u(x,t)$, $m = 0, 1, 2, 3, ...$ are continuous on $\mathbb{R} \times (t_0, t_0 + R)$ then the coefficient c_m are given by $f_m(x) = \frac{D^{m\alpha}u(x,t_0)}{\Gamma(1+m\alpha)}$, $m = 0, 1, 2, ...$ where

$$D^{m\alpha} = D^{\alpha} \cdot D^{\alpha} \cdot D^{\alpha} \cdot \cdots \cdot D^{\alpha}\ (m\,\text{times}).$$

Thus, the generalized Taylor series formula for fractional residual power series at $t = t_0$ can be expressed as

$$u(x,t) = \sum_{m=0}^{\infty} \frac{D^{m\alpha}u(x,t_0)}{\Gamma(1+m\alpha)}(t-t_0)^{m\alpha}, \tag{2.6}$$

where $x \in \mathbb{R}$, $t_0 \le t < t_0 + R$, $0 \le m-1 < \alpha \le m$.

If $\alpha = 1$, then classical Taylor series is obtained as

$$u(x,t) = \sum_{m=0}^{\infty} \frac{D^{m}u(x,t_0)}{\Gamma(1+m)}(t-t_0)^{m}, \text{where } x \in \mathbb{R}\,,\ t_0 \le t < t_0 + R. \tag{2.7}$$

Corollary 2.1. Suppose that $u(x,y,t)$ has a multiple fractional residual power series representation at $t = t_0$ of the form

$$u(x,y,t) = \sum_{m=0}^{\infty} f_m(x,y)(t-t_0)^{m\alpha}, \tag{2.8}$$

where $(x,y) \in \mathbb{R} \times \mathbb{R}\,, t_0 \le t < t_0 + R, 0 \le m-1 < \alpha \le m$.

If $D^{m\alpha}u(x,y,t)$, $m = 0,1,2,3,...$are continuous on $\mathbb{R}\times\mathbb{R}\times(t_0, t_0+R)$ then the coefficient c_m are given by

$$f_m(x,y) = \frac{D^{m\alpha}u(x,y,t_0)}{\Gamma(1+m\alpha)},\ m = 0,1,2,...$$

2.3 Analysis of Fractional Residual Power Series Method

To illustrate the essential concept of FRPSM, the generalized fractional differential equation of non-linear form is considered as follows:

$$D_t^{\alpha}[u(x,t)] = R(u) + N(u), 0 \le m-1 < \alpha \le m. \tag{2.9}$$

where $R(u)$ and $N(u)$ are linear and non-linear terms respectively, subject to initial conditions

$$D_t^{n\alpha}\left[u\left(x,0\right)\right] = f_n\left(x\right), \left(n = 0,1,2,...\right); \text{ where with } n=0, \text{ it is } u(x,0) = f_0(x) = f(x). \tag{2.10}$$

The FRPSM presents the solution for (2.9) at t = 0, which is

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \text{ where } x \in \mathbb{R}\,,\ 0 \leq t < R,\ 0 < \alpha \leq 1. \tag{2.11}$$

Let $u_k(x,t)$ denote as $k^{th}-$ truncated series

$$u_k(x,t) = \sum_{n=0}^{k} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \tag{2.12}$$

where $x \in \mathbb{R}\,,\ 0 \leq t < R,\ 0 < \alpha \leq 1,\ k = 1, 2, 3, ...$

The solution of FDE (2.9) namely $u(x,t)$ satisfies the initial conditions as given in (2.10). Moreover, applying $t = 0$ in equation (2.11), we obtain

$$u_0(x,0) = u(x,0) = f_0(x) = f(x). \tag{2.13}$$

Using series (2.12) for $k = 1$, we have

$$u_1(x,t) = f_0(x) + f_1(x) \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{2.14}$$

and in general

$$u_k(x,t) = f_0(x) + f_1(x) \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \sum_{n=2}^{k} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \tag{2.15}$$

where $k = 2, 3, 4, ...$

Subsequently, using FRPSM we can evaluate $f_n(x)$, $n = 1, 2, 3, ..., k$ in the equation (2.15).

Now, we define the residual function to generalized FDE (2.9) as

$$\text{Res}\, u(x,t) = D_t^{\alpha}[u(x,t)] - R(u) - N(u). \tag{2.16}$$

Thus, $k^{th}-$ Residual function is

$$\text{Res } u_k(x,t) = D_t^{\alpha}[u_k(x,t)] - R(u_k) - N(u_k). \tag{2.17}$$

As mentioned in El-Ajouh et al.[46], we can easily see that

$$\lim_{k\to\infty} \text{Res } u_k(x,t) = \text{Res } u(x,t) = 0.$$

$$D_t^{n\alpha}\left[\text{Res}\, u(x,t)\right] = 0. \tag{2.18}$$

In Caputo sense, fractional differentiation is

$$\mathrm{D}_t^{n\alpha}\left[\text{Res } u(x,0)\right] = \mathrm{D}_t^{n\alpha}\left[\text{Res } u_k(x,0)\right] = 0; \quad n = 0, 1, 2, ..., k. \tag{2.19}$$

To evaluate $f_i(x)$ where $i = 1, 2, ...,$ we calculate for $k = 1, 2, ...,$ in (2.15) then replace it in (2.17), taking $D_t^{(k-1)\alpha}$on both the sides, we have $f_i(x)$ where $i = 1, 2, ...,$ using

$$D_t^{(k-1)\alpha}\left[\text{Res } u_k(x,0)\right] = 0, \quad k = 1, 2, 3, ... \tag{2.20}$$

Substitution of $f_i(x)$, $i = 1, 2, ...,$ in (2.11) provide us with the series solution of (2.9).

2.3.1 Convergence Analysis of FRPSM

Theorem 2.3. [46] For any $\sum_{m=0}^{\infty} f_m(x)t^{m\alpha}, t \geq t_0$, there exist three possibilities

(i) The series converges only when $t = 0$,

(ii) The series converges for each $t \geq 0$,

(iii) There is a positive real number R such that the series converges whenever $0 \leq t < R$ and diverges whenever $t > R$.

The number R in case-3 is Radius of Convergence of the fractional residual power series (FRPS). By convention $R = 0$ in case-1 and $R \to \infty$ in case-2.

Theorem 2.4. [46] The power series $\sum_{m=0}^{\infty} f_m(x)t^{m\alpha}, -\infty < t < \infty$ has radius of convergence R, if and only if the FRPS, $\sum_{m=0}^{\infty} f_m(x)t^{m\alpha}, t \geq 0$ has radius of convergence $R^{1/\alpha}$. Here, radius of convergence $R = \lim_{m\to\infty} \left|\frac{f_m}{f_{m+1}}\right|$.

2.4 Application of the FRPSM and Numerical Discussions

2.4.1 Solution of Fractional KdV Equation

To demonstrate the one-dimensional non-linear homogeneous time fractional KdV (Korteweg-de Vries) equation, we used the concept of FRPSM, let us examine the question.

$$D_t^{\alpha}[u(x,t)] + 6uu_x + u_{xxx} = 0, \tag{2.21}$$

with initial conditions

$$u(x,0) = \frac{1}{2}\text{sech}^2\left(\frac{x}{2}\right). \tag{2.22}$$

The exact solution[163] of KdV equation is given by

$$u(x,t) = \frac{1}{2}\text{sech}^2\left(\frac{t+x}{2}\right). \tag{2.23}$$

Explicating the residual function for (2.21) as

$$\text{Res } u(x,t) = D_t^{\alpha}u(x,t) + 6uu_x + u_{xxx}. \tag{2.24}$$

Thus, $k^{th}-$ Residual function Res $u_k(x,t)$,

$$\text{Res } u_k(x,t) = D_t^{\alpha}u_k(x,t) + 6u_k u_{kx} + u_{kxxx}. \tag{2.25}$$

For $k = 1$, equations (2.15) and (2.25) yields

$$\begin{aligned}\text{Res } u_1(x,t) =& f_1 + 6ff_x + f_{xxx} + (6ff_{1x} + 6f_1f_x + f_{1xxx})\frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ &+ 6f_1f_{1x}\frac{t^{2\alpha}}{\Gamma(1+\alpha)^2}.\end{aligned} \tag{2.26}$$

Using initial condition (2.20), we have

$$f_1(x) = \frac{1}{2} \tanh\left(\frac{x}{2}\right) \operatorname{sech}^2\left(\frac{x}{2}\right). \tag{2.27}$$

Similarly, for $k = 2$ we get

$$\text{Res } u_2(x,t) = D_t^\alpha u_2 + 6u_2 u_{2x} + u_{2xxx}. \tag{2.28}$$

Now, from (2.15) at $k = 2$,

$$u_2(x,t) = f(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{2.29}$$

Residual function Res $u_2(x,t)$ is given by

$$\begin{aligned}\text{Res } u_2(x,t) =& f_1 + 6ff_x + f_{xxx} + (f_2 + 6ff_{1x} + 6f_1 f_x + f_{1xxx})\frac{t^\alpha}{\Gamma(1+\alpha)} \\ &+ (6ff_{2x} + 6f_2 f_x + f_{2xxx})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + 6f_1 f_{1x}\frac{t^{2\alpha}}{\Gamma(1+\alpha)^2} \\ &+ (6f_1 f_{2x} + 6f_2 f_{1x})\frac{t^{3\alpha}}{\Gamma(1+\alpha)(1+2\alpha)} + f_2 f_{2x}\frac{t^{4\alpha}}{\Gamma(1+2\alpha)^2}.\end{aligned} \tag{2.30}$$

Taking D_t^α on both side and calculating the equation $D_t^\alpha\left[\text{Res}u_2(x,0)\right] = 0$,

$$f_2 = -6ff_{1x} - 6f_1 f_x - f_{1xxx}. \tag{2.31}$$

Implementing the known values, we achieve

$$f_2(x) = \frac{1}{4}\operatorname{sech}^2\left(\frac{x}{2}\right)\left(2 - 3\operatorname{sech}^2\left(\frac{x}{2}\right)\right). \tag{2.32}$$

And for $k = 3$,

$$\text{Res }[u_3(x,t)] = D_t^\alpha u_3 + 6u_3 u_{3x} + u_{3xxx}. \tag{2.33}$$

Now, from (2.15) at $k = 3$,

$$u_3(x,t) = f(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}. \tag{2.34}$$

Residual function Res $u_3(x,t)$ is given by

$$\begin{aligned}\text{Res } u_3(x,t) =& f_1 + 6ff_x + f_{xxx} + (f_2 + 6ff_{1x} + 6f_1f_x + f_{1xxx})\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\ &+ (f_3 + 6ff_{2x} + 6f_2f_x + f_{2xxx})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\ &+ 6f_1f_{1x}\frac{t^{2\alpha}}{\Gamma(1+\alpha)^2} + (6ff_{3x} + 6f_3f_x + f_{3xxx})\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\\ &+ (6f_1f_{2x} + 6f_2f_{1x})\frac{t^{3\alpha}}{\Gamma(1+\alpha)(1+2\alpha)} + 6f_2f_{2x}\frac{t^{4\alpha}}{\Gamma(1+2\alpha)^2}\\ &+ (6f_3f_{1x} + 6f_1f_{3x})\frac{t^{4\alpha}}{\Gamma(1+\alpha)(1+3\alpha)}\\ &+ (6f_2f_{3x} + 6f_3f_{2x})\frac{t^{5\alpha}}{\Gamma(1+2\alpha)(1+3\alpha)} + 6f_3f_{3x}\frac{t^{6\alpha}}{\Gamma(1+3\alpha)^2}.\end{aligned} \tag{2.35}$$

Taking $D_t^{2\alpha}$ on both side and calculating the equation $D_t^{2\alpha}\left[\text{Res}u_3(x,0)\right] = 0$, then we get

$$f_3 = -6ff_{2x} - 6f_2f_x - f_{2xxx} - 6f_1f_{1x}. \tag{2.36}$$

Implementing the known values, we achieve

$$f_3(x) = \frac{1}{2}\tanh\left(\frac{x}{2}\right)\text{sech}^2\left(\frac{x}{2}\right)\left(1 - 3\text{sech}^2\left(\frac{x}{2}\right)\right). \tag{2.37}$$

Similarly, for $k = 4$ we have

$$f_4 = -6ff_{3x} - 18f_2f_{1x} - f_{3xxx} - 18f_1f_{2x} - 6f_3f_x. \tag{2.38}$$

with

$$f_4(x) = \frac{1}{2}\text{sech}^2\left(\frac{x}{2}\right) - \frac{15}{4}\text{sech}^4\left(\frac{x}{2}\right) + \frac{15}{4}\text{sech}^6\left(\frac{x}{2}\right). \tag{2.39}$$

Subsequently, values of $f_5, f_6, ...$can be obtained.

