Chapter 6

Hard Water-Induced Renal Disease: A Fractional-Order Model

6.1 Introduction

Hard water disease is a global issue that affects numerous regions, causing detrimental impacts on human health, industrial processes, and infrastructure. It is characterized by the accumulation of dissolved minerals, particularly calcium and magnesium, in water sources. These minerals can lead to the formation of mineral deposits, commonly known as limescale, which can clog pipes, reduce water flow, and impact the efficiency of appliances and equipment [93].

Using hard water for an extended period of time can lead to a variety of health problems, including kidney issues and other diseases [146, 161]. Here are some potential impacts: (1) Kidney stones: Hard water contains high levels of minerals like calcium and magnesium. Consuming these minerals in excess can lead to the formation of kidney stones. These stones can cause extreme pain and discomfort when passing through the urinary system. (2) Decreased kidney function: The minerals present in hard water can accumulate in the kidneys over time, leading to decreased kidney function. This can put additional stress on the kidneys and increase the risk of developing chronic kidney disease. (3) Urinary tract infections (UTIs): Hard water can also contribute to the development of urinary tract infections. The minerals in hard water can create an alkaline environment in the urinary tract, which is more favorable for bacterial growth. (4) Dermatological issues: Washing with hard water can dry out the skin and hair, leading to various dermatological problems such as dry skin, eczema, and brittle hair. Hard water can also worsen conditions like acne and psoriasis. (5) Cardiovascular disease: Some studies have suggested a link between hard water consumption and an increased risk of cardiovascular disease. Elevated levels of calcium and magnesium in hard water may contribute to the development of atherosclerosis, a condition where plaque builds up in the arteries.

To mitigate these risks, it is important to consider water softening options such as using a water softener or installing a water treatment system, or using bottled or filtered water for drinking and cooking purposes [66].To better understand the dynamics and implications of hard water disease, mathematical modelling approaches have gained recognition as valuable tools for studying and predicting its effects. By formulating models that capture the essential factors contributing to the disease's emergence and spread, researchers can gain insights into its behavior and guide decision-making processes for prevention and mitigation.

We develop to examine a comprehensive fractional mathematical model that incorporates some significant parameters, such as the concentration of magnesium and calcium level in hard water W, the probability of getting chronic kidney disease due to hard water consumptions, $\lambda(W) = \dfrac{W}{K+W}$, the absorbing rate of magnesium and calcium from hard water β, death rate μ, recovery rate γ, the highest concentration level of minerals in hard water K, magnesium and calcium concentration growth rate b, and water treatment effectiveness rate c. By involving these parameters, Tambaru et al. [135] has constructed hard water disease model as follows:

$$\frac{dS}{dt} = A - \left[\beta\left(\frac{W}{K+W}\right) + \mu\right] S, \tag{6.1}$$

$$\frac{dI}{dt} = \beta\left(\frac{W}{K+W}\right) S - (\gamma - \mu)I, \tag{6.2}$$

$$\frac{dR}{dt} = \gamma I - \mu R, \tag{6.3}$$

$$\frac{dW}{dt} = bW\left(1 - \frac{W}{K}\right) - cW. \tag{6.4}$$

This chapter is summarized as follows: Section 6.2 states the definitions related to fractional calculus. Section 6.3 covers the presented hard water model with fractional-order by involving Caputo derivative. The existence and uniqueness of the system is demonstrated in section 6.4. Local and Global stability of the model is discussed in section 6.5. Section 6.6 is a showcase of the solution of the hard water model by using Adomian Decomposition General Transform method. Sections 6.7 and 6.8 are describe the numerical results and conclusion respectively.

6.2 Preliminaries

Definition 6.2.1. *For order $\alpha > 0$ and $e : \mathbb{R}^+ \to \mathbb{R}$, Riemann-Liouvillie [106] established a fractional integral operator as:*

$$ {}_0I^{\alpha}(e(\varkappa)) = \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} e(\nu) d\nu, \ \ \varkappa > 0. \tag{6.5}$$

Definition 6.2.2. *Riemann-Liouville [106] established a fractional differential operator for the function $e(\varkappa)$ with order $\alpha > 0$ is defined to be:*

$$ {}_0D^{\alpha}(e(\varkappa)) = \frac{1}{\Gamma(\mathrm{j} - \alpha)} \left(\frac{d}{d\varkappa}\right)^{\mathrm{j}} \int_0^{\varkappa} (\varkappa - \nu)^{\mathrm{j}-\alpha-1} e(\nu) d\nu, \ \ \varkappa > 0, \tag{6.6}$$

where, $\mathrm{j} - 1 < \alpha \leq \mathrm{j}, \mathrm{j} \in \mathbb{N}$.

Definition 6.2.3. *Caputo [106] contributed in the field of fractional calculus by defining the derivative of fractional order $\alpha > 0$ for the function $e(\varkappa) \in \mathbb{C}^{\mathrm{j}}$ as*

$$ {}_0^CD^{\alpha}(e(\varkappa)) = \frac{1}{\Gamma(\mathrm{j} - \alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\mathrm{j}-\alpha-1} e^{\mathrm{j}}(\nu) d\nu, \ \ \varkappa > 0, \tag{6.7}$$

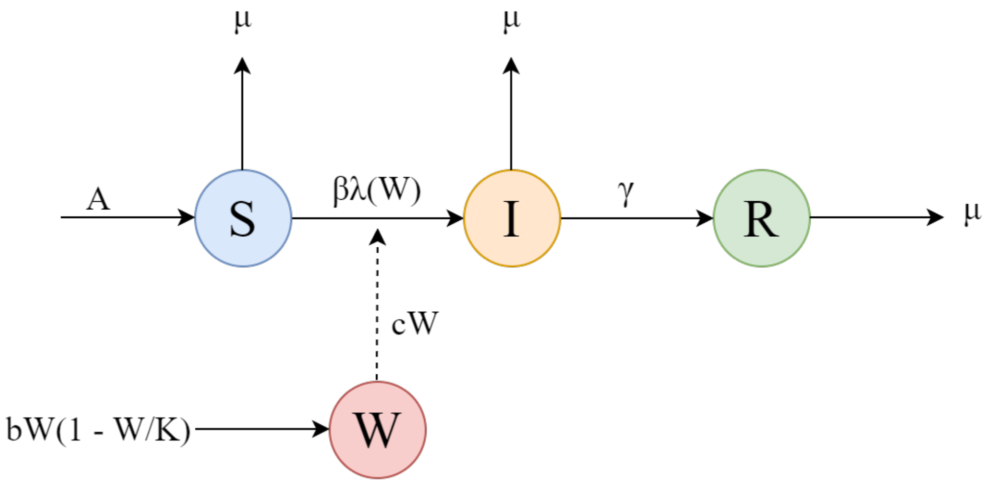
and is defined for absolute continuous functions where, $\mathrm{j} - 1 < \alpha \leq \mathrm{j}$, $\mathrm{j} \in \mathbb{N}$.

