Chapter 1

Introduction

1.1 Fractional Calculus

In a letter to L'Hospital on September 30^{th}, 1695, Leibniz raised the possibility of generalizing the operation of differentiation to non-integer orders, that is finding of $\frac{d^n}{dx^n} f(x)$, and L'Hospital asked what would be the result of half differentiating the function $f(x) = x$; that is $\frac{d^{1/2}}{dx^{1/2}}[x]$. Leibniz replied: "It leads to a paradox, from which one day useful consequences will be drawn". The paradoxical aspects arise because there are several different ways of generalizing the differentiation operator into non-integer orders, leading to inequivalent results. We can say with accuracy that this query was dated on September 30^{th}, 1695 and that incident gave the birth to "Fractional Calculus"; therefore this subject of fractional calculus with half derivatives and half integrals is as old as conventional Newtonian or Leibniz's calculus. However, this subject of fractional-calculus was dormant until the beginning of the century, and only now it has started finding the applications in science and engineering [37, 107].

we already know about $\frac{d}{dx} f(x)$ or $\int_0^x f(y)dy$ i.e. the usual classical derivative and classical integration operation in order to get expressions for n fold differentiation and integration. The concept of the generalisation is like that from natural numbers i.e. $n = 1, 2, 3, ...$ we generalise the number line to the negative side i.e. $n = -1, -2, -3, ...$ and call them integer numbers. Then we have, in between these numbers: $\pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{3}{4}, \pm\frac{3}{2}, ...$ or even irrational ones like $\pi, \sqrt{2}, -\sqrt{3}$ etc. and we call them the real line. Similarly, we extend one whole differentiation i.e. $\frac{d}{dx}[f(x)]$ to have twice the differentiation i.e. $\frac{d^2}{dx^2}[f(x)]$, then thrice i.e. $\frac{d^3}{dx^3}[f(x)]$ and generalise to get the expression for n folds, that is $\frac{d^n}{dx^n}[f(x)]$. Similarly we generalise one whole integration i.e. $\int_0^x f(y)dy$ to n folds integration, that is

$\int_0^x f(y)(dy)^n$, or $\int_0^x dx_{n-1} \int_0^{x_{n-1}} dx_{n-2} ... \int_0^{x_2} dx_1 \int_0^{x_1} f(y)dy$. The extension of our classical calculus to fractional calculus is a generalisation of the theory of calculus. This extension in terms of mathematics is an 'analytic-continuation' of the operation i.e. $n-$ fold differentiation and integration from the n integer to the entire complex plane z. So we can have the derivatives (or integrals) in between like $\frac{d^{1/2}}{dx^{1/2}} f(x)$. That is, we have operation $\frac{d^\alpha}{dx^\alpha} f(x)$, where α is arbitrary, a 'real number', a 'complex number', even a 'function' as a continuous distribution. A positive α will signify a differentiation process and a negative α will give us integration. This is a generalised theory of calculus, a subject as old as Leibniz's or Newtonian calculus, and is called fractional calculus. Like further generalisation, we can have α as a complex number, a continuous distribution function, and a system. Therefore, there is fun in learning the subject of fractional calculus [37].

The definition of the fractional derivative is in itself a developing concept. Numerous definitions have been suggested for the fractional derivative, with almost each definition possessing some form of deficiency. It is generally believed that the choice of the derivative used is dictated by the situation that is being modeled. Since it was discovered that the fractional derivative can be successfully applied to practical problems, it is the Caputo derivative that has been used the most. The only shortcoming of the Caputo derivative is the singularity issue. Other fractional derivatives that have this singularity problem are Riemann–Liouville, Caputo–Hadamard, and Riesz. In a bid to address the singularity concern, the Caputo–Fabrizio derivative was proposed, this derivative eliminated the singularity problem through the use of an exponential kernel. Atangana and Baleanu replaced the exponential kernel with the Mittag-Leffler function to create another non-singular kernel derivative called the Atangana–Baleanu derivative. More detailed discussions encompassing both theory and applications of the non-singular derivatives are found in literatures [106, 137].

In the last few decades, Fractional Calculus has become an important tool in many areas of engineering [28], magnetohydrodynamics [50], solid-state physics [132], wind turbine control [141], diabetes therapies [134], option pricing [168], porous media [99], and fluid mechanics [114] by converting these problems into mathematical models through fractional orders and several models using fractional differential operators still need to be solved.