Now, substituting values of $f, f_1, f_2, ...$ in equation (2.15), $f(x,t)$ is expressed in terms of series as

$$
\begin{aligned}
u(x,t) =& \frac{1}{2}\text{sech}^2\left(\frac{x}{2}\right) + \frac{1}{2}\tanh\left(\frac{x}{2}\right)\text{sech}^2\left(\frac{x}{2}\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} \\
&+ \frac{1}{4}\text{sech}^2\left(\frac{x}{2}\right)\left(2 - 3\text{sech}^2\left(\frac{x}{2}\right)\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&+ \frac{1}{2}\tanh\left(\frac{x}{2}\right)\text{sech}^2\left(\frac{x}{2}\right)\left(1 - 3\text{sech}^2\left(\frac{x}{2}\right)\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\
&+ \left(\frac{1}{2}\text{sech}^2\left(\frac{x}{2}\right) - \frac{15}{4}\text{sech}^4\left(\frac{x}{2}\right) + \frac{15}{4}\text{sech}^6\left(\frac{x}{2}\right)\right)\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \ldots
\end{aligned} \tag{2.40}
$$

By considering the different values of $\alpha \in (0,1]$, corresponding values of $u(x,t)$ are discussed in Table (2.6).

Table 2.1: The Absolute Error in Solution of KdV Equation by FRPSM Method and ADM Method [150] when $x = -20$ and $\alpha = 1$.

t	$\lvert u_{exact} - u_{rpsm}\rvert$	$\lvert u_{exact} - u_{adm}\rvert$
0.1	8.25131E-10	3.73002E-09
0.2	1.65416E-09	3.37506E-09
0.3	2.49062E-09	3.05388E-09
0.4	3.33775E-09	2.76327E-09
0.5	4.19844E-09	2.50031E-09

Table 2.2: The Absolute Error in Solution of KdV Equation by FRPSM Method and ADM Method [150] when $x = -10$ and $\alpha \to 1$.

t	$\lvert u_{exact} - u_{rpsm}\rvert$	$\lvert u_{exact} - u_{adm}\rvert$
0.1	2.86113E-05	8.21524E-05
0.2	3.64285E-05	7.43351E-05
0.3	5.48495E-05	6.72617E-05
0.4	7.35051E-05	6.08613E-05
0.5	9.24593E-05	5.50699E-05

Table 2.3: The Absolute Error in Solution of KdV Equation by FRPSM Method and ADM Method [150] when $x = 0$ and $\alpha \to 1$.

t	$\lvert u_{exact} - u_{rpsm}\rvert$	$\lvert u_{exact} - u_{adm}\rvert$
0.1	2.90000E-09	2.08040E-06
0.2	1.87933E-07	3.31454E-05
0.3	2.12660E-06	1.66623E-04
0.4	1.18419E-05	5.21491E-04
0.5	-4.46589E-05	1.25742E-03

Table 2.4: The Absolute Error in Solution of KdV Equation by FRPSM Method and ADM Method [150] when $x = 10$ and $\alpha \to 1$.

t	$\lvert u_{exact} - u_{rpsm}\rvert$	$\lvert u_{exact} - u_{adm}\rvert$
0.1	1.59158E-05	1.00339E-04
0.2	3.64406E-05	1.10891E-04
0.3	5.49109E-05	1.22552E-04
0.4	7.36995E-05	1.35439E-04
0.5	9.29355E-05	1.49681E-04

Table 2.5: The Absolute Error in Solution of KdV Equation by FRPSM Method and ADM Method [150] when $x = 20$ and $\alpha \to 1$.

t	$\lvert u_{exact} - u_{rpsm}\rvert$	$\lvert u_{exact} - u_{adm}\rvert$
0.1	8.25166E-10	4.55585E-09
0.2	1.65471E-09	5.03500E-09
0.3	2.49341E-09	5.56453E-09
0.4	3.34659E-09	6.14976E-09
0.5	4.22009E-09	6.79654E-09

Table 2.6: Value of $u(x,t)$ with different fractional order α when $x = 2$.

t	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0.1	0.354750978	0.29496648	0.259395686	0.238566145	0.226368188
0.2	0.382232355	0.327692525	0.28835476	0.261507702	0.243526238
0.3	0.400996543	0.353185341	0.313530228	0.283441821	0.261461322
0.4	0.415702913	0.375006055	0.336761221	0.305029944	0.280173439
0.5	0.427988593	0.394504163	0.358769892	0.326536438	0.299662589

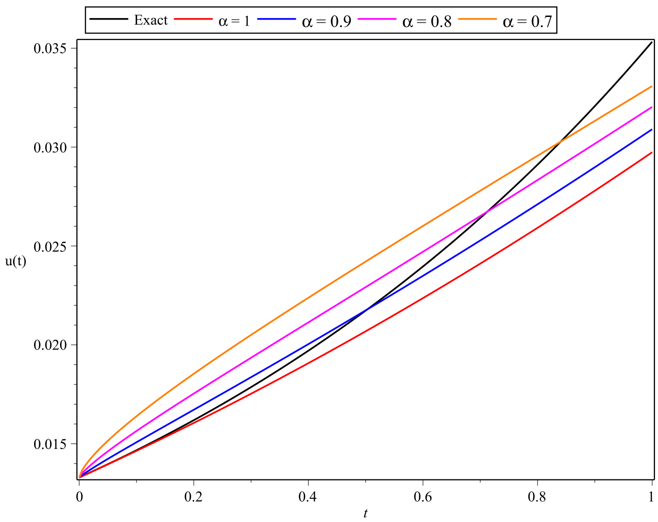
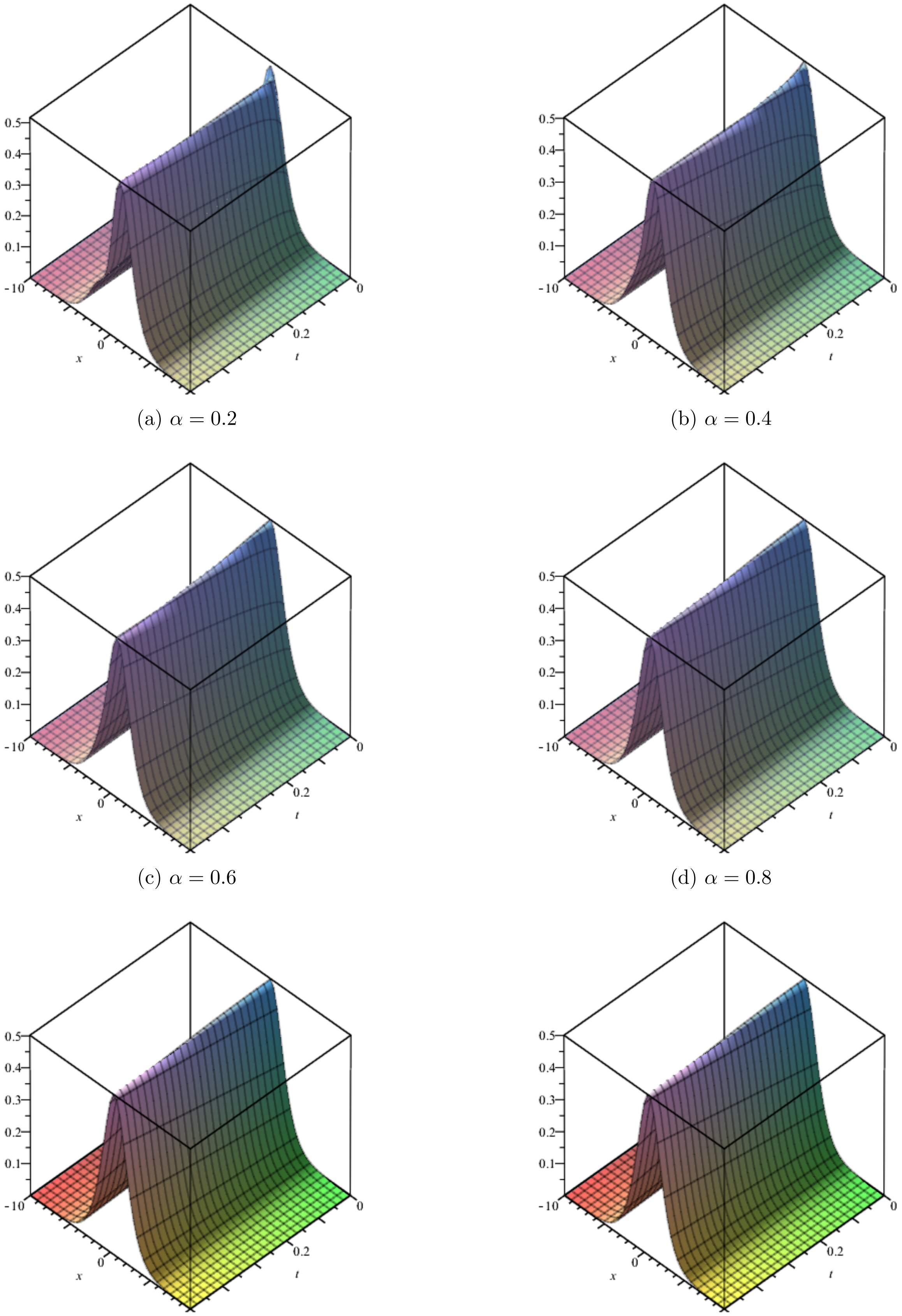


Figure 2.1: Comparison of fractional KdV equation with different value of α with exact solution.



(a) $\alpha = 0.2$ (b) $\alpha = 0.4$

(c) $\alpha = 0.6$ (d) $\alpha = 0.8$

(e) $\alpha = 1$ (f) Exact solution

Figure 2.2: Solution of KdV equation by FRPSM with different order α.