6.3 Fractional hard water model

We present fractional-order mathematical model by replacing the classical-order model (6.1) with the Caputo fractional derivative operator. As we are aware, the fundamental notion of fractional-order modification is that fractional-order derivatives are simply used in place of integer-order derivatives is unlikely to be appropriate since such generalization might result in an unphysical model. In such instance, a correct generalization requires absolute equal dimensionality. As a consequence, the Caputo fractional model for hard water disease resembles this into initial value problem (IVP):

$$\begin{aligned}
{}_0^C D_t^\alpha (S(t)) &= A - \left[\beta\left(\frac{W}{K+W}\right) + \mu\right] S(t), \\
{}_0^C D_t^\alpha (I(t)) &= \beta\left(\frac{W}{K+W}\right) S - (\gamma - \mu) I(t), \\
{}_0^C D_t^\alpha (R(t)) &= \gamma I(t) - \mu R(t), \\
{}_0^C D_t^\alpha (W(t)) &= bW(t)\left(1 - \frac{W(t)}{K}\right) - cW(t).
\end{aligned} \tag{6.8}$$

The initial conditions for the proposed model are $S(0) = S_0, I(0) = I_0, R(0) = R_0$ and $W(0) = W_0$ with S_0, I_0, R_0 and $W_0 > 0$.



Hard water disease model diagram.

6.4 Qualitative analysis

Using Sadovskii's fixpoint theorem, we discuss the existence and uniqueness of the epidemiological fractional order model (6.8) by considering banach space $\mathfrak{S}$ with $\varkappa \in [0, \mathfrak{T}]$, $S, I, R, W \in \mathbb{C}(\mathfrak{R}, \mathfrak{S}) \cap L^1{}_{loc}(\mathfrak{R}, \mathfrak{S})$ and $\mathfrak{R} = \{(\varkappa, K) : \varkappa \in [0, \mathfrak{T}]\}$, $K \in B(0, \bar{\mathfrak{r}})$ for some $\mathfrak{T}, \bar{\mathfrak{r}} > 0$ will be analyzed in this section.

Theorem 6.4.1. *Consider $\mathbb{B} \subset \mathfrak{S}$ and let $\Phi : \mathbb{B} \to \mathbb{B}$ be a mapping with condensation property. Then, there exists a fixed point of the mapping Φ within the set $\mathbb{B}$.*

Examine IVP on the cylinder $\mathfrak{R} = \{(\varkappa, K) \in \mathbb{R} \times \mathfrak{S} : \varkappa \in [0, \mathfrak{T}], K \in B(0, \bar{\mathfrak{r}})\}$ for some fixed $\mathfrak{T}, \bar{\mathfrak{r}} > 0$. Assume that $\exists\ \mathfrak{p} \in (0, \bar{\mathfrak{r}})$, $\mathfrak{M}_1, \mathfrak{M}_2, L, L_1 \in L_{1/\mathfrak{p}}([0, \mathfrak{T}], \mathbb{R}^+)$ & S_1, I_1, R_1, W_1, $S_2, I_2, R_2, W_2 \in \mathbb{C}(\mathbb{R}, \mathfrak{S}) \cap L^1{}_{loc}(\mathbb{R}, \mathfrak{S})$ such that $S = S_1 + S_2, I = I_1 + I_2, R = R_1 + R_2, W = W_1 + W_2$ and the following assumptions hold,

Assumption 1– *$S_1, I_1, R_1,$ and W_1 are bounded and Lipschitz.*

Assumption 2– *$S_2, I_2, R_2,$ and W_2 are compact and bounded.*

Assumption 3– *$|\mathbb{R}(\varkappa, K) - \mathbb{R}(\varkappa, J)| \leq L_1(\varkappa) \|K - J\|$, $\forall\ (\varkappa, K), (\varkappa, J) \in \mathbb{R}$.*

Applying (6.2.1) on (7.2), the Lemma (6.4.1) can be formulated as follows:

Lemma 6.4.1. *The given IVP is equivalent to the following set of integral equations:*

$$
\begin{aligned}
S(\varkappa) &= S(0) + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} S_1(\nu, S(\nu)) d\nu + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} S_2(\nu, S(\nu)) d\nu, \\
I(\varkappa) &= I(0) + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} I_1(\nu, I(\nu)) d\nu + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} I_2(\nu, I(\nu)) d\nu, \\
R(\varkappa) &= R(0) + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} R_1(\nu, R(\nu)) d\nu + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} R_2(\nu, R(\nu)) d\nu, \\
W(\varkappa) &= W(0) + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} T_1(\nu, T(\nu)) d\nu + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} T_2(\nu, T(\nu)) d\nu.
\end{aligned} \tag{6.9}
$$

As demonstrated by the next two theorems, it is now relatively straightforward to establish the existence and uniqueness of solution (7.2)*.*

Theorem 6.4.2. *In accordance with **Assumption 1** and **2**, the given IVP has a minimum of one solution in $[0, \mathfrak{T}]$, given that,*

$$\Theta = \frac{C\|L\|_{1/\mathfrak{p}}(\mathfrak{T})^{\Delta}}{\Gamma(\alpha)} < 1, \qquad where\ \ \Delta = (\alpha - \mathfrak{p})\,, C = \left(\frac{1-\mathfrak{p}}{\alpha - \mathfrak{p}}\right)^{1-\mathfrak{p}}$$

Proof. Select $\bar{\mathfrak{r}} \ni |S_0| + \Gamma(\alpha)^{-1} C \left(\|\mathfrak{M}_1\|_{1/\mathfrak{p}} + \|\mathfrak{M}_2\|_{1/\mathfrak{p}}\right)(\mathfrak{T})^{\Delta} \leq \bar{\mathfrak{r}}$ and let $B_{\bar{\mathfrak{r}}} = \{K : \|K\| \leq \bar{\mathfrak{r}}\}$ be the closed $\bar{\mathfrak{r}}$-ball in $\mathbb{BC}([0,\mathfrak{T}], \mathfrak{S})$ with sup norm $\|.\|$. Using Lemma (6.4.1) one can obtain that point of $S : B_{\bar{\mathfrak{r}}} \to \mathbb{BC}([0,\mathfrak{T}], \mathfrak{S}), K \mapsto S_1K + S_2K$ with the following:

$$S_1K(\varkappa) = S(0) + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} S_1(\nu, K(\nu))\, d\nu,$$

$$S_2K(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} S_2(\nu, K(\nu))\, d\nu,$$

as a solution of the model (7.2). In three phases, we demonstrate that $S(\varkappa)$ is condensing, and hence the presence of a fixed point for $S(\varkappa)$ holds from Theorem (6.4.1).