1.2 Fractional Differential equations

A fractional differential equation (FDE) is an equation that involves derivatives of a function of a non-integer order. The concept of fractional calculus, which deals with fractional order derivatives and integrals, has been gaining the attention due to its applications in various fields such as physics [132, 135], engineering [28, 109], finance [168], and biology [134].

Fractional differential equations can be expressed in various forms, such as Caputo, Riemann-Liouville, or Grunwald-Letnikov forms. A Caputo FDE of order α is given by:

$$D_t^{\alpha} y(t) = f(t, y(t))$$

where $y(t)$ is the unknown function, $f(t, y(t))$ is a given function, and α is a real number between 0 and 1. The Caputo derivative is defined as the fractional integral of the derivative of order α of $y(t)$ with respect to t. The Riemann-Liouville and Grunwald-Letnikov forms are alternative definitions of fractional derivatives.

Solving FDEs can be challenging due to its non-locality and non-integer order of the derivatives. However, there are various analytical, semi-analytical, and numerical methods are available for the exact or approximate solutions of FDEs. FDEs have applications in various fields, such as the study on dynamic problems [83], the modeling of the piping system of a nuclear power plant [170], the analysis of anomalous diffusion processes [164], the control of fractional-order systems [28], and the modeling of many important diseases [57, 84, 85]. Therefore, the study of fractional calculus and FDEs is an active research area with many open problems and challenges.

1.2.0 Why Fractional Differential Equation?

During the modeling of various physical phenomena [36, 37, 120], there arise differential equations of various types, which need to be solved and solutions are interpreted in the relevance of physical conditions.

During the 1960's and thereafter, researchers realized the significance of fractional calculus due to the characteristics of fractional derivative about, carrying behavior for the memory and hereditary properties in the neighbourhood (within the domain) of the point. This characteristic dramatically improves the results/output related to physical phenomena, when modeled it through FDE as governing differential equation; particularly in the time domain applications, the obtained results are quite astonishing while dealing with continuous processes, for e.g. Atomic reactor application [36, 37], DC motor controller model [36, 37], Gas in Fluid application [120], Cooling of the semi-infinite body in radiation [120], in the usage of PID controller [120], Electrical analog model of porous duke [120], etc.

This fact motivated a large number of researchers to carry out and work with different and diversified approaches to solve FDEs, which eventually leads to improved outcomes. Thus, In continuation of these global attempts, this thesis is a humble contribution towards the knowledge extension by innovating a few novel methods and approaches, where combinations of certain already known approaches are merged artistically with various integral transforms.

In a nutshell, The memory and hereditary properties in fractional calculus refer to the fact that the fractional derivative or integral of a function depends not only on the present value of the function but also on its past values. In other words, the history of the function affects in its fractional derivative or integral. It allows for the modeling and analysis of systems with long-term memory or non-local behavior, which cannot be captured by classical calculus.

For example, in Fractional differential equations, the memory effect allows for the inclusion of fractional order derivatives that describe systems with memory or long-term dependence on past values. Similarly, in fractional integral equations, the memory effect allows for the inclusion of fractional order integrals that describe systems with non-local or long-range interactions.

One well-known example of the memory effect in fractional calculus is the governing fractional harmonic oscillator differential equation [120]. Unlike the classical harmonic oscillator, where the system's response depends solely on the current position and velocity, the fractional harmonic oscillator incorporates the influence of past positions and velocities into its current behavior. Consequently, the system exhibits a memory-like response and may show behavior such as oscillatory transients, power-law decays, and long-term relaxation.

Thus, fractional calculus in general, and fractional differential equation in particular, provides a powerful mathematical tool to describe and analyze complex systems that exhibit memory or non-local behavior, leading to more accurate and comprehensive models in various scientific and engineering applications.

1.2.1 Analytical and Numerical methods

Fractional Differential equations form the backbone of various physical systems occurring in a wide range of science and engineering disciplines viz. physics [68, 121], chemistry [126], biology [116], structural mechanics [31, 114], computer network [44], biomechanics [59], etc. Generally, these physical systems are modeled either using fractional ordinary or partial differential equations (FODEs or FPDEs). In order to know the behavior of the system, we need to investigate the solutions of the governing FDEs. The exact solution of differential equations may be obtained using well-known classical methods. Generally, the physical systems occurring in nature comprise of complex phenomena for which computation of exact results may be quite challenging. In such cases, numerical or semi-analytical methods may be preferred [35, 67].