2.4.2 Solution of fractional Kawahara type - 1 equation

To demonstrate the non-linear homogeneous time-fractional Kawahara equation with $a = 1, b = 1, m = 1$ and $\lambda = 1$ in (2.2). We used the concept of FRPSM, let us examine the equation

$$D_t^{\alpha}[u(x,t)] + uu_x + u_{xxx} - u_{xxxxx} = 0, 0 < \alpha \leq 1, t > 0, x \in R, \tag{2.41}$$

with initial conditions

$$u(x,0) = \frac{105}{169}\text{sech}^4\left(\frac{x}{2\sqrt{13}}\right). \tag{2.42}$$

The exact solution [102] at ($\alpha = 1$) of (2.41), is given by

$$u(x,t) = \frac{105}{169}\text{sech}^4\left(\frac{1}{2\sqrt{13}}\left(x - \frac{36}{169}t\right)\right). \tag{2.43}$$

Explicating the residual function for (2.41) as

$$\text{Res}[u(x,t)] = D_t^{\alpha}[u(x,t)] + uu_x + u_{xxx} - u_{xxxxx} = 0, \tag{2.44}$$

thus, for the k^{th} residual function $\text{Res}[u_k(x,t)]$,

$$\text{Res}[u_k(x,t)] = D_t^{\alpha}[u_k] + u_k u_{kx} + u_{kxxx} - u_{kxxxxx} = 0. \tag{2.45}$$

For $k = 1$, equations (2.12) and (2.45) yields

$$\begin{aligned}\text{Res}[u_1(x,t)] =& f_1 + ff_x + f_{xxx} - f_{xxxxx} + (ff_{1x} + f_1 f_x + f_{1xxx} \\ &- f_{1xxxxx})\frac{t^{\alpha}}{\Gamma(\alpha+1)} + f_1 f_{1x}\frac{t^{2\alpha}}{\Gamma(\alpha+1)^2},\end{aligned} \tag{2.46}$$

using initial condition (2.42), we have

$$f_1(x) = \frac{7560}{28561\sqrt{13}}\tanh\left(\frac{x}{2\sqrt{13}}\right)\text{sech}^4\left(\frac{x}{2\sqrt{13}}\right). \tag{2.47}$$

Similarly, for $k = 2$ we have

$$\text{Res}[u_2(x,t)] = D_t^\alpha u_2 + u_2 u_{2x} + u_{2xxx} - u_{2xxxxx} = 0. \tag{2.48}$$

Now, from (2.12) at $k = 2$,

$$u_2(x,t) = f(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{2.49}$$

and

$$\begin{aligned}\text{Res}[u_2(x,t)] =& f_1 + ff_x + f_{xxx} - f_{xxxxx} \\ &+ (f_2 + ff_{1x} + f_1 f_x + f_{1xxx} - f_{1xxxxx})\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &+ (f_2 f_x + f_x f_{2x} + f_{2xxx} - f_{2xxxxx})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f_1 f_{1x}\frac{t^{2\alpha}}{\Gamma(\alpha+1)^2} \\ &+ (f_2 f_{1x} + f_{2x} f_1)\frac{t^{3\alpha}}{\Gamma(\alpha+1)(2\alpha+1)} + f_2 f_{2x}\frac{t^{4\alpha}}{\Gamma(2\alpha+1)^2}.\end{aligned} \tag{2.50}$$

Taking D_t^α on both side and calculating the equation $D_t^\alpha(\text{Res}[u_2(x,0)]) = 0$, then we get

$$f_2 = -ff_{1x} - f_1 f_x - f_{1xxx} + f_{1xxxxx}. \tag{2.51}$$

Implementing the known values, we achieve

$$f_2(x) = \frac{68040}{62748517}\text{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\left(4 - 5\text{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right). \tag{2.52}$$

And, for $k = 3$ we have

$$\text{Res}[u_3(x,t)] = D_t^\alpha u_3 + u_3 u_{3x} + u_{3xxx} - u_{3xxxxx} = 0. \tag{2.53}$$

Now, from (2.12) at $k = 3$,

$$u_3(x,t) = f(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \tag{2.54}$$

and

$$
\begin{aligned}
\mathrm{Res}[u_3(x,t)] =& f_1 + f f_x + f_{xxx} - f_{xxxxx} \\
&+ (f_2 + f f_{1x} + f_1 f_x + f_{1xxx} - f_{1xxxxx})\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\
&+ (f_3 + f_2 f_x + f f_{2x} + f_{2xxx} - f_{2xxxxx})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
&+ f_{1x} f_{1xxx}\frac{t^{2\alpha}}{\Gamma(\alpha+1)^2} + (f f_{3x} + f_3 f_x + f_{3xxx} - f_{3xxxxx})\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\
&+ (f_1 f_{2x} + f_2 f_{1x})\frac{t^{3\alpha}}{\Gamma(\alpha+1)(2\alpha+1)} + f_2 f_{2x}\frac{t^{4\alpha}}{\Gamma(2\alpha+1)^2} \\
&+ (f_3 f_{1x} + f_1 f_{3x})\frac{t^{4\alpha}}{\Gamma(\alpha+1)(3\alpha+1)} \\
&+ (f_2 f_{3x} + f_3 f_{2x})\frac{t^{5\alpha}}{\Gamma(2\alpha+1)(3\alpha+1)} + f_3 f_{3x}\frac{t^{6\alpha}}{\Gamma(3\alpha+1)^2}. \qquad (2.55)
\end{aligned}
$$

Again, taking $D_t^{2\alpha}$ on both side and calculating the equation $D_t^{2\alpha}(\mathrm{Res}[\mathrm{u}_3(x,0)]) = 0$, then we get with

$$f_3 = -f_2 f_x - f f_{2x} - f_{2xxx} + f_{2xxxxx} - f_{1x} f_{1xxx}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}. \qquad (2.56)$$

By repeating the same process next approximation will be

$$f_4(x) = \frac{4762800\sqrt{13}}{137858491849}\mathrm{sech}^8\left(\frac{x}{2\sqrt{13}}\right)\left(-32 + 86\mathrm{sech}^2\left(\frac{x}{2\sqrt{13}}\right) - 55\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\right). \qquad (2.57)$$

Similarly, the values of $f_5, f_6, ...$ can be obtained.
Substituting values of $f_i,\ i = 1, 2, 3, 4, \ldots$, into equation (2.12), we obtained the approximate solution of time-fractional Kawahara equation (2.41).

$$
\begin{aligned}
&u(x,t) = \\
&\frac{105}{169}\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right) + \frac{7560}{28561\sqrt{13}}\tanh\left(\frac{x}{2\sqrt{13}}\right)\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} \\
&+ \frac{68040}{62748517}\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\left(4 - 5\mathrm{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&+ \frac{9525600}{10604499373}\mathrm{sech}^8\left(\frac{x}{2\sqrt{13}}\right)\tanh\left(\frac{x}{2\sqrt{13}}\right)\left(4 - 5\mathrm{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\
&+ \frac{4762800\sqrt{13}}{137858491849}\mathrm{sech}^8\left(\frac{x}{2\sqrt{13}}\right)\left(-32 + 86\mathrm{sech}^2\left(\frac{x}{2\sqrt{13}}\right) - 55\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\right)\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \\
&+ \ldots. \qquad (2.58)
\end{aligned}
$$

Table 2.7: The Absolute Error in the solution of Kawahara Equation (2.41) by RPSM method, VIM method [100], HPM method [100], and ADM method [161].

x	t	$\mid u_{exact} - u_{rpsm} \mid$	$\mid u_{exact} - u_{vim} \mid$	$\mid u_{exact} - u_{hpm} \mid$	$\mid u_{exact} - u_{adm} \mid$
-20	0.1	1.58361E-11	3.97274E-11	1.01469E-08	6.04582E-09
	0.2	8.28473E-11	3.13089E-10	4.04332E-08	2.67826E-08
	0.3	5.38679E-10	1.05406E-09	9.06246E-08	7.30563E-08
	0.4	2.82485E-09	2.49070E-09	1.60493E-07	1.14693E-07
	0.5	7.28460E-09	4.85110E-09	2.49809E-07	6.02567E-07
-10	0.1	8.29485E-10	2.46593E-09	1.53191E-06	4.34952E-08
	0.2	4.83572E-09	1.92434E-08	6.11064E-06	9.53984E-08
	0.3	1.63892E-08	6.48336E-08	1.37103E-05	3.03457E-07
	0.4	7.04823E-08	1.53132E-07	2.43056E-05	7.24586E-07
	0.5	3.47678E-07	2.98332E-07	3.78709E-05	2.19450E-06
0	0.1	2.00732E-10	1.74165E-10	5.42153E-06	1.49535E-08
	0.2	8.48329E-10	1.77754E-09	2.16847E-05	5.04568E-08
	0.3	3.18473E-09	9.03168E-09	4.87858E-05	2.38543E-07
	0.4	9.48372E-09	2.82581E-08	8.67179E-05	6.45683E-07
	0.5	4.03846E-08	6.90256E-08	1.35472E-04	1.39024E-06
10	0.1	6.03826E-10	2.41376E-09	1.54059E-06	5.83492E-08
	0.2	2.94573E-09	1.94624E-08	6.17975E-06	1.00438E-07
	0.3	9.89347E-09	6.58172E-08	1.39436E-05	4.89372E-07
	0.4	4.28345E-08	1.56456E-07	2.48584E-05	9.45935E-07
	0.5	1.38277E-07	3.06352E-07	3.89507E-05	5.34853E-06
20	0.1	2.28467E-11	3.98825E-11	1.02266E-08	4.93485E-09
	0.2	8.68913E-11	3.16631E-10	4.10637E-08	9.56823E-09
	0.3	5.03682E-10	1.07259E-09	9.27540E-08	3.48934E-08
	0.4	2.00384E-09	2.54866E-09	1.65539E-07	7.03948E-08
	0.5	7.28395E-09	4.99298E-09	2.59666E-07	2.94583E-07

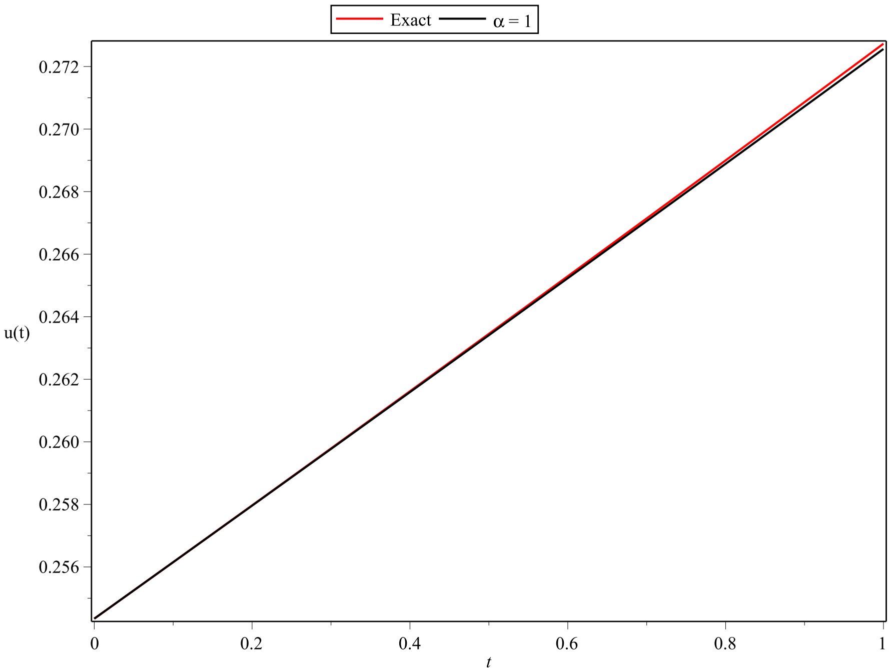


Figure 2.3: Comparison of solution of Fractional Kawahara Equation at fractional order of $\alpha = 1$ with Exact solution at $x = 5$.