Step 1: We must demonstrate $S(B_{\bar{\mathfrak{r}}}) \subset B_{\bar{\mathfrak{r}}}$. For $K \in B_{\bar{\mathfrak{r}}}$, we have

$$\begin{aligned}
|S(\varkappa)| &\leq |S_0| + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} |S(\nu, K(\nu))| d\nu \\
&\leq |S_0| + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} |S_1(\nu, K(\nu))| d\nu + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} |S_2(\nu, K(\nu))| d\nu \\
&\leq |S_0| + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} \mathfrak{M}_1(\nu)\, d\nu + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} \mathfrak{M}_2(\nu)\, d\nu \\
&\leq |S_0| + \frac{1}{\Gamma(\alpha)} \left(\int_0^{\varkappa} (\varkappa - \nu)^{\frac{\alpha-1}{1-\mathfrak{p}}} d\nu\right)^{1-\mathfrak{p}} \left(\int_0^{\varkappa} (\mathfrak{M}_1(\nu))^{\frac{1}{\mathfrak{p}}} d\nu\right)^{\mathfrak{p}} \\
&+ \frac{1}{\Gamma(\alpha)} \left(\int_0^{\varkappa} (\varkappa - \nu)^{\frac{\alpha-1}{1-\mathfrak{p}}} d\nu\right)^{1-\mathfrak{p}} \left(\int_0^{\varkappa} (\mathfrak{M}_2(\nu))^{\frac{1}{\mathfrak{p}}} d\nu\right)^{\mathfrak{p}} \\
&\leq |H_0| + \frac{C\left(\|\mathfrak{M}_1\|_{\frac{1}{\mathfrak{p}}} + \|\mathfrak{M}_2\|_{\frac{1}{\mathfrak{p}}}\right)}{\Gamma(\alpha)} (\mathfrak{T})^{\Delta} \leq r, \text{and thus } S(B_{\bar{\mathfrak{r}}}) \subset B_{\bar{\mathfrak{r}}}.
\end{aligned}$$

Step 2: We illustrate that S_1 is a contraction. If $K, J \in B_{\bar{\mathfrak{r}}}$ we lead to

$$\begin{aligned} |S_1(K(\varkappa)) - S_1(J(\varkappa))| &\le \frac{1}{\Gamma(\alpha)} \int\limits_0^{\varkappa} (\varkappa-\nu)^{\mathfrak{r}-1} L(\nu) |K(\nu) - J(\nu)| d\nu \\ &\le \frac{1}{\Gamma(\alpha)} \left(\int\limits_0^{\varkappa} (\varkappa-\nu)^{\frac{\alpha-1}{1-\mathfrak{p}}} d\nu \right)^{1-\mathfrak{p}} \left(\int\limits_0^{\varkappa} L^{\frac{1}{\mathfrak{p}}}(\nu) d\nu \right)^{\mathfrak{p}} \|K-J\| \\ &\le \frac{C\|L\|_{\frac{1}{\mathfrak{p}}} (\mathfrak{T})^{\Delta}}{\Gamma(\alpha)} \|K-J\|, \end{aligned}$$

and hence $S_1(\varkappa)$ is a contraction with, $\|S_1(K) - S_1(J)\| \le \Theta \|K-J\|$.

Step 3: We want to show that $\mathfrak{S}_2$ is compact. For $0 \le \wp_1 \le \wp_2 \le \mathfrak{T}$, we have

$$\begin{aligned} |S_2(K(\wp_2)) - S_2(K(\wp_1))| &\le \frac{1}{\Gamma(\alpha)} \left| \int\limits_0^{\wp_2} (\wp_2-\nu)^{\alpha-1} S_2(\nu, K(\nu)) d\nu \right| \\ &- \frac{1}{\Gamma(\alpha)} \left| \int\limits_0^{\wp_1} (\wp_1-\nu)^{\alpha-1} S_2(\nu, K(\nu)) d\nu \right| \\ &\le \frac{1}{\Gamma(\alpha)} \left| \int\limits_0^{\wp_1} (\wp_2-\nu)^{\alpha-1} S_2(\nu, K(\nu)) d\nu \right| \\ &+ \frac{1}{\Gamma(\alpha)} \left| \int\limits_{\wp_1}^{\wp_2} (\wp_2-\nu)^{\alpha-1} S_2(\nu, K(\nu)) d\nu \right| \\ &- \frac{1}{\Gamma(\alpha)} \left| \int\limits_0^{\wp_1} (\wp_1-\nu)^{\alpha-1} S_2(\nu, K(\nu)) d\nu \right| \end{aligned}$$

$$\begin{aligned} |S_2K(\wp_2) - S_2K(\wp_1)| &\le \frac{1}{\Gamma(\alpha)} \int\limits_0^{\wp_1} ((\wp_1-\nu)^{\alpha-1} - (\wp_2-\nu)^{\alpha-1}) |S_2(\nu, K(\nu))| d\nu \\ &+ \frac{1}{\Gamma(\alpha)} \int\limits_{\wp_1}^{\wp_2} (\wp_2-\nu)^{\alpha-1} |S_2(\nu, K(\nu))| d\nu \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \int_{0}^{\wp_1} ((\wp_1-\nu)^{\alpha-1} - (\wp_2-\nu)^{\alpha-1})\mathfrak{M}_2(\nu)d\nu \\
&+ \frac{1}{\Gamma(\alpha)} \int_{\wp_1}^{\wp_2} (\wp_2-\nu)^{\alpha-1}\mathfrak{M}_2(\nu)d\nu \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\left(\int_{0}^{\wp_1} \left((\wp_1-\nu)^{\frac{\alpha-1}{1-\mathfrak{p}}} - (\wp_2-\nu)^{\frac{\alpha-1}{1-\mathfrak{p}}} \right) d\nu \right)^{1-\mathfrak{p}} \right) \\
&\times \left(\int_{0}^{\wp_1} (\mathfrak{M}_2(\nu))^{\frac{1}{\mathfrak{p}}} d\nu \right)^{\mathfrak{p}} \\
&+ \frac{1}{\Gamma(\alpha)} \left(\int_{\wp_1}^{\wp_2} (\wp_2-\nu)^{\frac{\alpha-1}{1-\mathfrak{p}}} d\nu \right)^{1-\mathfrak{p}} \left(\int_{0}^{\wp_1} (\mathfrak{M}_2(\nu))^{\frac{1}{\mathfrak{p}}} d\nu \right)^{\mathfrak{p}} \\
&\leq \frac{C}{\Gamma(\alpha)} \|\mathfrak{M}_2\|_{\frac{1}{\mathfrak{p}}} \left[\left((\wp_2-\wp_1)^{\frac{\alpha-1}{1-\mathfrak{p}}} \right)^{1-\mathfrak{p}} + (\wp_2-\wp_1)^{\alpha-\mathfrak{p}} \right] \\
&\leq \frac{2C\|\mathfrak{M}_2\|_{\frac{1}{\mathfrak{p}}}}{\Gamma(\alpha)} (\wp_2-\wp_1)^{\alpha-\mathfrak{p}}.
\end{aligned}
$$

The right side of this inequality is unaffected by K. We establish the compactness of S_2 by its relative compactness in $S_2(B_{\bar{\mathfrak{r}}})$ using the Arzela-Ascoli theorem. Due to the fact that S_1 and S_2 have distinct features (contraction and compactness respectively), their composite, represented as S, forms a condensing map on $B_{\bar{\mathfrak{r}}}$. This, along with Theorem (6.4.1), ensures a fixpoint for S, which is relevant to remaining variables such as $I(\varkappa), R(\varkappa), W(\varkappa)$. □