1.2.2 Integral transform

To fully harness the capability of fractional differential equations in modeling problems that arise in the real world, it is imperative that we have methods of solutions that are computationally inexpensive, easily accessible, and highly accurate. Integral transforms are some of the techniques that have proven their worth, as they are regarded to be easy to implement and demand reasonable labour in terms of computations. Integral transforms offer an alternative to integration in the solution of differential equations. The integral transform maps the domain of the original problem into a different domain consisting of an algebraic equation that is normally easy to manipulate. Taking the inverse of the new domain results in the solution of the original problem. There are different types of integral transforms that are used in the solution of differential equations, but it is the Laplace transform that is mostly applied. Most of the integral transforms that have been suggested are extensions of the Laplace transform. Some of the integral transforms that are closely related to the Laplace transform are the Elzaki transform [48], Sawi transform [101], Kamal transform [79], Sumudu transform [159], Natural transform [89], Formable transform [130], Shehu transform [103], General transform [70], etc.

1.2.3 Semi-Analytical methods

It is more difficult to obtain solutions of nonlinear fractional ordinary or partial differential equations. Therefore, it may not always be possible to obtain analytical solutions of those nonlinear FODEs or FPDEs. In this case, we use semi-analytical methods who provide the solution in the terms of series. Semi-analytical methods are based on finding the other terms of the series from the given initial conditions for the problem being considered. At this point, we encounter the concept of convergence of the series. So, it is necessary to perform a convergence analysis of these methods. As this convergence analysis can be carried out theoretically, one can gain information about the convergence of the series solution by looking at the absolute error between the numerical solution and the analytical solution. In some semi-analytic methods, a very good convergence can be achieved with only a few terms of the series, but more terms may be needed in some problems. That is, if the terms of the series increase, this provides better convergence to the analytical solution [35, 67, 154].

To find an approximate or analytical solutions of nonlinear fractional differential equations various methods are available in literature like Adomian decomposition methods [4], Laplace decomposition method [53], homotopy perturbation method [62, 63], homotopy analysis method [98], homotopy analysis transform method [117], Differential transform method [146], Wavelet collocation method [27], the tanh-coth method [162], exp-function method [66] and many more.

1.3 Mathematical Modeling

Many research scholars have reported that modeling with fractional calculus concept is very suitable and reliable to give an accurate description of memory and some physical properties of various materials and processes, which are completely missing in classical or integer-order equations, and a fractional mathematical model can give more reliable information about real life phenomena [145]. In addition, many physical systems encountered in various disciplines have been described by fractional differential equations, which include highly dispersive optical solitons numerical study with differential delay group [54], fractional boussinesq equation in a gradient unconfined aquifers [125], fractional model for thermal activity of conventional casson nano-particles with ramped temperature due to an infinite vertical plate [126], Machine learning based fault diagnosis system in nuclear reactor with medium voltage power cable [129], Fuzzy fractional black-scholes European option pricing model [168], PI-controller system for active vibration control of complicated piping system in nuclear power plant [170].

1.3.1 Epidemic Models

Epidemiology is the investigation of how illnesses disseminate in a live entity in relation to its surroundings. The epidemiology of an illness can be studied using numerical simulations. Several studies have attempted to predict and simulate the transmission of contagious ailments in the past, including measles [122], hepatitis b [143], HIV/AIDS [72], dengue fever [148], papillomavirus in human [167], and more recently, Covid19 [166], leptospirosis [111], and the Zika virus [15]. Mathematical simulation is being employed to investigate not only the transmission of contagious ailments, but also increasingly non-communicable diseases, as research advances. Medications and other environmental ailments can often be modeled. This is possible owing to the characteristics of how it spreads, namely via intimate communication as the media spreads.

1.4 Contribution towards knowledge extension

As a part of contributing towards knowledge extension, we successfully tried to develop a rich theory that eventually results in applications to various and diversified real-world phenomena; and the very purpose of this section itself is to make realize the reader glance with this sole beauty of content.

Chapter − 2: deals with Time-fractional (i) Korteweg-de Vries (KdV) and (ii) Kawahara equations.
Korteweg-de Vries (KdV) equation arises during the modeling, in the study of the behavior of solitary waves in shallow water. Original modeled governing PDE is $u_t + 6uu_x + u_{xxx} = 0$, popularly known as KdV equation; and its fractional analogue $D_t^\alpha u + 6uu_x + u_{xxx} = 0$ has been studied by various authors for the solution. As a part of the literature survey, we came across a few authors viz. Syam, Arqub, etc., who gave the solution $u(x,t)$ for the fractional analogue of the above IVP in [150], [22] through ADM and HPM respectively. The present work is all about continuing the study for the solution via different approaches of residual power series method (RPSM) [2, 22] and trying successfully to improve the comparative accuracy of results.