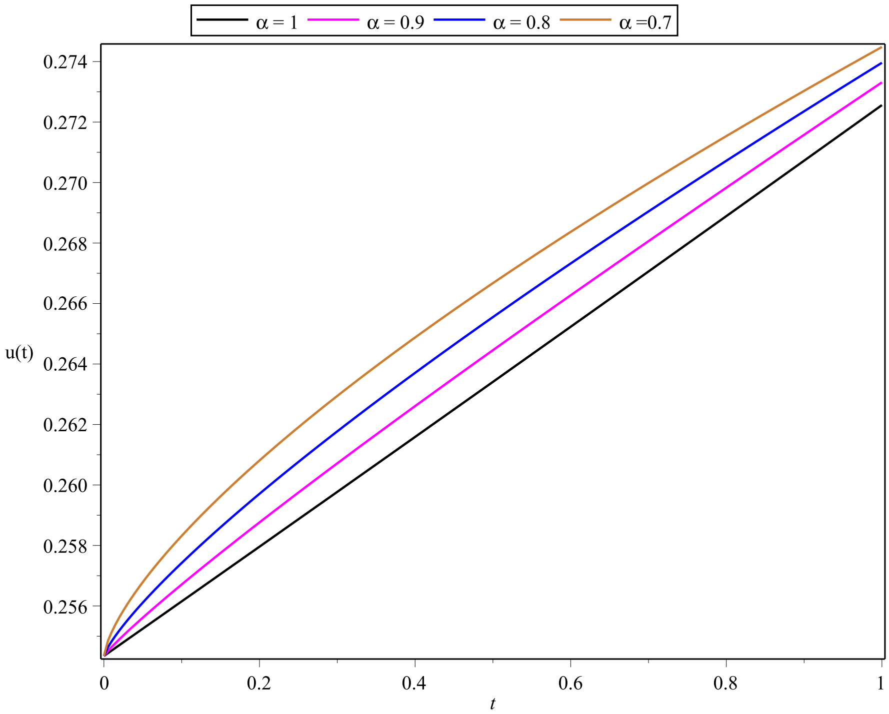


Figure 2.4: Comparison of solution of Fractional Kawahara Equation with different fractional order of α at $x = 5$.

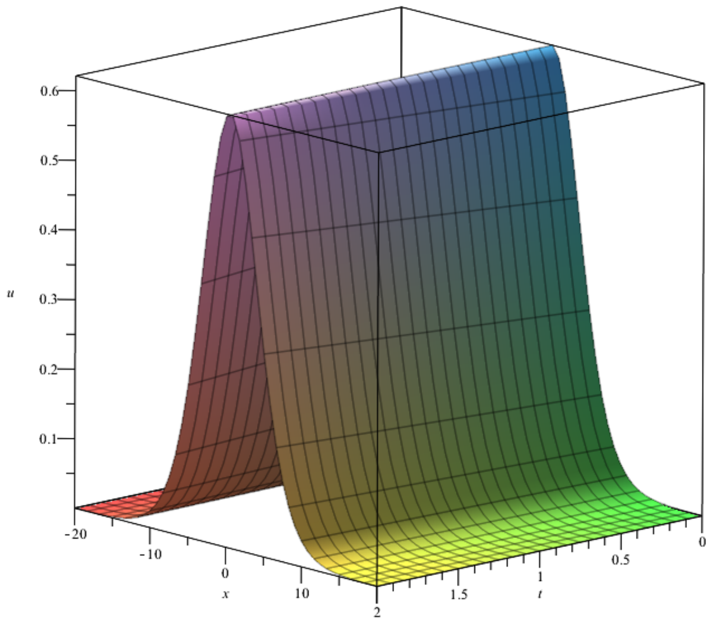


Figure 2.5: Behaviour of Fractional Kawahara Equation by FRPSM Method at $\alpha = 1$

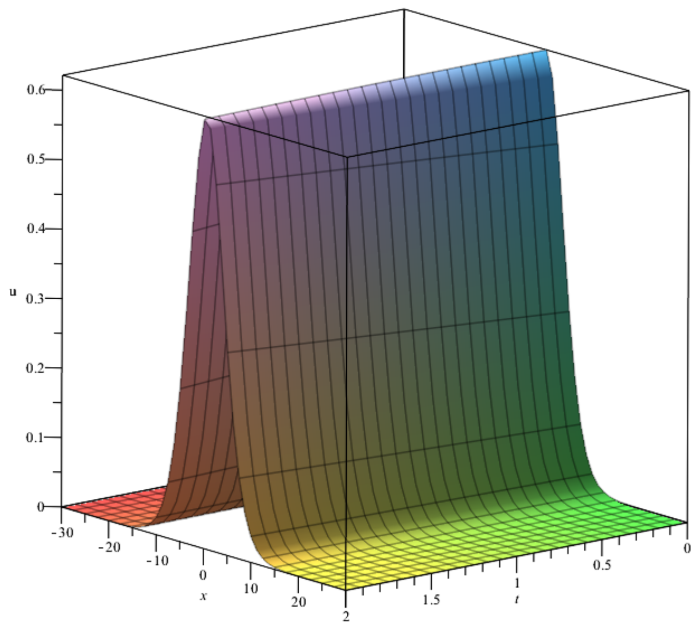


Figure 2.6: Behaviour of Exact Solution of Kawahara Equation

2.4.3 Solution of fractional Kawahara type - 2 equation

To demonstrate the non-linear homogeneous time-fractional Kawahara equation with $a = 1, b = 1, m = 1$ and $\lambda = 1$ in (2.2), let us examine the equation

$$D_t^\alpha[u(x,t)] + uu_x + u_{xxx} - u_{xxxxx} = 0, 0 < \alpha \leq 1, t > 0, x \in R, \tag{2.59}$$

with initial conditions

$$u(x,0) = -\frac{72}{169} + \frac{420}{169} \frac{\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)}{\left(1 + \operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)^2}. \tag{2.60}$$

The exact solution [102] at ($\alpha = 1$) of (2.59) is given by

$$u(x,t) = -\frac{72}{169} + \frac{420}{169} \frac{\operatorname{sech}^2\left(\frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)\right)}{\left(1 + \operatorname{sech}^2\left(\frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)\right)\right)^2}. \tag{2.61}$$

By implementing the same methodology of FRPSM, we leads to the approximate solution of time-fractional Kawahara equation (2.59),

$$\begin{aligned}
&u(x,t) = \\
&-\frac{72}{169} + \frac{420}{169}\frac{\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)}{\left(1+\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)^2} \\
&+\frac{420\sqrt{13}}{2197}\frac{\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\tanh\left(\frac{x}{2\sqrt{13}}\right)\left(\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)-1\right)}{\left(1+\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)^3}\frac{t^\alpha}{\Gamma(1+\alpha)} \\
&+\frac{210}{2197}\frac{\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\left(1-11\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)+10\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)-\operatorname{sech}^6\left(\frac{x}{2\sqrt{13}}\right)\right)}{\left(1+\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)^4}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&-\frac{420\sqrt{13}}{28561}\frac{\operatorname{sech}^3\left(\frac{x}{2\sqrt{13}}\right)\tanh\left(\frac{x}{2\sqrt{13}}\right)\left(1-14\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)+26\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)-7\operatorname{sech}^6\left(\frac{x}{2\sqrt{13}}\right)\right)}{\left(1+\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)^5}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\
&+\ldots.
\end{aligned} \tag{2.62}$$

Table 2.8: The Absolute Error in the solution of Kawahara Equation (2.59) by RPSM method, VIM method, HPM method, and ADM method.

x	t	$\mid u_{exact} - u_{rpsm} \mid$	$\mid u_{exact} - u_{vim} \mid$	$\mid u_{exact} - u_{hpm} \mid$	$\mid u_{exact} - u_{adm} \mid$
-20	0.1	7.95296E-10	7.82552E-09	7.95395E-09	3.88526E-08
	0.2	1.61492E-09	1.56394E-08	1.61571E-08	6.88312E-08
	0.3	2.45886E-09	2.34416E-08	2.46153E-08	9.88098E-08
	0.4	3.32711E-09	3.12321E-08	3.33346E-08	2.87882E-07
	0.5	4.21967E-09	3.90108E-08	4.23207E-08	5.87664E-07
-10	0.1	4.53022E-10	4.52809E-09	4.52965E-09	4.61140E-08
	0.2	9.06454E-10	9.05602E-09	9.05996E-09	6.48881E-08
	0.3	1.36030E-09	1.35838E-08	1.35875E-08	8.36620E-08
	0.4	1.81456E-09	1.81115E-08	1.81090E-08	2.24359E-07
	0.5	2.26924E-09	2.26392E-08	2.26209E-08	4.12097E-07
0	0.1	2.00000E-10	2.00000E-10	2.00000E-10	2.00000E-10
	0.2	2.00000E-10	2.00000E-10	2.00000E-10	2.00000E-10
	0.3	1.20000E-09	1.20000E-09	1.20000E-09	1.20000E-09
	0.4	2.90000E-09	2.90000E-09	2.90000E-09	2.90000E-09
	0.5	7.40000E-09	7.40000E-09	7.40000E-09	7.40000E-09
10	0.1	4.52616E-10	4.52828E-09	4.52558E-09	1.85653E-08
	0.2	9.04828E-10	9.05680E-09	9.04371E-09	5.97907E-08
	0.3	1.35664E-09	1.35856E-08	1.35510E-08	8.10158E-08
	0.4	1.80806E-09	1.81146E-08	1.80440E-08	2.22406E-07
	0.5	2.25908E-09	2.26440E-08	2.25193E-08	4.34650E-07
20	0.1	7.70965E-10	7.83709E-09	7.71064E-09	3.88950E-08
	0.2	1.51759E-09	1.56857E-08	1.51839E-08	7.89160E-08
	0.3	2.23988E-09	2.35457E-08	2.24256E-08	1.89369E-07
	0.4	2.93782E-09	3.14172E-08	2.94417E-08	3.89577E-07
	0.5	3.61140E-09	3.92999E-08	3.62381E-08	5.89784E-07