Theorem 6.4.3. *Under consideration of* ***Assumption 3*** *and* $\Theta = \dfrac{C\|L_1\|_{1/\mathfrak{p}}(\mathfrak{T})^{\Delta}}{\Gamma(\alpha)} < 1$*, the IVP has unique solution on* $[0, \mathfrak{T}]$*.*

Proof. Define the mapping F by

$$FS(\varkappa) = S(0) + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} S(\nu, S(\nu))\, d\nu.$$

For $S(\varkappa), S_1(\varkappa) \in B_{\bar{\mathfrak{r}}}$, we have

$$\begin{aligned}
|FS(\varkappa) - FS_1(\varkappa)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha-1} L_1(\nu)\, |S(\nu) - S_1(\nu)| d\nu \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^{\varkappa} (\varkappa - \nu)^{\frac{\alpha-1}{1-\mathfrak{p}}} d\nu \right)^{1-\mathfrak{p}} \left(\int_0^{\varkappa} L_1{}^{\frac{1}{\mathfrak{p}}}(\nu)\, d\nu \right)^{\mathfrak{p}} \|S - S_1\| \\
&\leq \frac{C \|L_1\|_{1/\mathfrak{p}} (\mathfrak{T})^{\Delta}}{\Gamma(\alpha)} \|S - S_1\|.
\end{aligned}$$

Thus, the condition $\Theta = \dfrac{C \|L_1\|_{1/\mathfrak{p}} (\mathfrak{T})^{\Delta}}{\Gamma(\alpha)} < 1$ ensure the existence of a unique solution. Similarly unique solution exists for other model equations. □

6.5 Stability analysis

6.5.1 Local stability

We consider two equilibrium points to explore local stability for hard water disease model (6.1).

1. Disease-free (normal water) equilibrium point E^0 :

$$E^0 = \left(\frac{A}{\mu}, 0, 0, 0 \right). \tag{6.10}$$

It means minerals concentration level are normal in water with $I = R = W = 0$. Hence, Equilibrium point E^0 is settled as disease-free system.

2. Endemic equilibrium point E^{end} :

$$E^{end} = \left(S^{end}, E^{end}, I^{end}, R^{end}\right), \tag{6.11}$$

where

$$S^{end} = \frac{(2b-c)A}{(b-c)\beta + (2b-c)\mu},$$

$$I^{end} = \frac{(b-c)A\beta}{(\mu+\gamma)\left((b-c)\beta + (2b-c)\mu\right)},$$

$$R^{end} = \frac{(b-c)A\beta\gamma}{(\mu^2+\mu\gamma)\left((b-c)\beta + (2b-c)\mu\right)},$$

$$W^{end} = \frac{(b-c)K}{b}.$$

We acquire the endemic equilibrium point E^{end}, by equating system (6.1) with zero. Note that when we take $b = c$, results lead to the disease-free equilibrium point E^0.

Theorem 6.5.1. *Disease-free equilibrium point E^0 for hard water model* (6.1) *is locally stable if condition $b < c$ is satisfied.*

Proof. First, we construct a Jacobian matrix J for hard water system (6.1).

$$J = \begin{bmatrix} -\mu & 0 & 0 & -\frac{bA}{K\mu} \\ 0 & -\gamma-\mu & 0 & \frac{bA}{K\mu} \\ 0 & \gamma & -\mu & 0 \\ 0 & 0 & 0 & b-c \end{bmatrix}. \tag{6.12}$$

To find the eigenvalues of matrix J, the characteristic polynomial is

$$\lambda^4 + a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{6.13}$$

with the basic simplification, coefficients of (6.13) can be defined as

$$\begin{aligned} a_0 &= c - b + 3\mu + \gamma, \\ a_1 &= \mu(2\gamma + 3\mu) - (\gamma + 3\mu)b + (\gamma + 3\mu)c, \\ a_2 &= \mu^3 - 3(b - c)\mu^2 + \gamma\mu^2 - 2(b - c)\mu\gamma, \\ a_3 &= -\mu^2(b - c)(\mu + \gamma). \end{aligned} \tag{6.14}$$

Using Routh-Hurwitz criteria, the disease-free equilibrium point E^0 is locally stable if $a_0 > 0, a_2 > 0, a_3 > 0$, and $a_0a_1a_2 - a_2^2 - a_0^2a_3 > 0$.
From (6.14), we can say that if $b < c$, then the value of coefficients a_0, a_2, a_3 are positive.
And for

$$\begin{aligned} a_0a_1a_2 - a_2^2 - a_0^2a_3 = &(c - b + 3\mu + \gamma)\left(\mu(2\gamma + 3\mu) - (\gamma + 3\mu)(b - c)\right)\big(\mu^3 - 3(b - c)\mu^2 \\ &+ \gamma\mu^2 - 2(b - c)\mu\gamma\big) - \big(\mu^3 - 3(b - c)\mu^2 + \gamma\mu^2 \\ &- 2(b - c)\mu\gamma\big)^2\left(-\mu^2(b - c)(\mu + \gamma)\right), \\ = &\ 2\mu(\gamma + 2\mu)^2(b - c - \mu)^2(\gamma + \mu - b + c). \end{aligned} \tag{6.15}$$

From (6.15), $a_0a_1a_2 - a_2^2 - a_0^2a_3$ is also positive for $b < c$.
So, If $b < c$, then E^0 is stable, else unstable.
We can also discuss the local stability from eigenvalues. By solving Jacobian matrix J, we get eigenvalues $-\mu, -\mu, -\gamma - \mu$, and $b - c$. And all are negative for $b < c$.
Thus, we can claim that point E^0 is locally stable for $b < c$. □

Theorem 6.5.2. *Endemic equilibrium point E^{end} for hard water model* (6.1) *is locally stable if condition $b > c$ is satisfied.*

Proof. To discuss the Endemic equilibrium point E^{end}, we follow the same process.
As we obtained earlier in (6.14), we have $a_0 > 0, a_2 > 0, a_3 > 0$, and $a_0a_1a_2 - a_2^2 - a_0^2a_3 > 0$.
For condition $b > c$, all the eigenvalues of Jacobian matrix J are negative.
Hence, the point E^{end} is locally stable for $b > c$. □

6.5.2 Global stability

Let

$$ {}_0^C D_{\varkappa}^{\alpha} \mathfrak{u}_1(\varkappa) = \mathfrak{g}(\varkappa, \mathfrak{u}_1(\varkappa)), \qquad where, 0 < \alpha < 1. \tag{6.16}$$

with initial condition $\mathfrak{u}_1(0) = \mathfrak{u}_{10}$, where, $\mathfrak{u}_1 \in \mathbb{R}, \varkappa \in (0, +\infty)$.

