Chapter 4

Novel Fractional Mathematical Models for Chikungunya Virus Dynamics

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4.1 Introduction

The Chikungunya virus is an arbovirus that is primarily transmitted to humans through the bite of infected Aedes mosquitoes. The virus was first discovered in 1952, during an outbreak in Tanzania. The name "Chikungunya" derives from a word in the Makonde language, meaning "to become contorted," reflecting the severe joint pain that is a common symptom of the disease [122]. For several years after its discovery, Chikungunya remained mostly confined to regions of Africa, with occasional outbreaks reported in Asia. However, in 2004, the virus started to spread in an unprecedented manner, causing a large-scale outbreak in the Indian Ocean islands of Réunion and Mauritius. This outbreak marked the first instance of Chikungunya virus being transmitted locally in large numbers outside of Africa [149].

From 2005 onwards, Chikungunya continued to spread rapidly, affecting several countries in Asia, Europe, and the Americas. It became a significant global public health concern, with millions of cases reported and a high burden of illness [86]. The virus was able to adapt to new Aedes mosquito species present in these regions, allowing for its sustained transmission. The symptoms of Chikungunya virus infection include fever, severe joint pain, muscle pain, headache, fatigue, and rash [50]. While most individuals recover after a few days or weeks, joint pain can persist for months or even years in some cases. Efforts to control and prevent the spread of Chikungunya virus focus on mosquito control, as there is no specific antiviral treatment available. Mosquito breeding sites need to be eliminated, and personal protective measures like using insect repellents and wearing long sleeves and pants are recommended [68]. The history of the Chikungunya virus serves as a reminder of the ability of mosquito-borne viruses to rapidly spread across continents and cause significant public health challenges [80]. Monitoring and surveillance of mosquito-borne diseases, as well as research and development of effective vaccines and treatments, are crucial in preventing and mitigating future outbreaks [31].

This chapter is organized as follows: Section 4.2 provides an overview of the Chikungunya virus models and its parametric description. Section 4.3 introduces some definitions that we

have used in our work. Section 4.4 discusses the qualitative and stability analysis of the both suggested Chikungunya virus systems. Section 4.5 introduces the approximate solution of the present model through a novel numerical scheme. Sections 4.6 and 4.7 present the results and discussion, as well as the conclusion for the fractional-order Chikungunya system.

4.2 Model formation

We formulate the two unique Chikungunya virus models with adoptive immune response and with latently infected cells.

4.2.1 Chikungunya virus model with adoptive immune response

Consider the system of differential equations

$$\begin{aligned}
\frac{d\mathbb{S}(t)}{dt} =& \lambda - d\mathbb{S}(t) - b\mathbb{S}(t)\mathbb{V}(t),\\
\frac{d\mathbb{I}(t)}{dt} =& b\mathbb{S}(t)\mathbb{V}(t) - a\mathbb{I}(t) - \epsilon\mathbb{I}(t)\mathbb{Z}(t),\\
\frac{d\mathbb{V}(t)}{dt} =& m\mathbb{I}(t) - r\mathbb{V}(t) - q\mathbb{V}(t)\mathbb{B}(t),\\
\frac{d\mathbb{B}(t)}{dt} =& \eta + c\mathbb{B}(t)\mathbb{V}(t) - \delta\mathbb{B}(t),\\
\frac{d\mathbb{Z}(t)}{dt} =& \gamma + \omega\mathbb{I}(t)\mathbb{Z}(t) - \mu\mathbb{Z}(t),
\end{aligned} \tag{4.1}$$

with initial conditions $\mathbb{S}(0) = \mathbb{S}_0, \mathbb{I}(0) = \mathbb{I}_0, \mathbb{V}(0) = \mathbb{V}_0, \mathbb{B}(0) = \mathbb{B}_0, \mathbb{Z}(0) = \mathbb{Z}_0$.