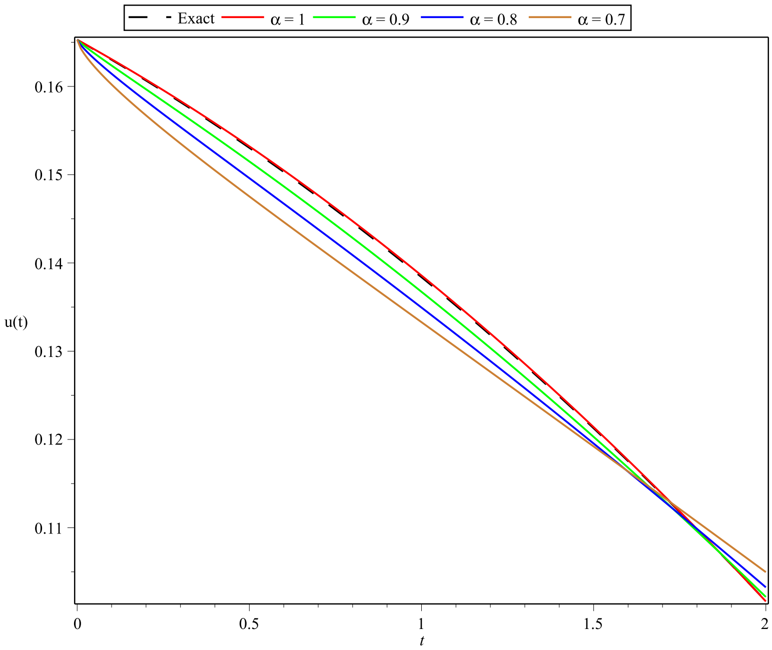
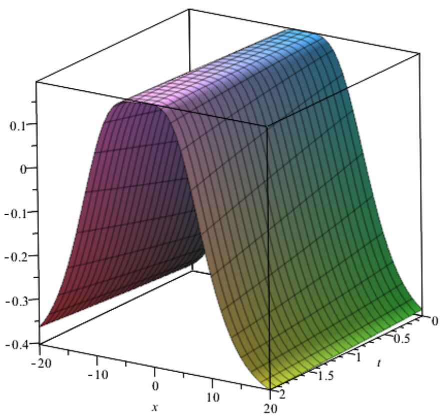
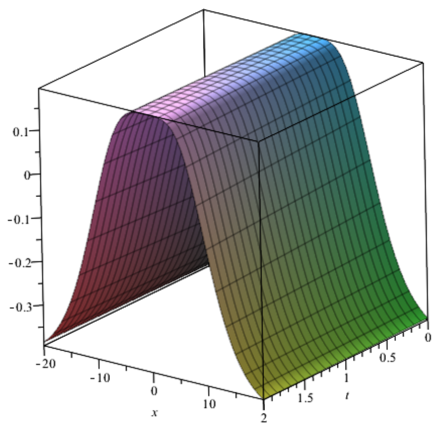


Figure 2.7: Comparison of solution of Fractional Kawahara Equation (2.59) with different fractional order of α with exact solution at $x = 5$.



(a) $\alpha = 1$



(b) Exact solution

Figure 2.8: Solution of Kawahara equation (2.59) by FRPSM with order $\alpha = 1$ and Exact solution.

2.4.4 Solution of fractional Modified Kawahara type-1 equation

Let us consider the non-linear homogeneous time-fractional Modified Kawahara equation with $a = 1, b = p, \lambda = -q$ and $m = 2$ in equation (2.2) as

$$D_t^{\alpha} u + u^2 u_x + p u_{xxx} + q u_{xxxxx} = 0, \tag{2.63}$$

with the initial condition

$$u(x,0) = \frac{3p}{\sqrt{-10q}} sech^2(kx), \text{ where } k = \frac{1}{2}\sqrt{\frac{-p}{5q}}. \tag{2.64}$$

The exact solution of (2.63) for spacial case $\alpha = 1, p > 0$, and $q < 0$ is

$$u(x,t) = \frac{3p}{\sqrt{-10q}} sech^2 \left[k \left(x - \left(\frac{25q - 4p^2}{25q} \right) t \right) \right]. \tag{2.65}$$

Illuminating the residual function for (2.63) as

$$\text{Res}[u(x,t)] = D_t^{\alpha} + u^2 u_x + p u_{xxx} + q u_{xxxxx} = 0, \tag{2.66}$$

thus, for the k^{th} residual function $\text{Res}[u_k(x,t)]$ defined as

$$\text{Res}[u_k(x,t)] = D_t^{\alpha} u_k + u_k^2 u_{kx} + p u_{kxxx} + q u_{kxxxxx} = 0. \tag{2.67}$$

For $k = 1$, equation (2.12) and (2.67) yields

$$\begin{aligned} \text{Res}[u_1(x,t)] =& f_1 + f^2 f_x + p f_{xxx} + q f_{xxxxx} \\ &+ \left(f^2 f_{1x} + 2 f f_1 f_x + p f_{1xxx} + q f_{1xxxxx} \right) \frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ &+ \left(2 f f_1 f_{1x} + f_1^2 f_x \right) \frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2} \\ &+ f_1^2 f_{1x} \frac{t^{3\alpha}}{(\Gamma(1+\alpha))^3}, \end{aligned} \tag{2.68}$$

using initial condition (2.64), we get

$$f_1(x) = \frac{-6\sqrt{2}p^{\frac{7}{2}}}{125q^2}\tanh\left(kx\right)\operatorname{sech}^2\left(kx\right). \tag{2.69}$$

Similarly, for $k = 2$ we have

$$\operatorname{Res}[u_2(x,t)] = D_t^{\alpha}u_2 + u_2^2 u_{2x} + pu_{2xxx} + qu_{2xxxxx} = 0. \tag{2.70}$$

Now, from equations (2.12) and (2.65) yields

$$\begin{aligned}\operatorname{Res}[u_2(x,t)] =& f_1 + f^2 f_x + pf_{xxx} + qf_{xxxxx} \\ &+ \left(f_2 + f^2 f_{1x} + 2ff_1f_x + pf_{1xxx} + qf_{1xxxxx}\right)\frac{t^{\alpha}}{\Gamma\left(1+\alpha\right)} \\ &+ \left(2ff_{1x}f_1 + f_1^2 f_x\right)\frac{t^{2\alpha}}{\left(\Gamma\left(1+\alpha\right)\right)^2} \\ &+ \left(2ff_2f_x + f^2 f_{2x} + pf_{2xxx} + qf_{2xxxxx}\right)\frac{t^{2\alpha}}{\Gamma\left(1+2\alpha\right)} \\ &+ \left(2f_1f_2f_x + 2ff_2f_{1x} + 2ff_1f_{2x}\right)\frac{t^{3\alpha}}{\Gamma\left(1+\alpha\right)\Gamma\left(1+2\alpha\right)} \\ &+ f_1^2 f_{1x}\frac{t^{3\alpha}}{\left(\Gamma\left(1+\alpha\right)\right)^3} + \left(f_x f_2^2 + 2ff_2f_{2x}\right)\frac{t^{4\alpha}}{\left(\Gamma\left(1+2\alpha\right)\right)^2} \\ &+ \left(2f_1f_2f_{1x} + f_1^2 f_{2x}\right)\frac{t^{4\alpha}}{\left(\Gamma\left(1+\alpha\right)\right)^2\Gamma\left(1+2\alpha\right)} \\ &+ f_{1x}f_2^2\frac{t^{5\alpha}}{\Gamma\left(1+\alpha\right)\left(\Gamma\left(1+2\alpha\right)\right)^2} + f_2^2 f_{2x}\frac{t^{6\alpha}}{\left(\Gamma\left(1+2\alpha\right)\right)^2},\end{aligned} \tag{2.71}$$

Operating D_t^{α} on both side and computing the equation $D_t^{\alpha}(\operatorname{Res}[u_2(x,0)]) = 0$, we have

$$f_2 = -f^2 f_{1x} - 2ff_1f_x - pf_{1xxx} - qf_{1xxxxx}. \tag{2.72}$$

Using equation (2.64) and (2.69)

$$f_2(x) = \frac{12\sqrt{2}p^{\frac{11}{2}}}{15625q^3}\sqrt{\frac{-5p}{q}}\operatorname{sech}^4\left(kx\right)\left(2\cosh^2\left(kx\right) - 3\right). \tag{2.73}$$

Similarly, the values of $f_3, f_4, ...$ can be obtained.

Substituting values of $f_i,\ i = 1, 2, 3, \ldots,$ into equation (2.12) yields

$$\begin{aligned} u(x,t) =& \frac{3p}{\sqrt{-10q}} sech^2(kx) + \frac{-6\sqrt{2}p^{\frac{7}{2}}}{125q^2} \tanh(kx) \operatorname{sech}^2(kx) \frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ &+ \frac{12\sqrt{2}p^{\frac{11}{2}}}{15625q^3} \sqrt{\frac{-5p}{q}} \operatorname{sech}^4(kx) \left(2\cosh^2(kx) - 3\right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \ldots \end{aligned} \tag{2.74}$$

Table 2.9: The Absolute Error in the solution of modified Kawahara Equation (2.63) by RPSM method, VIM method [76], HPM method [76] and ADM method [161].

x	t	$\mid u_{exact} - u_{rpsm} \mid$	$\mid u_{exact} - u_{vim} \mid$	$\mid u_{exact} - u_{hpm} \mid$	$\mid u_{exact} - u_{adm} \mid$
-20	0.1	8.51590E-07	2.14652E-05	2.14652E-05	4.56893E-06
	0.2	2.54525E-06	4.16408E-05	4.16408E-05	7.78534E-06
	0.3	4.73921E-06	6.05908E-05	6.05908E-05	1.45834E-05
	0.4	5.79092E-06	7.83757E-05	7.83757E-05	4.57893E-05
	0.5	4.94688E-06	9.50531E-05	9.50531E-05	8.29452E-05
-10	0.1	6.94927E-05	1.79880E-03	1.79880E-03	7.40245E-04
	0.2	1.95225E-04	3.49407E-03	3.49407E-03	9.52489E-04
	0.3	3.43100E-04	5.09051E-03	5.09051E-03	2.58935E-03
	0.4	4.61860E-04	6.59263E-03	6.59263E-03	5.45924E-03
	0.5	8.52801E-04	8.00473E-03	8.00473E-03	8.04589E-03
0	0.1	2.42340E-05	6.50131E-04	6.50131E-04	1.58935E-04
	0.2	5.25436E-05	2.59710E-03	2.59710E-03	3.85782E-04
	0.3	8.26822E-05	5.83065E-03	5.83065E-03	6.02482E-04
	0.4	3.33831E-04	1.03338E-02	1.03338E-02	9.45839E-04
	0.5	6.77303E-04	1.60832E-02	1.60832E-02	3.49305E-03
10	0.1	8.96249E-06	1.90724E-03	1.90724E-03	2.04824E-04
	0.2	2.80261E-05	3.92803E-03	3.92803E-03	4.24364E-04
	0.3	6.44425E-05	6.06769E-03	6.06769E-03	6.34567E-04
	0.4	7.06877E-05	8.33178E-03	8.33178E-03	9.00523E-04
	0.5	1.72788E-04	1.07261E-02	1.07261E-02	2.56345E-03
20	0.1	1.31295E-07	2.28219E-05	2.28219E-05	5.23589E-06
	0.2	2.99425E-07	4.70715E-05	4.70715E-05	8.98472E-06
	0.3	2.82314E-06	7.28231E-05	7.28231E-05	3.56394E-05
	0.4	7.15552E-06	1.00156E-04	1.00156E-04	6.13824E-05
	0.5	8.95801E-06	1.29151E-04	1.29151E-04	8.95723E-05

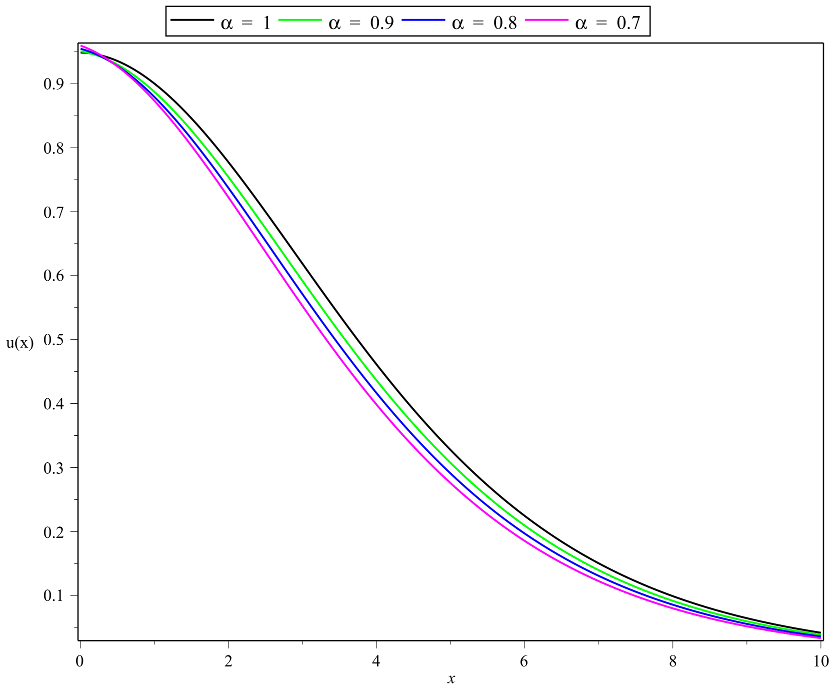


Figure 2.9: Comparison of solution of Time-fractional Modified Kawahara Equation with different fractional order of α at time $t = 1$.