We are currently examining the global stability of (6.16). Given $\varepsilon > 0$, and considering a continuous function φ defined over the interval $[0, +\infty)$ with values in the positive real numbers. Consider following relations:

$$ |{}_0^C D_{\varkappa}^{\alpha} \mathfrak{u}_1(\varkappa) - \mathfrak{g}(\varkappa, \mathfrak{u}_1(\varkappa))| \leq \varepsilon; \tag{6.17}$$

$$ |{}_0^C D_{\varkappa}^{\alpha} \mathfrak{u}_1(\varkappa) - \mathfrak{g}(\varkappa, \mathfrak{u}_1(\varkappa))| \leq \varphi(\varkappa); \tag{6.18}$$

$$ |{}_0^C D_{\varkappa}^{\alpha} \mathfrak{u}_1(\varkappa) - \mathfrak{g}(\varkappa, \mathfrak{u}_1(\varkappa))| \leq \varepsilon\varphi(\varkappa). \tag{6.19}$$

Definition 6.5.1. *The IVP* (6.16) *exhibits Ulam-Hyers(U-H) stability if* $\exists\ c_{\mathfrak{g}} > 0$ *such that* $\forall\ \varepsilon > 0$ *and each* $\mathfrak{u}_1 \in C[0, +\infty)$ *satisfying* (6.17)*, a solution* $\mathfrak{u}_2 \in C[0, +\infty)$ *of* (6.16) *is present, satisfying*

$$|\mathfrak{u}_1(\varkappa) - \mathfrak{u}_2(\varkappa)| \leq \varepsilon c_{\mathfrak{g}}.$$

Definition 6.5.2. *The IVP* (6.16) *demonstrates Generalized Ulam-Hyers(G-U-H) stability if* $\exists\ c_{\mathfrak{g}} \in \mathbb{R}^+$ *such that* $c_{\mathfrak{g}}(0) = 0$, $\forall\ \varepsilon > 0$ *and* $\mathfrak{u}_1 \in C$ *of* (6.18)*, there exists a solution* $\mathfrak{u}_2 \in C$ *of* (6.16) *fullfilling*

$$|\mathfrak{u}_1(\varkappa) - \mathfrak{u}_2(\varkappa)| \leq c_{\mathfrak{g}}(\varepsilon).$$

Definition 6.5.3. *The IVP* (6.16) *is Ulam-Hyers-Rassias(U-H-R) stable if* $\exists\ c_{\mathfrak{g},\varphi} \in \mathbb{R}$, $\forall$ $\varepsilon > 0$ *and every* $\mathfrak{u}_1 \in C$ *of* (6.19), *there exists a solution* $\mathfrak{u}_2 \in C$ *of* (6.16) *addressing*

$$|\mathfrak{u}_1(\varkappa) - \mathfrak{u}_2(\varkappa)| \leq \varepsilon c_{\mathfrak{g},\varphi}\varphi(\varkappa).$$

Definition 6.5.4. *The IVP* (6.16) *reveals Generalized Ulam-Hyers-Rassias(G-U-H-R) stability if* $c_{\mathfrak{g},\varphi} \in \mathbb{R}$, *for every* $\mathfrak{u}_1 \in C$ *of* (6.18), *there exists a solution* $\mathfrak{u}_2 \in C$ *of* (6.16) *with*

$$|\mathfrak{u}_1(\varkappa) - \mathfrak{u}_2(\varkappa)| \leq c_{\mathfrak{g},\varphi}\varphi(\varkappa).$$

Hypothesis 2. *Let* $\varphi \in C[0, +\infty)$ *is an accumulating function, then* $\exists\ \chi_\varphi > 0$, *implying that*

$$\frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - \nu)^{\alpha - 1} \varphi(\nu)\, d\nu \leq \chi_\varphi \varphi(\varkappa), \qquad \varkappa > 0.$$

Lemma 6.5.1. *Let us construct two continuous functions namely* $\mathfrak{x}$ *&* $\mathfrak{y}$ *over* $[0, \mathfrak{T}] \times [0, +\infty)$ *where* $\mathfrak{T} \leq \infty$. *If* $\mathfrak{y}$ *is increasing and* $\exists\ \mu, \zeta > 0$, *implying that*

$$\mathfrak{x}(\varkappa) \leq \mathfrak{y}(\varkappa) + \mu \int_0^{\varkappa} (\varkappa - \nu)^{\zeta - 1} \mathfrak{x}(\nu)\, d\nu, \qquad \varkappa \geq 0.$$

then

$$\mathfrak{x}(\varkappa) \leq \mathfrak{y}(\varkappa) + \int_0^{\varkappa} \left[\sum_{k=0}^{\infty} \frac{(\mu\Gamma(\zeta))^k}{\Gamma(k\zeta)} (\varkappa - \nu)^{\zeta - 1} \mathfrak{y}(\nu) \right] d\nu, \qquad \varkappa \geq 0.$$

If $\mathfrak{y}(\varkappa) = a$ *(constant) on* $\varkappa \in [0, \mathfrak{T})$, *then*

$$\mathfrak{x}(\varkappa) \leq a\mathrm{E}_\zeta\left(\mu\Gamma(\zeta)\varkappa^\zeta\right), \qquad \varkappa \geq 0,$$

where, $\mathrm{E}_\zeta(\bullet)$ *is the Mittag Leffler function.*

Theorem 6.5.3. *If the criteria of Hypothesis* (2) *is fulfilled, then the IVP* (6.16) *is the G-U-H-R stable.*

Proof. Assuming $\mathfrak{u}_1$ is a solution of (6.18) over the interval $C[0,\mathfrak{T})$, we consider $\mathfrak{u}_2$ as a solution of (6.16). Thus,

$$|\mathfrak{u}_1(\varkappa) - \mathfrak{u}_{10}(\varkappa) - \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa}(\varkappa-\nu)^{\alpha-1}\mathfrak{g}(\nu,\mathfrak{u}_1(\nu))d\nu| \leq \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa}(\varkappa-\nu)^{\alpha-1}\varphi(\nu)\,d\nu$$
$$\leq \chi_{\varphi}\varphi(\varkappa).$$

It follows from these connections:

$$|\mathfrak{u}_1(\varkappa) - \mathfrak{u}_2(\varkappa)| \leq |\mathfrak{u}_1(\varkappa) - \mathfrak{u}_{10}(\varkappa) - \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa}(\varkappa-\nu)^{\alpha-1}\mathfrak{g}(\nu,\mathfrak{u}_1(\nu))\,d\nu|$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{\varkappa}(\varkappa-\nu)^{\alpha-1}|\mathfrak{g}(\nu,\mathfrak{u}_1(\nu)) - \mathfrak{g}(\nu,\mathfrak{u}_2(\nu))|d\nu$$
$$\leq \chi_{\varphi}\varphi(\varkappa) + \frac{N}{\Gamma(\alpha)}\int_0^{\varkappa}(\varkappa-\nu)^{\alpha-1}|\mathfrak{u}_1(\nu) - \mathfrak{u}_2(\nu)|d\nu.$$

As per Lemma (6.5.1), $\exists$ constant $N^* > 0$ separated from $\chi_{\varphi}\varphi(\varkappa)$. This constant satisfies $|\mathfrak{u}_1(\varkappa) - \mathfrak{u}_2(\varkappa)| \leq N^*\chi_{\varphi}\varphi(\varkappa) := a_{\mathfrak{g},\varphi}\varphi(\varkappa)$. As a result, the IVP (6.16) can be classified as having G-U-H-R stability. □