In the study of Plasma theory, there arise Magneto-Acoustic waves and for which the governing differential equation gives rise to a nonlinear PDE (Kawahara equation) of the form $u_t + au^m u_x + bu_{xxx} - \lambda u_{xxxxx} = 0$; and to demonstrate the behavior of this continuous process more accurately, it's resulting fractional analogue is in terms of $D_t^\alpha u + au^m u_x + bu_{xxx} - \lambda u_{xxxxx} = 0$, which has been considered and studied by various authors Jin [76], Lu [100], Mahmood [102], etc., who obtained the solution of same FPDE by VIM, HPM, and ADM respectively. The solution $u(x,t)$ of fractional Kawahara equation through novel approach adopted in this chapter, by residual power series method (RPSM) [2, 22] results in more powerful comparative outputs.

Chapter-3: introduces, the homotopy perturbation Sawi transform method, in order to deal with Fractional ODEs and PDEs; which is yet another feather in the sequence of methods developed in the recent past, where the key idea is the combination of a certain type of integral transform with the classical approach. Here the semi-analytic method namely HPSTM (homotopy perturbation Sawi transform method) is invented and successfully implemented by merging the Sawi transform with the homotopy perturbation method. The efficiency of the proposed method (over RPSM) has been established by employing it on the fractional logistic equation as nonlinear FODE and the fractional Fornberg-Whitham equation as nonlinear FPDE.

Chapter-4: presents a newly developed homotopy perturbation general transform method (which we call it as HPGTM), and in the same spirit as that of chapter-3, combines Jafari's powerful General transform [70] with homotopy perturbation method. The efficacy of the method is observed by dealing successfully with the problem of solving various linear and nonlinear FODEs and FPDEs, viz. fractional radioactive decay model as linear FODE, fractional Riccati equation as nonlinear FODE, fractional Kolmogorov equation as linear FPDE, fractional Klein–Gordon equation as nonhomogeneous linear FPDE, and fractional Rosenau–Hyman equation as nonlinear FPDE. The superiority of the method by examining comparative study with ADM and RPSM methods is established.

Chapter-5: uncovers the study of sub-diffusion dynamics of nano-precipitate growth and destruction. The governing nonlinear Swada-Kotera-Ito PDE representing the said phenomena is $u_t + 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} + 21uu_{xxxxx} + u_{xxxxxxx} = 0$ [162]. The fractional analogue of this celebrated Swada-Kotera-Ito equation $D_t^{\alpha}u + 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} + 21uu_{xxxxx} + u_{xxxxxxx} = 0$ is extensively been treated by several authors [10, 11, 12, 46, 75, 164], by employing various methods and approaches like q-HAM, $\frac{G}{G'}-$expansion method, RPSM, FRDTM, ADTM or HPTM, etc. respectively. The Swada-Kotera-Ito equation in its fractional form is treated in the Present work and attains its aim to reveal the successful application of blending the Adomian decomposition method with the Shehu transform and using Caputo as well as Atangana-Baleanu derivatives. And hence name this new method as Adomian decomposition Shehu transform method (ADShTM).

Chapter-6: showcases the study of the COVID-19 model [16], as a part of addressing epidemic models with depth in the light of fractional calculus (in general), revealing the strength of fractional differential equations as long as it takes to demonstrate the significance towards real-world problems touching human life existence directly. The idea is to use the Atangana-Baleanu fractional derivative in place of the Caputo derivative, in the well-established semi-analytical method – "Homotopy Perturbation Laplace Transform Method (HPLTM) [88]", and thus achieving for more accurate comparative results. The existence, uniqueness, and stability analysis are all well discussed with the help of the Banach's fixed point theorem. Many comparative graphical studies are incorporated in support of the significance of the present fractional-ordered model. The objective behind the work is to present a model which fits the actual data more accurately with those obtained by simulating with the sample data related to COVID-19, from the WHO official website.

Chapter-7: which as a part of studying one more epidemic model, represents another major disease diabetes, the human race has been facing and striving from ancient times [136]. The aim of this study is to analyze a comprehensive regulatory framework for managing glucose and insulin in blood in the presence of diabetes mellitus more effectively. Mathematical model of diabetes [136] what we considered, is dealt with and examined in the framework of the fractional ordered system by involving ABC fractional derivative. This whole problem has been addressed using a semi-analytical technique, namely the "Adomian decomposition Laplace transform method (ADLTM)" [71]. The usefulness of this ADLTM technique is discussed by comparing the results with other classical methods, viz. homotopy perturbation Laplace transform method (HPLTM) [88] and modified homotopy analysis transform method (MHATM) [155]. Using the Banach fixed point theorem, the existence and stability analysis of the solution has been discussed. Certain figures and tables illustrating this diabetes model are given with few fractional orders. We also have used the Maple software to generate the numeric and graphical plots. This detailed investigation also explores how well the level of glucose and insulin affects the dynamics of disease infection.