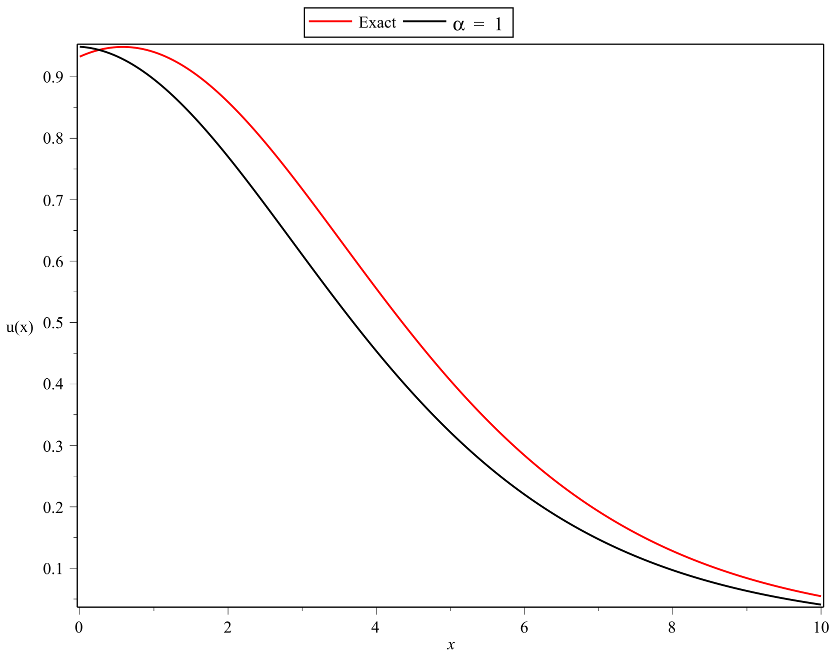


Figure 2.10: Comparison of solution of Fractional Modified Kawahara Equation with fractional order of $\alpha = 1$ with an exact solution at $t = 1$.

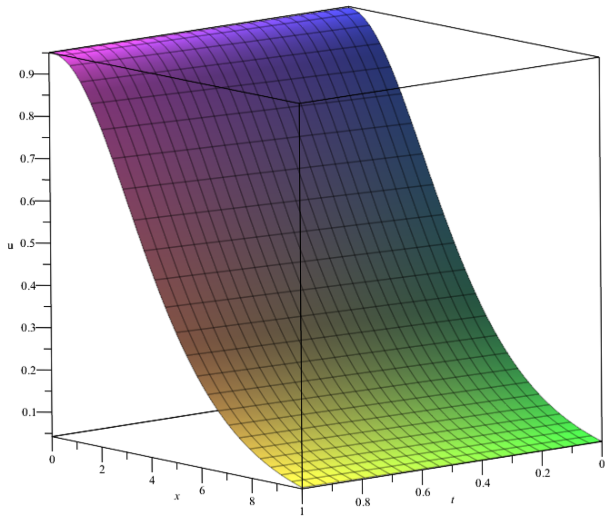


Figure 2.11: Behaviour of Fractional Modified Kawahara Equation by FRPSM Method at $\alpha = 1$.

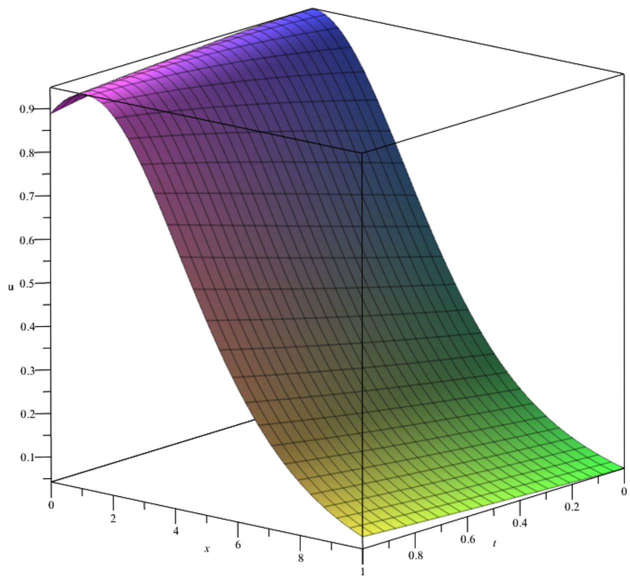


Figure 2.12: Behaviour of Exact Solution of Modified Kawahara Equation.

2.4.5 Solution of fractional Modified Kawahara type-2 equation

Let us consider the non-linear homogeneous time-fractional Modified Kawahara equation with $a = 1, b = p, \lambda = -q$ and $m = 2$ in equation (2.2) as

$$D_t^\alpha u + u^2 u_x + p u_{xxx} + q u_{xxxxx} = 0, \tag{2.75}$$

with the initial condition

$$u(x,0) = \frac{3p}{\sqrt{-10q}} sech^2(k_2 x), \text{ where } k_2 = \frac{1}{2}\sqrt{\frac{-p}{5q}}. \tag{2.76}$$

The exact solution of (2.75) for spacial case $\alpha = 1, p > 0$, and $q < 0$ is

$$u(x,t) = \frac{3p}{\sqrt{-10q}} sech^2 \left[k_2 \left(x - \left(\frac{25q - 4p^2}{25q} \right) t \right) \right]. \tag{2.77}$$

By implementing the same methodology of FRPSM, we obtain the approximate solution of time-fractional modified Kawahara equation (2.75),

$$\begin{aligned} u(x,t) = & \\ & \frac{6p}{\sqrt{-10q}} \frac{\operatorname{sech}(k_2 x)}{(1+\operatorname{sech}(k_2 x))} \\ & - \frac{6\sqrt{2}p^{\frac{3}{2}}}{q} \frac{\operatorname{sech}(k_2 x)\tanh(k_2 x)}{(1+\operatorname{sech}(k_2 x))^2} \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & + \frac{6p^2}{5\sqrt{-10}q^{\frac{3}{2}}} \frac{\operatorname{sech}(k_2 x)\left(1+\operatorname{sech}(k_2 x) - 2\tanh^2(k_2 x)\right)}{(1+\operatorname{sech}(k_2 x))^3} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \frac{6p^{\frac{5}{2}}}{25\sqrt{2}q^2} \frac{\operatorname{sech}(k_2 x)\tanh(k_2 x)\left(-5-4\operatorname{sech}(k_2 x)+\operatorname{sech}^2(k_2 x)+6\tanh^2(k_2 x)\right)}{(1+\operatorname{sech}(k_2 x))^4} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ & + \ldots . \end{aligned} \tag{2.78}$$

Table 2.10: The Absolute Error in the solution of modified Kawahara Equation (2.75) by RPSM method, VIM method [76], HPM method [76] and ADM method [161].

x	t	$\mid u_{exact} - u_{rpsm} \mid$	$\mid u_{exact} - u_{vim} \mid$	$\mid u_{exact} - u_{hpm} \mid$	$\mid u_{exact} - u_{adm} \mid$
	0.1	4.76456E-09	4.76530E-07	1.53243E-07	1.35453E-07
	0.2	9.50163E-09	9.50753E-07	6.23543E-07	6.34343E-07
-20	0.3	1.42176E-08	1.42375E-06	1.26435E-06	2.53453E-06
	0.4	1.89185E-08	1.89656E-06	4.63456E-06	5.33366E-06
	0.5	2.36101E-08	2.37022E-06	8.84245E-06	7.22210E-06
	0.1	3.98788E-08	3.98842E-06	1.27345E-06	1.23843E-07
	0.2	7.95333E-08	7.95761E-06	4.85632E-06	5.38434E-07
-10	0.3	1.19010E-07	1.19155E-05	7.83475E-06	8.80034E-07
	0.4	1.58355E-07	1.58697E-05	1.23854E-05	3.45454E-06
	0.5	1.97611E-07	1.98279E-05	3.58237E-05	6.44565E-06
	0.1	1.63646E-08	1.63646E-06	4.33563E-07	3.11456E-07
	0.2	6.51157E-08	6.51157E-06	7.74384E-07	5.33387E-07
0	0.3	1.45228E-07	1.45228E-05	1.84754E-06	2.23454E-06
	0.4	2.55007E-07	2.55007E-05	3.55465E-06	6.58473E-06
	0.5	3.92107E-07	3.92107E-05	6.99343E-06	8.44584E-06
	0.1	4.01523E-08	4.01576E-06	1.39453E-06	2.37436E-07
	0.2	8.06291E-07	8.06719E-06	4.53453E-06	6.84473E-07
10	0.3	1.21484E-07	1.21628E-05	7.53463E-06	9.53453E-07
	0.4	1.62772E-07	1.63114E-05	1.23992E-05	3.56564E-06
	0.5	2.04551E-07	2.05219E-05	4.53444E-05	7.00349E-06
	0.1	4.79878E-09	4.79952E-07	3.44848E-07	7.11234E-08
	0.2	9.63886E-09	6.64476E-07	5.11234E-07	2.00456E-07
20	0.3	1.45277E-08	1.45476E-06	8.45454E-07	5.22203E-07
	0.4	1.94732E-08	1.95203E-06	2.34854E-06	8.00345E-07
	0.5	2.44835E-08	2.45756E-06	5.11456E-06	2.33456E-06