Corollary 6.5.1. *Using the same logic as in Theorem* (6.5.3)*, one can demonstrate that IVP* (6.16)*, coupled by* (6.19)*, achieves U-H-R stability.*

Corollary 6.5.2. *By employing the similar methods of Theorem* (6.5.3) *with* (6.17)*, one is able to illustrate IVP* (6.16) *is U-H stable.*

6.6 Adomian Decomposition General Transform Method

To solve the suggested fractional-order hard water disease model (6.8), we utilized semi-analytical approach ADGTM [27]. By employing the ADGTM [27] technique, the approximate solution of system (6.8) can be

$$
\begin{aligned}
S(t) =& S_0(t) + S_1(t)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + S_2(t)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + S_3(t)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ...\\
=& S_0 + (A - \beta\lambda S_0 - \mu S_0)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \big(A(1-\mu) + \mu^2 S_0 + \beta^2\lambda^2 S_0\\
& - \beta\lambda(A - 2\mu S_0)\big)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \big(A(1-\mu) + \mu^2 S_0 + \beta^2\lambda^2 S_0\\
& + \beta^2\lambda^2(A - 3\mu S_0) + 2\beta\lambda\left(2\mu A - A - 3\mu^2 S_0\right)\big)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ...,\\
I(t) =& I_0(t) + I_1(t)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + I_2(t)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + I_3(t)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ...\\
=& I_0 + (\beta\lambda S_0 - \gamma I_0 - \mu I_0)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \big(-\beta^2\lambda^2 S_0 + (\mu+\gamma)^2 I_0\\
& + \beta\lambda(A - \gamma S_0 - 2\mu S_0)\big)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \big(\beta^3\lambda^3 S_0 - (\mu+\gamma)^3 I_0\\
& - 2\beta\gamma\left(A(2\mu-1) - \gamma^2 S_0 - 3\mu^2 S_0 - (3\mu S_0 - A)\gamma\right)\big)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ...,\\
R(t) =& R_0(t) + R_1(t)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + R_2(t)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + R_3(t)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ...\\
=& R_0 + (\gamma I_0 - \mu R_0)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + ((\beta\lambda S_0 - \gamma I_0 - \mu I_0)\gamma\\
& - (\gamma I_0 - \mu R_0)\mu)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \big(\gamma^3 I_0 - (\beta\lambda S_0 - 3\mu I_0)\gamma^2 - \mu^3 R_0\\
& + \left(3\mu^2 I_0 - \beta^2\lambda^2 S_0 + \beta\lambda(A - 3\mu S_0)\right)\big)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ...,\\
W(t) =& W_0(t) + W_1(t)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + W_2(t)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + W_3(t)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ...\\
=& W_0 + \left(bW_0\left(1 - \frac{W_0}{K}\right) - cW_0\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\
& + \frac{((b-c)K - bW_0)\left((K^2 - bKW_0)(b-c) + b^2W_0^2\right)}{K^3}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\
& + \frac{W_0((b-c)K - bW_0)\left((b-c)K^2 - (b^2-bc)KW_0 + b^2W_0^2\right)}{K^7}\\
& \cdot\frac{((b-c)(K^4 - bK^3(b-c)W_0 + b^2K^2(b-c+1)W_0^2 - 2b^3KW_0^3) + b^4W_0^4)t^{3\alpha}}{\Gamma(1+3\alpha)}\\
& + ... \qquad (6.20)
\end{aligned}
$$

6.7 Results and Discussion

We illustrated the simulation for the solution of the system using equation (6.20). Where initial conditions of system are $S_0 = 999, I_0 = 1, R_0 = 0, W_0 = 20$ with parametric values $A = 15, \beta = 0.1, \mu = \frac{1}{65}, K = 60, \gamma = \frac{1}{45}, b = 0.1, c = 0.4$ for $b < c$ and $b = 0.2, c = 0.1$ for $b > c$. We determine the solution of fractional-order hard water disease model (6.2.3) using Adomian decomposition general transform method (ADSTM) as (6.20). Figures – 6.1 to 6.4 reveal the simulation of the solution of S, I, R, and W when growth rate b of calcium and magnesium is less than the rate of water treatment effectiveness c at some fractional order $\alpha = 0.7, 0.8, 0.9$ and 1. Similarly, Figures – 6.5 to 6.8 display the simulation of the solution of S, I, R, and W when growth rate b of calcium and magnesium is higher than the rate of water treatment effectiveness c at some fractional order $\alpha = 0.7, 0.8, 0.9$ and 1. Figure – 6.9 is the graph of infected population at different rate of water treatment effectiveness $c = 0.15, 0.2, 0.25, 0.3$. And figure – 6.10 is the plot of calcium and magnesium growth rate at different concentration level of minerals $b = 0.2, 0.4, 0.6, 0.8$ and 1 in hard water.

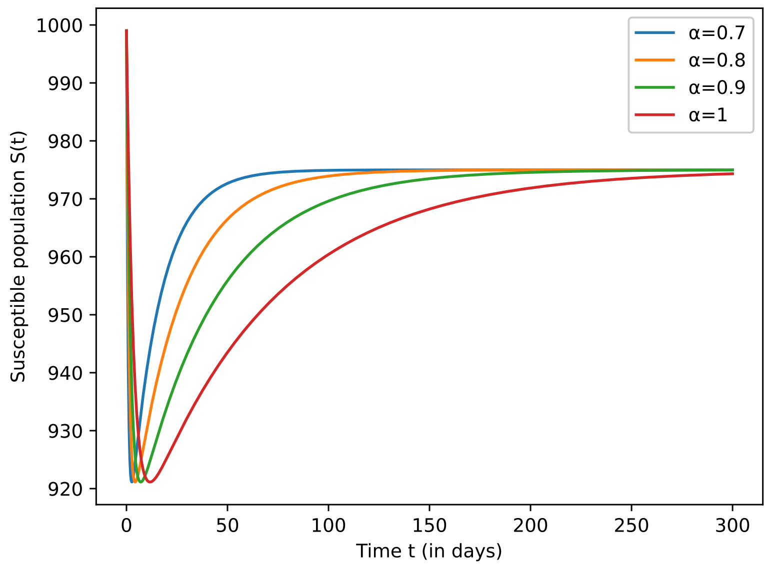


Figure 6.1: Simulation for Susceptible when $b < c$ at some fractional-order α.