where, $\mathbb{S}$ denotes the susceptible cells in blood concentration of human body, $\mathbb{I}$ is the infected cells in concentration, $\mathbb{V}$ represents the Chikungunya virus germs in concentration, $\mathbb{B}$ is the antibodies in blood concentration, and $\mathbb{Z}$ is the Cytotoxic T-lymsists phocytes (CTL) cells in human body. Whole model (4.1) is formulated using various parameters such as λ is a birth rate constant and d indicates the death rate constant of susceptible cells, b is the rate of transmission from $\mathbb{S}$ to $\mathbb{V}$ due to Chikungunya virus, a is the death rate parameter of infect cells, ϵ is Chikungunya virus killing rate due to CTL cells immune response, m represent the

average count of Chikungunya virus germs, r stand for the death rate of Chikungunya virus, q shows the antibodies attack rate in concentration, η is the number of antibodies in blood concentration, c denotes the antibodies increasing rate, δ is the antibodies reducing rate, γ indicates the average count of CTL cells in body, ω denotes the CTL cells increasing rate, and μ is the CTL cells death rate. The flow chrt of the system (4.1) and (4.2) are given in Figures 4.1 and 4.2.

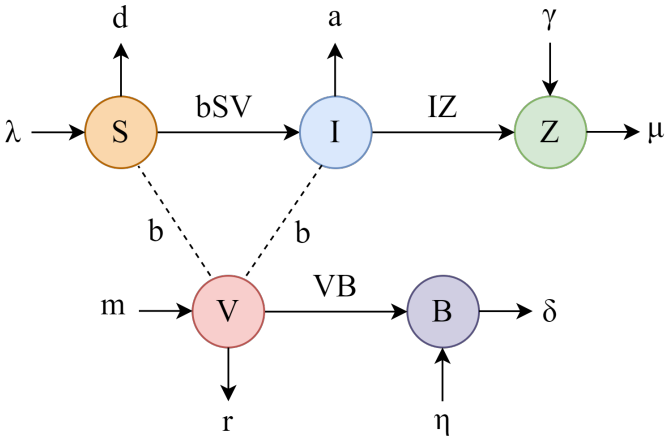


Figure 4.1: Diagram for the Chikungunya virus model with adoptive immune response.

4.2.2 Chikungunya virus model with latently infected cells

Consider the Chikungunya virus mathematical model:

$$\begin{aligned}
\frac{d\mathbb{S}(t)}{dt} &= \lambda - d\mathbb{S}(t) - b\mathbb{S}(t)\mathbb{V}(t),\\
\frac{d\mathbb{L}(t)}{dt} &= (1-\rho)b\mathbb{S}(t)\mathbb{V}(t) - (\theta+\pi)\mathbb{L}(t),\\
\frac{d\mathbb{I}(t)}{dt} &= \rho b\mathbb{S}(t)\mathbb{V}(t) - \pi\mathbb{L}(t) - a\mathbb{I}(t) - \epsilon\mathbb{I}(t)\mathbb{Z}(t),\\
\frac{d\mathbb{V}(t)}{dt} &= m\mathbb{I}(t) - r\mathbb{V}(t) - q\mathbb{V}(t)\mathbb{B}(t),\\
\frac{d\mathbb{B}(t)}{dt} &= \eta + c\mathbb{B}(t)\mathbb{V}(t) - \delta\mathbb{B}(t),\\
\frac{d\mathbb{Z}(t)}{dt} &= \gamma + \omega\mathbb{I}(t)\mathbb{Z}(t) - \mu\mathbb{Z}(t),
\end{aligned} \tag{4.2}$$

with initial conditions $\mathbb{S}(0) = \mathbb{S}_0, \mathbb{L}(0) = \mathbb{L}_0, \mathbb{I}(0) = \mathbb{I}_0, \mathbb{V}(0) = \mathbb{V}_0, \mathbb{B}(0) = \mathbb{B}_0, \mathbb{Z}(0) = \mathbb{Z}_0$.