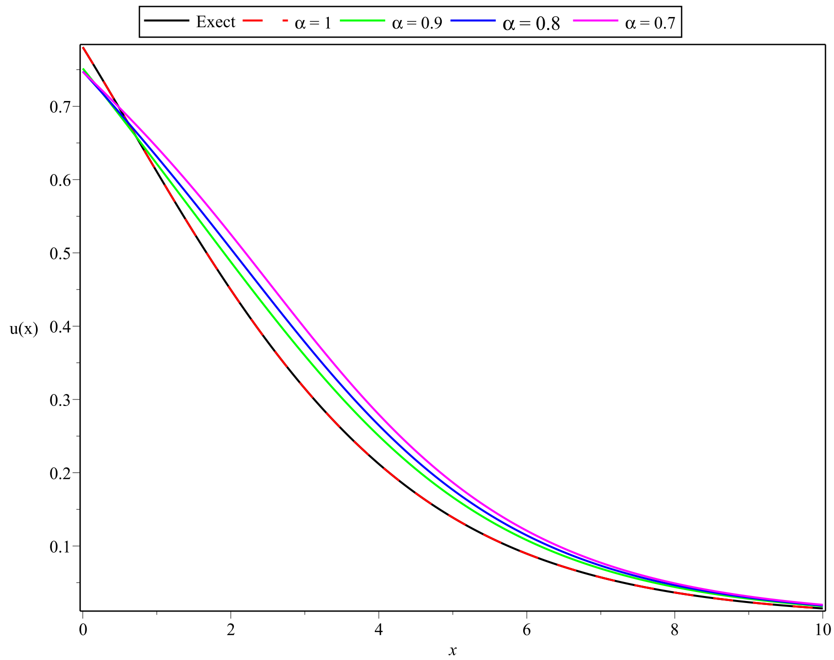
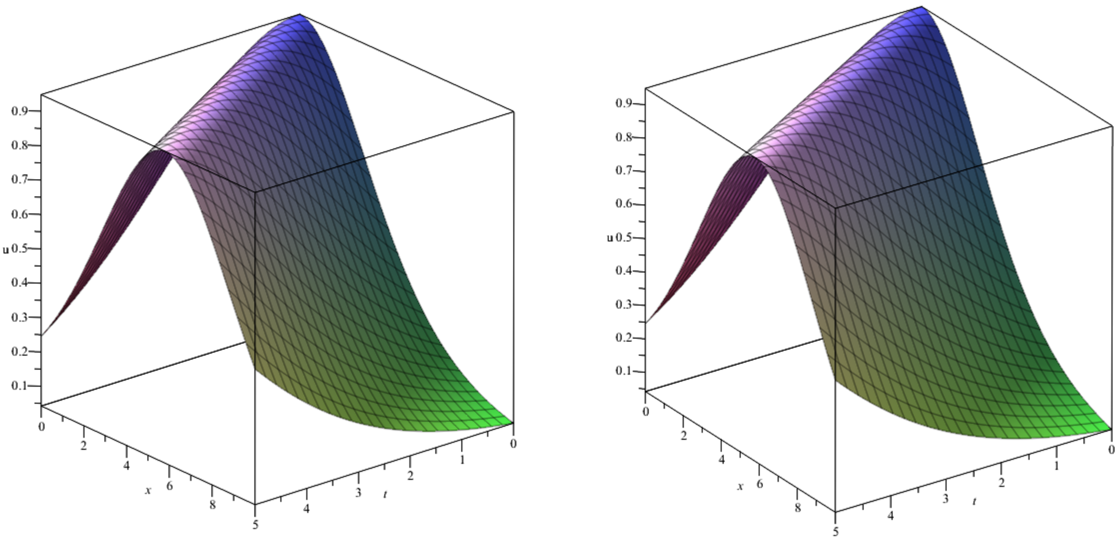


Figure 2.13: Comparison of solution of Fractional Modified Kawahara Equation as equation (2.75) with different fractional order of α with exact solution at $t = 1$.



(a) $\alpha = 1$ (b) Exact solution

Figure 2.14: Solution of Modified Kawahara equation (2.75) by FRPSM with order $\alpha = 1$ and Exact solution.

2.5 Results and Conclusion

The present work exhibits the solution of fractional ordered KdV, Kawahara, and Modified Kawahara equations by FRPSM. The results carried out during this article are predicted to be useful for several complicated non-linear physical problems. The calculations of this technique are straightforward. The absolute errors (in Tables – 2.1 to 2.5 and 2.7 to 2.10) indicate that FRPSM achieves better accuracy as compared to ADM and HPM. The behaviour of the equations using different fractional orders i.e., $\alpha = 0.7, 0.8, 0.9$ along with $\alpha = 1$ and exact solution are seen in Figures – 2.1, 2.3, 2.4, 2.7, 2.9, 2.10 and 2.13. Moreover, Figures – 2.2(a) to 2.2(f), 2.5, 2.6, 2.8, 2.11, 2.12 and 2.14 describe the geometrical behavior of the solutions obtained by FRPSM and the exact solution respectively. The results so obtained state the convergence of the method. Consequently, we can extend this approach to a similar type of space-time-fractional differential equations occurring in nature to obtain.

2.6 A Maple implementation and graphs for Korteweg-de Vries and Kawahara Equations

2.6.1 A Maple code with 2D plot for the solution of KdV equation by RPSM at x = 5 and some fractional order α

```
v0 := (1/2)*sech((1/2)*x)^2;
```

$$\frac{1}{2}\left(\operatorname{sech}\left(\frac{x}{2}\right)\right)^2$$

```
v1 := (1/2)*tanh((1/2)*x)*sech((1/2)*x)^2;
```

$$\frac{1}{2}\tanh\left(\frac{1}{2}x\right)\operatorname{sech}\left(\frac{1}{2}x\right)^2$$

```
v2 := (1/4)*sech((1/2)*x)^4*(2*cosh((1/2)*x)^2-3);
```

$$\frac{1}{4}\operatorname{sech}\left(\frac{1}{2}x\right)^4\left(2\cosh\left(\frac{1}{2}x\right)^2-3\right)$$

```
u := v0+v1*t^α/GAMMA(1+α)+v2*t^(2*α)/GAMMA(1+2*α);
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{1}{2}x\right)^2+\frac{1}{2}\frac{\tanh\left(\frac{1}{2}x\right)\operatorname{sech}\left(\frac{1}{2}x\right)^2t^{\alpha}}{\Gamma(1+\alpha)}+\frac{1}{4}\frac{\operatorname{sech}\left(\frac{1}{2}x\right)^4\left(2\cosh\left(\frac{1}{2}x\right)^2-3\right)t^{2\alpha}}{\Gamma(1+2\alpha)}$$

```
u1 := eval(u,[x = 5,α = 0.7]);
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{5}{2}\right)^2+0.5502737025\tanh\left(\frac{5}{2}\right)\operatorname{sech}\left(\frac{5}{2}\right)^2t^{0.7}+0.2012608031\operatorname{sech}\left(\frac{5}{2}\right)^2t^{1.4}$$

```
u2 := eval(u,[x = 5,α = 0.8]);
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{5}{2}\right)^2+0.5368356370\tanh\left(\frac{5}{2}\right)\operatorname{sech}\left(\frac{5}{2}\right)^2t^{0.8}+0.1748710866\operatorname{sech}\left(\frac{5}{2}\right)^2t^{1.6}$$

```
u3 := eval(u,[x = 5,α = 0.9]);
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{5}{2}\right)^2+0.5198770670\tanh\left(\frac{5}{2}\right)\operatorname{sech}\left(\frac{5}{2}\right)^2t^{0.9}+0.1491210102\operatorname{sech}\left(\frac{5}{2}\right)^2t^{1.8}$$

```
u4 := eval(u,[x = 5,α = 1]);
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{5}{2}\right)^2+\frac{1}{2}\tanh\left(\frac{5}{2}\right)\operatorname{sech}\left(\frac{5}{2}\right)^2t+\frac{1}{8}\operatorname{sech}\left(\frac{5}{2}\right)^2t^2$$

```
u := (1/2)*sech(-(1/2)*x+(1/2)*t)^2;
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{1}{2}t-\frac{1}{2}x\right)^2$$

```
u0 := eval(u,[x = 5]);
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{1}{2}t-\frac{5}{2}\right)^2$$

```
plot([u0,u4,u3,u2,u1], t = 0..1, color = ["black", "red", "blue", "magenta",
"coral"]);
```

See figure – (2.1)

2.6.2 A Maple code for Exact Solution of KdV equation and its 3D plot

```
pde := diff(u(x,t),t)+6*u(x,t)*(diff(u(x,t),x))+diff(u(x,t),x,x,x) = 0;
```

$$\frac{\partial}{\partial t}u(x,t) + 6u(x,t)\left(\frac{\partial}{\partial x}u(x,t)\right) + \frac{\partial^3}{\partial x^3}u(x,t) = 0$$

```
pdsolve(pde);
```

$$u(x,t) = -2\_C2^2 \tanh(\_C2x + \_C3t + \_C1)^2 + \frac{1}{6}\frac{8\_C2^3 - \_C3}{\_C2}$$

```
ivp := u(x, 0) = (1/2)*sech((1/2)*x)^2;
```

$$u(x,0) = \frac{1}{2}\mathrm{sech}\left(\frac{1}{2}x\right)^2$$

```
pdsolve([pde, ivp]);
```

$$u(x,t) = \frac{1}{2\cosh\left(-\frac{1}{2}x + \frac{1}{2}t\right)^2}, u(x,t) = \frac{1}{2\cosh\left(\frac{1}{2}x + \frac{1}{2}t\right)^2}$$

```
plot3d((1/2)*sech(-(1/2)*x+(1/2)*t)^2, t = 0..0.5, x = -10..10);
```

See figure – (2.2f)

2.6.3 A Maple code for the solution of KdV equation by RPSM technique which represent the 3D plot at different fractional order α

```
v0 := (1/2)*sech((1/2)*x)^2;
```

$$\frac{1}{2}\left(\operatorname{sech}\left(\frac{x}{2}\right)\right)^2$$

```
v1 := (1/2)*tanh((1/2)*x)*sech((1/2)*x)^2;
```

$$\frac{1}{2}\tanh\left(\frac{1}{2}x\right)\operatorname{sech}\left(\frac{1}{2}x\right)^2$$

```
v2 := (1/4)*sech((1/2)*x)^4*(2*cosh((1/2)*x)^2-3);
```

$$\frac{1}{4}\operatorname{sech}\left(\frac{1}{2}x\right)^4\left(2\cosh\left(\frac{1}{2}x\right)^2-3\right)$$

```
u := v0+v1*t^α/GAMMA(1+α)+v2*t^(2*α)/GAMMA(1+2*α);
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{1}{2}x\right)^2+\frac{1}{2}\frac{\tanh\left(\frac{1}{2}x\right)\operatorname{sech}\left(\frac{1}{2}x\right)^2t^{\alpha}}{\Gamma(1+\alpha)}+\frac{1}{4}\frac{\operatorname{sech}\left(\frac{1}{2}x\right)^4\left(2\cosh\left(\frac{1}{2}x\right)^2-3\right)t^{2\alpha}}{\Gamma(1+2\alpha)}$$

```
v := eval(u, α = 1);
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{1}{2}x\right)^2+\frac{1}{2}\tanh\left(\frac{1}{2}x\right)\operatorname{sech}\left(\frac{1}{2}x\right)^2t+\frac{1}{8}\operatorname{sech}\left(\frac{1}{2}x\right)^4\left(2\cosh\left(\frac{1}{2}x\right)^2-3\right)t^2$$

```
plot3d(v, t = 0..0.5, x = -10..10);
```

See figure – (2.2e)

```
w1 := eval(u, α = .2);
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{1}{2}x\right)^2+0.5445622105\tanh\left(\frac{1}{2}x\right)\operatorname{sech}\left(\frac{1}{2}x\right)^2t^{0.2}+0.2817651245\operatorname{sech}\left(\frac{1}{2}x\right)^4\left(2\cosh\left(\frac{1}{2}x\right)^2-3\right)t^{0.4}$$

```
plot3d(w1, t = 0..0.5, x = -10..10);
```