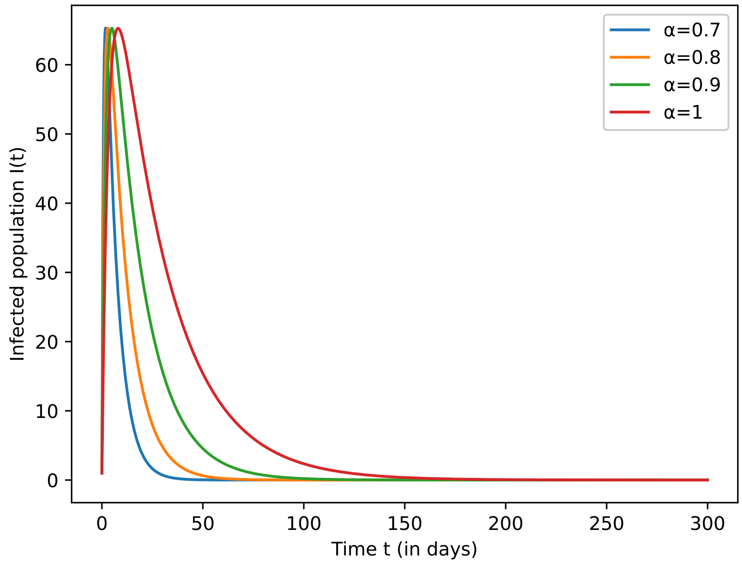


Figure 6.2: Concentration of minerals in water when $b < c$ at some fractional-order α.

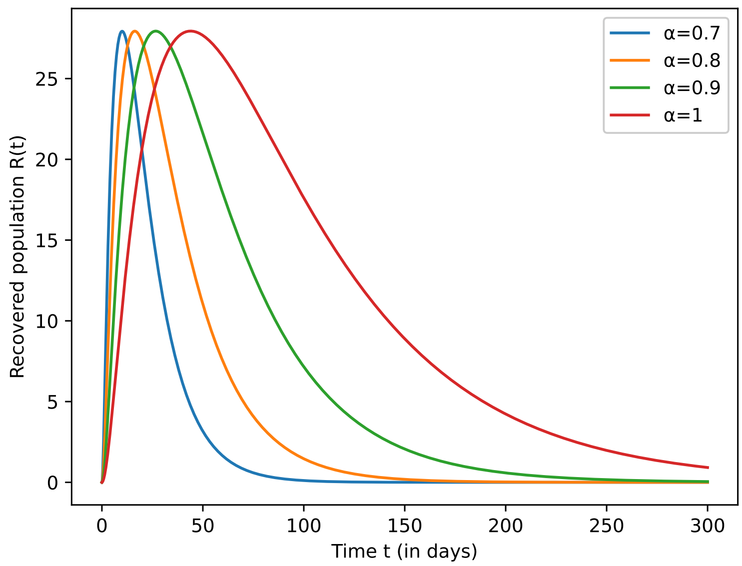


Figure 6.3: Simulation for Recovered population when $b < c$ at some fractional-order α.

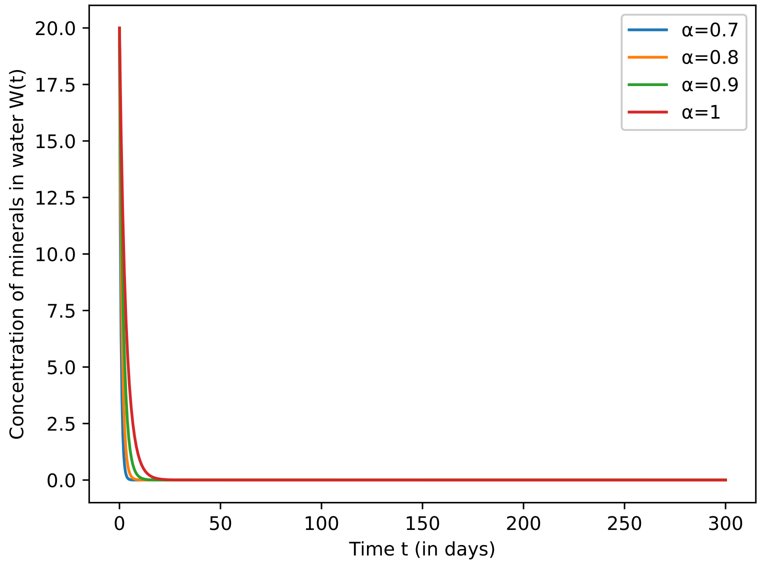


Figure 6.4: Concentration of minerals in water when $b < c$ at some fractional-order α.

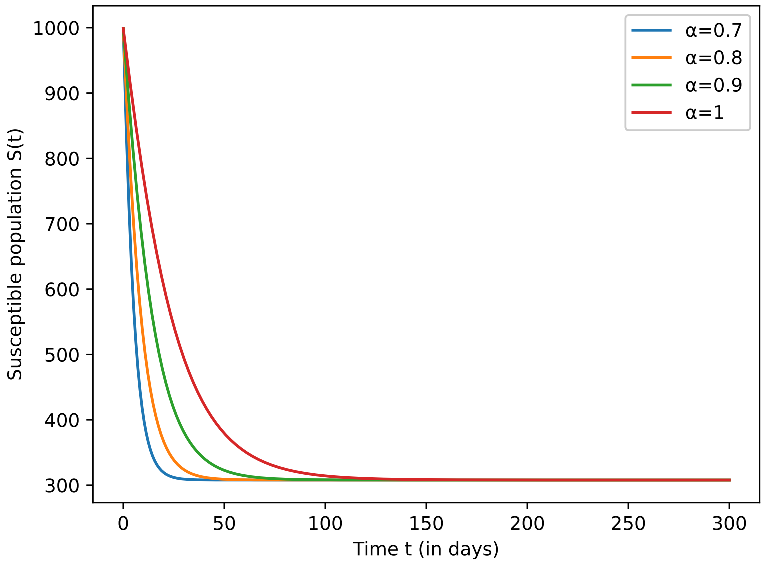


Figure 6.5: Simulation for Infected population when $b > c$ at some fractional-order α.

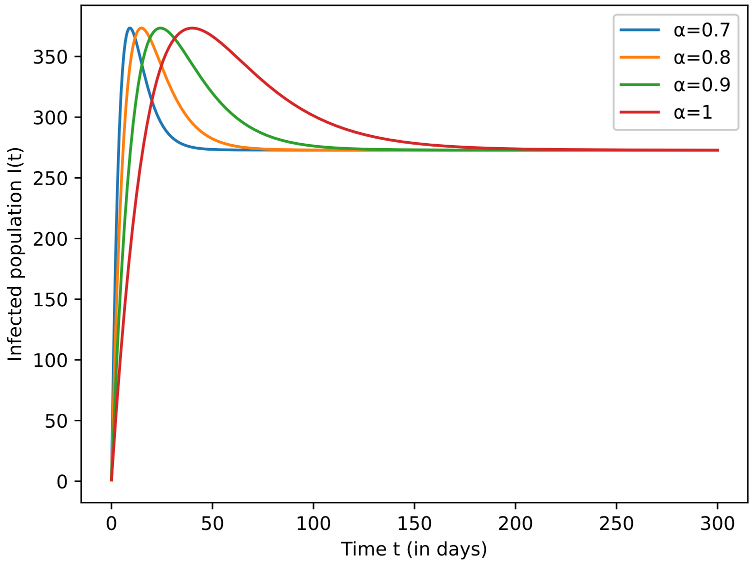


Figure 6.6: Concentration of minerals in water when $b > c$ at some fractional-order α.