1.5 Preliminaries

Definition 1.1. [107] The left-sided Riemann-Liouville fractional integral operator of order α is given by

$$ {}_aD_t^{-\alpha}\left[u(x,t)\right] = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}u(x,\tau)d\tau, \quad n-1<\alpha\leq n. \tag{1.1}$$

Definition 1.2. [120] The Caputo time-fractional derivative of order $\alpha > 0$ of $u(x,t)$ is defined as

$$ {}_0^CD_t^{\alpha}\left[u(x,t)\right] = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-a-1}\frac{\partial^n u(x,\tau)}{\partial\tau^n}d\tau, & n-1<\alpha<n \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in N. \end{cases} \tag{1.2}$$

Definition 1.3 [120] The one-parameter Mittag-Leffler function for $\alpha > 0$ and $z \in C$ is

$$ E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \text{for } |z|<1. \tag{1.3}$$

Definition 1.4. The time-fractional derivative of $u(x,t)$ with order α in Caputo-Fabrizio sense, known as Caputo-Fabrizio (CF) [34] derivative of order α is defined as

$$ {}_a^{CF}D_t^{\alpha}\left[u(x,t)\right] = \begin{cases} \frac{M(\alpha)}{1-\alpha}\int_a^t \exp\left(\frac{-\alpha(\mathrm{t}-\tau)}{(1-\alpha)}\right)\frac{\partial^n u(x,\tau)}{\partial\tau^n}d\tau, \; u\in H^1(a,b), \, b>a, \\ \frac{\alpha M(\alpha)}{1-\alpha}\int_{-\infty}^t \exp\left(\frac{-\alpha(\mathrm{t}-\tau)}{(1-\alpha)}\right)\left(u(x,t)-u(x,\tau)\right)d\tau, \; u\notin H^1(a,b), \end{cases} \tag{1.4}$$

where $n-1<\alpha\leq n$ if $\alpha=\beta+n-1$ for any $\beta\in(0,1]$ and $u\in N$. Also, if $u\notin H^1(a,b)$, in that case $u(x,t)\in L^1(-\infty,b)$ for a given x. Also, $M(\alpha)$ holds the properties $M(0)=M(1)=1$

Definition 1.5. The time-fractional derivative of $u \in H^1(a,b), b > a$ with order α in Atangana-Baleanu sense, known as Atangana-Baleanu fractional derivative in Caputo sense (ABC) [24, 19] with order α is defined as

$$ {}_{a}^{ABC}D_t^{\alpha}\left[u(x,t)\right] = \frac{M(\alpha)}{n-\alpha}\int_a^t E_{\alpha}\left(\frac{-\alpha(t-\tau)^{\alpha}}{(n-\alpha)}\right)\frac{\partial^n u(x,\tau)}{\partial \tau^n}d\tau, \tag{1.5}$$

where $n-1 < \alpha \leq n$ if $\alpha = \beta + n - 1$ for any $\beta \in (0,1]$ and $n \in N$. Also, $M(\alpha)$ holds the properties $M(0) = M(1) = 1$ and $E_{\alpha}(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ is Mittag-Leffler[120] function in one-parameter.

Theorem 1.1. [19] Laplace Transform of ABC fractional derivative of $u(x,t)$ with conditions $u(x,0), \frac{\partial}{\partial t}u(x,0), \frac{\partial^2}{\partial t^2}u(x,0), ..., \frac{\partial^{n-1}}{\partial t^{n-1}}u(x,0)$ and order $n-1 < \alpha \leq n$ is

$$ \mathcal{L}\left\{{}_{0}^{ABC}D_t^{\alpha}u(x,t)\right\} = \frac{M(\alpha)}{(n-\alpha)}\left[\frac{s^{\alpha}L\left\{u(x,t)\right\} - \sum\limits_{k=0}^{n-1} s^{\alpha-k-1}D_t^k u(x,0)}{s^{\alpha} + \left(\frac{\alpha}{n-\alpha}\right)}\right]. \tag{1.6}$$