See figure – (2.2a)

```
w2 := eval(u, α = .4);
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{1}{2}x\right)^2 + 0.5635302490\tanh\left(\frac{1}{2}x\right)\operatorname{sech}\left(\frac{1}{2}x\right)^2 t^{0.4} + 0.2684178185\operatorname{sech}\left(\frac{1}{2}x\right)^4 \left(2\cosh\left(\frac{1}{2}x\right)^2 - 3\right) t^{0.8}$$

```
plot3d(w2, t = 0..0.5, x = -10..10);
```

See figure – (2.2b)

```
w3 := eval(u, α = .6);
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{1}{2}x\right)^2 + 0.5595874770\tanh\left(\frac{1}{2}x\right)\operatorname{sech}\left(\frac{1}{2}x\right)^2 t^{0.6} + 0.2269009210\operatorname{sech}\left(\frac{1}{2}x\right)^4 \left(2\cosh\left(\frac{1}{2}x\right)^2 - 3\right) t^{1.2}$$

```
plot3d(w3, t = 0..0.5, x = -10..10);
```

See figure – (2.2c)

```
w4 := eval(u, α = .8);
```

$$\frac{1}{2}\operatorname{sech}\left(\frac{1}{2}x\right)^2 + 0.5368356370\tanh\left(\frac{1}{2}x\right)\operatorname{sech}\left(\frac{1}{2}x\right)^2 t^{0.8} + 0.1748710866\operatorname{sech}\left(\frac{1}{2}x\right)^4 \left(2\cosh\left(\frac{1}{2}x\right)^2 - 3\right) t^{1.6}$$

```
plot3d(w4, t = 0..0.5, x = -10..10);
```

See figure – (2.2d)

2.6.4 A Maple code with 2D plot for the solution of Kawahara equation by RPSM at x = 5 and some fractional order α

```
v0 := (105/169)*sech(x/(2*sqrt(13)))^4;
```

$$\frac{105}{169}\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^4$$

```
v1 := 7560*tanh(x/(2*sqrt(13)))*sech(x/(2*sqrt(13)))^4/(28561*sqrt(13));
```

$$\frac{7560\sqrt{13}}{371293}\tanh\left(\frac{1}{26}x\sqrt{13}\right)\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^4$$

```
v2 := 68040/62748517.(sech(x/(2*sqrt(13)))^4).(4-5*sech(x/(2*sqrt(13)))^2);
```

$$\frac{68040}{62748517}\left(\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^4\right).\left(4-5\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^2\right)$$

```
v3 := 9525600/10604499373.(sech(x/(2*sqrt(13)))^8)(tanh(x/(2*sqrt(13))))
(4-5*sech(x/(2*sqrt(13)))^2);
```

$$\frac{9525600}{10604499373}\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^8\tanh\left(\frac{1}{26}x\sqrt{13}\right)\left(4-5\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^2\right)$$

```
u := v0+v1*t^α/GAMMA(1+α)+v2*t^(2*α)/GAMMA(1+2*α)+v3*t^(3*α)/GAMMA(1+3*α);
```

$$\frac{105}{169}\operatorname{sech}\left(\tfrac{1}{26}x\sqrt{13}\right)^4+\frac{7560\sqrt{13}}{371293}\tanh\left(\tfrac{1}{26}x\sqrt{13}\right)\operatorname{sech}\left(\tfrac{1}{26}x\sqrt{13}\right)^4 t$$
$$+\frac{68040}{62748517}\left(\operatorname{sech}\left(\tfrac{1}{26}x\sqrt{13}\right)^4\right).\left(4-5\operatorname{sech}\left(\tfrac{1}{26}x\sqrt{13}\right)^2\right)\frac{t^2}{2}$$
$$+\frac{9525600}{10604499373}\operatorname{sech}\left(\tfrac{1}{26}x\sqrt{13}\right)^8\tanh\left(\tfrac{1}{26}x\sqrt{13}\right)\left(4-5\operatorname{sech}\left(\tfrac{1}{26}x\sqrt{13}\right)^2\right)\frac{t^3}{6}+\ldots$$

```
u0 := eval(u, [x = 5, α = 1])
```

$$\frac{105}{169}\operatorname{sech}\left(\tfrac{5}{26}\sqrt{13}\right)^4+\frac{7560}{371293}\sqrt{13}\tanh\left(\tfrac{5}{26}\sqrt{13}\right)\operatorname{sech}\left(\tfrac{5}{26}\sqrt{13}\right)^4 t$$
$$+\frac{34020}{62748517}\operatorname{sech}\left(\tfrac{5}{26}\sqrt{13}\right)^4\left(4-5\operatorname{sech}\left(\tfrac{5}{26}\sqrt{13}\right)^2\right)t^2$$

```
u1 := eval(u,[x = 5, α = 0.7]);
```

$$\frac{105}{169}\mathrm{sech}\left(\frac{5}{26}\sqrt{13}\right)^4 + 0.02240855169\sqrt{13}\tanh\left(\frac{5}{26}\sqrt{13}\right)\mathrm{sech}\left(\frac{5}{26}\sqrt{13}\right)^4 t^{0.7}$$
$$+0.0008729312309\mathrm{sech}\left(\frac{5}{26}\sqrt{13}\right)^4\left(4-5\mathrm{sech}\left(\frac{5}{26}\sqrt{13}\right)^2\right)t^{1.4}$$

```
u2 := eval(u,[x = 5, α = 0.8]);
```

$$\frac{105}{169}\mathrm{sech}\left(\frac{5}{26}\sqrt{13}\right)^4 + 0.02186131931\sqrt{13}\tanh\left(\frac{5}{26}\sqrt{13}\right)\mathrm{sech}\left(\frac{5}{26}\sqrt{13}\right)^4 t^{0.8}$$
$$+0.0007584707526\mathrm{sech}\left(\frac{5}{26}\sqrt{13}\right)^4\left(4-5\mathrm{sech}\left(\frac{5}{26}\sqrt{13}\right)^2\right)t^{1.6}$$

```
u3 := eval(u,[x = 5,α = 0.9]);
```

$$\frac{105}{169}\mathrm{sech}\left(\frac{5}{26}\sqrt{13}\right)^4 + 0.02117072299\sqrt{13}\tanh\left(\frac{5}{26}\sqrt{13}\right)\mathrm{sech}\left(\frac{5}{26}\sqrt{13}\right)^4 t^{0.9}$$
$$+0.0006467845949\mathrm{sech}\left(\frac{5}{26}\sqrt{13}\right)^4\left(4-5\mathrm{sech}\left(\frac{5}{26}\sqrt{13}\right)^2\right)t^{1.8}$$

```
u4 := eval((105/169)*sech((x-(36/169)*t)/(2*sqrt(13)))^4,[x=5]);
```

$$\frac{105}{169}\mathrm{sech}\left(\frac{1}{26}\sqrt{13}\left(5-\frac{36}{169}t\right)\right)^4$$

```
plot([u4, u0, u1, u2, u3], t = 0 .. 1, color = ["black", "red", "magenta",
"blue", "gold"]);
```

See figures – (2.3) and (2.4)

2.6.5 A Maple code for the solution of Kawahara equation by RPSM and its 3D plot at $\alpha = 1$

```
v0 := (105/169)*sech(x/(2*sqrt(13)))^4;
```

$$\frac{105}{169}\mathrm{sech}\left(\frac{1}{26}x\sqrt{13}\right)^4$$

```
v1 := 7560*tanh(x/(2*sqrt(13)))*sech(x/(2*sqrt(13)))^4/(28561*sqrt(13));
```

$$\frac{7560\sqrt{13}}{371293}\tanh\left(\frac{1}{26}x\sqrt{13}\right)\mathrm{sech}\left(\frac{1}{26}x\sqrt{13}\right)^4$$

```
v2 := 68040/62748517.(sech(x/(2*sqrt(13)))^4).(4-5*sech(x/(2*sqrt(13)))^2);
```

$$\frac{68040}{62748517}\left(\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^4\right).\left(4-5\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^2\right)$$

```
v3 := 9525600/10604499373.(sech(x/(2*sqrt(13)))^8)(tanh(x/(2*sqrt(13))))
(4-5*sech(x/(2*sqrt(13)))^2);
```

$$\frac{9525600}{10604499373}\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^8\tanh\left(\frac{1}{26}x\sqrt{13}\right)\left(4-5\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^2\right)$$

```
u := v0+v1*t^α/GAMMA(1+α)+v2*t^(2*α)/GAMMA(1+2*α) + v3*t^(3*α)/GAMMA(1+3*α);
```

$$\frac{105}{169}\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^4+\frac{7560\sqrt{13}}{371293}\tanh\left(\frac{1}{26}x\sqrt{13}\right)\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^4t$$
$$+\frac{68040}{62748517}\left(\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^4\right).\left(4-5\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^2\right)\frac{t^2}{2}$$
$$+\frac{9525600}{10604499373}\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^8\tanh\left(\frac{1}{26}x\sqrt{13}\right)\left(4-5\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^2\right)\frac{t^3}{6}+\ldots$$

```
v := eval(u, α = 1);
```

$$\frac{105}{169}\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^4+\frac{7560\sqrt{13}}{371293}\tanh\left(\frac{1}{26}x\sqrt{13}\right)\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^4t$$
$$+\frac{68040}{62748517}\left(\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^4\right).\left(4-5\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^2\right)\frac{t^2}{2}$$
$$+\frac{9525600}{10604499373}\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^8\tanh\left(\frac{1}{26}x\sqrt{13}\right)\left(4-5\operatorname{sech}\left(\frac{1}{26}x\sqrt{13}\right)^2\right)\frac{t^3}{6}+\ldots$$

```
plot3d(v,t=0..2,x=-20..20);
```

See figure – (2.5)

2.6.6 A Maple code for Exact Solution of Kawahara equation and its 3D plot

```
pde := diff(u(x,t),t)+u(x,t)*(diff(u(x,t),x))+diff(u(x,t),x,x,x)
-(diff(u(x,t),x,x,x,x,x)) = 0;
```

$$pde := \frac{\partial}{\partial t}u(x,t)+u(x,t)\left(\frac{\partial}{\partial x}u(x,t)\right)+\frac{\partial^3}{\partial x^3}u(x,t)-\frac{\partial^5}{\partial x^5}u(x,t)=0$$

```
pdsolve(pde)
```

$$u(x,t) = \frac{105\tanh\left(\_C3t - \frac{\sqrt{13}x}{26} + \_C1\right)^4}{169} + \frac{69}{169} - \frac{210\tanh\left(\_C3t - \frac{\sqrt{13}x}{26} + \_C1\right)^2}{169} + 2\sqrt{13}\_C3$$

```
simplify (trig)
```

$$u(x,t) = \frac{105 + \left(338\sqrt{13}\_C3 - 36\right)\cosh\left(\_C3t - \frac{\sqrt{13}x}{26} + \_C1\right)^4}{169\cosh\left(\_C3t - \frac{\sqrt{13}x}{26} + \_C1\right)^4}$$

```
ivp := u(x,0) = (105/169)*sech(x/(2*sqrt(13)))^4;
```

$$ivp := u(x,0) = \frac{105\mathrm{sech}\left(\frac{\sqrt{13}x}{26}\right)^4}{169}$$

```
pdsolve([pde,ivp]);
```

$$u(x,t) = \frac{105}{169}\mathrm{sech}\left(\frac{x - \frac{36t}{169}}{2\sqrt{13}}\right)^4$$

```
plot3d((105/169)*sech((x-(36/169)*t)/(2*sqrt(13)))^4, t = 0..0.2, x = -20..20);
```

See figure – (2.6)