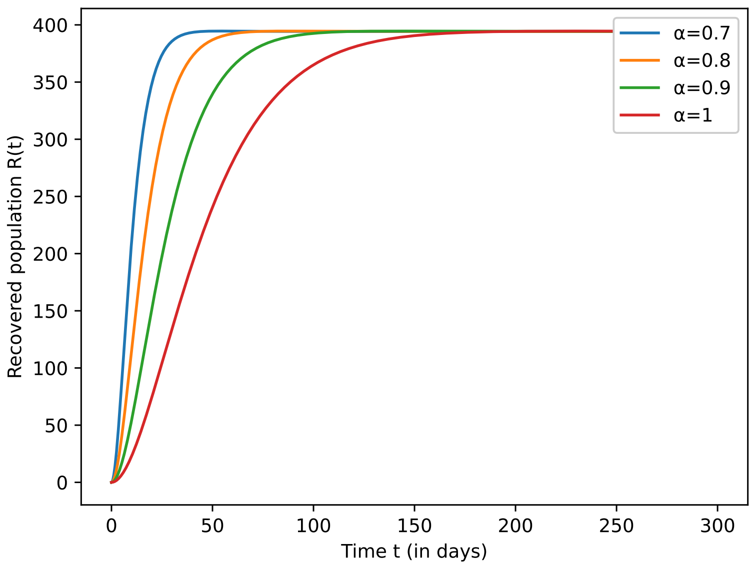


Figure 6.7: Simulation for Infected population when $b > c$ at some fractional-order α.

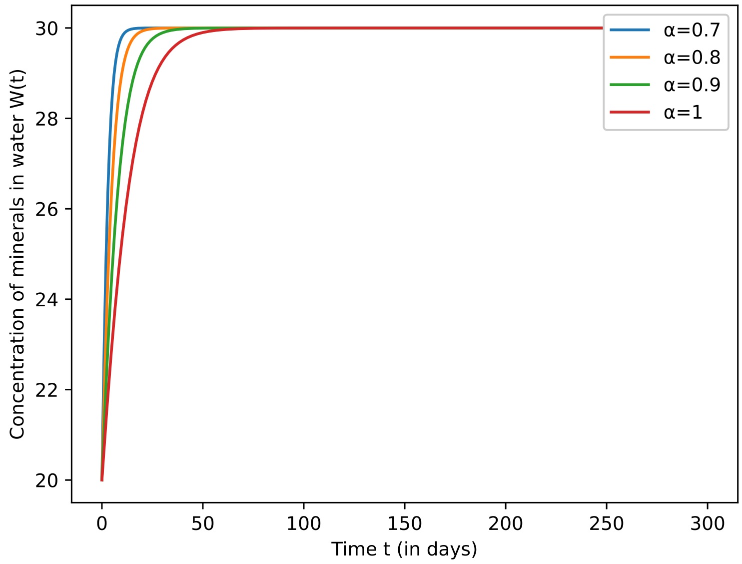


Figure 6.8: Concentration of minerals in water when $b > c$ at some fractional-order α.

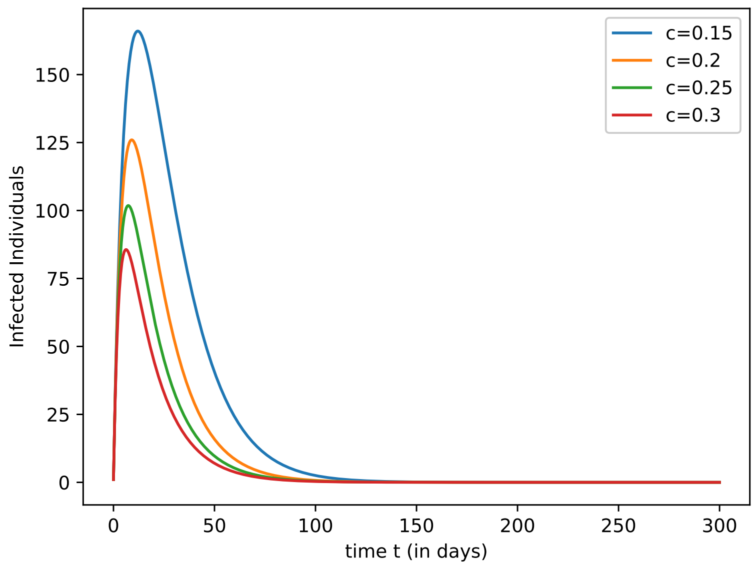


Figure 6.9: Infected individuals at different rate of water treatment effectiveness.

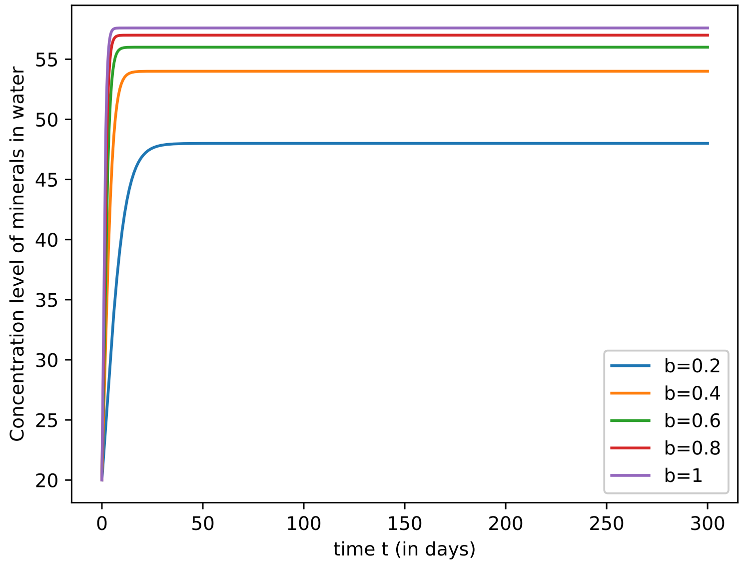


Figure 6.10: Calcium and magnesium growth rate at different concentration level of minerals in water.

6.8 Conclusion

We presented a fractional-order hard water mathematical model with four different class for the duration of 300 days. We achieved good level of accuracy in the solution of system by using Adomian decomposition general transform method (ADGTM). we obtained locally stable region of model for disease-free equilibrium for $b < c$ and endemic equilibrium for $b > c$. The global stability of the system is also examine with the help of Ulam-Hyers-Rassias(U-H-R) stability conditions. Qualitative analysis of the proposed fractional system is discussed through Sadovskii's fix point theory. From figures – 6.1 to 6.4, we can observe that when growth rate b of minerals like calcium and magnesium in water are lesser than the water treatment effectiveness rate c, very few individuals get infected and they recovered rapidly. While growth rate b of minerals like calcium and magnesium in water are higher than the water treatment effectiveness rate c, figures – 6.5 to 6.8 show that huge susceptible mass get infected and recovered very slow. We can claim from figure – 6.9 that better water treatment reduce the infected individuals. The fractional-order hard water disease model demonstrated the potential of mathematical modelling in studying and understanding the dynamics of the disease. The results provided crucial insights into the spatial and temporal patterns of the disease, as well as the impact of various factors on its spread. This knowledge can guide public health interventions and water management practices to reduce disease transmission and improve overall water quality.