The second suggested system (4.2) is the modification of the system (4.1) by including the latently infected cells class $\mathbb{L}$. Few more parameters are utilized like ρ denotes the actively infected cells in concentration, π is the transmission rate of latently infected cells to actively infected cells, and θ indicates the death rate of latently infected cells in concentration.

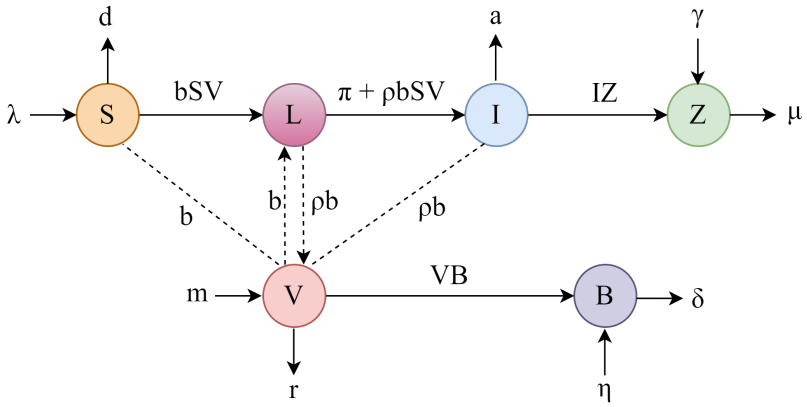


Figure 4.2: Diagram for the Chikungunya virus model with latently infected cells.

4.3 Preliminaries

Definition 4.3.1. *For order $\zeta > 0$ and $e : \mathbb{R}^+ \to \mathbb{R}$, Riemann-Liouvillie [106] established a fractional integral operator as:*

$$ {}_0\mathbb{I}^{\zeta}(e(\varkappa)) = \frac{1}{\Gamma(\zeta)} \int_0^{\varkappa} (\varkappa - \nu)^{\zeta - 1} e(\nu) d\nu, \ \ \varkappa > 0. \tag{4.3} $$

Definition 4.3.2. *Riemann-Liouville [106] established a fractional differential operator for the function $e(\varkappa)$ with order $\zeta > 0$ is defined to be:*

$$ {}_0D^{\zeta}(e(\varkappa)) = \frac{1}{\Gamma(\mathrm{j} - \zeta)} \left(\frac{d}{d\varkappa} \right)^{\mathrm{j}} \int_0^{\varkappa} (\varkappa - \nu)^{\mathrm{j} - \zeta - 1} e(\nu) d\nu, \ \ \varkappa > 0, \tag{4.4} $$

where, $\mathrm{j} - 1 < \zeta \leq \mathrm{j}, \mathrm{j} \in \mathbb{N}$.

Definition 4.3.3. *Caputo [106] contributed in the field of fractional calculus by defining the derivative of fractional order $\zeta > 0$ for the function $e(\varkappa) \in \mathbb{C}^{\mathrm{j}}$ as*

$$ {}_0^C D^{\zeta}(e(\varkappa)) = \frac{1}{\Gamma(\mathrm{j}-\zeta)} \int_0^{\varkappa} (\varkappa - \nu)^{\mathrm{j}-\zeta-1} e^{\mathrm{j}}(\nu) d\nu, \ \varkappa > 0, \tag{4.5} $$

and is defined for absolute continuous functions where, $\mathrm{j} - 1 < \zeta \leq \mathrm{j}$, $\mathrm{j} \in \mathbb{N}$.

4.4 Fractional Chikungunya virus model with qualitative and stability analysis

4.4.1 Fractional Chikungunya virus model with adoptive immune response

The proposed Chikungunya virus model (4.1) with adoptive immune response, which includes the well-know Caputo fractional derivative [24] is as follows:

$$ \begin{aligned} {}_0^{\mathsf{C}} D_t^{\zeta} \mathbb{S}(t) =& \lambda - d\mathbb{S}(t) - b\mathbb{S}(t)\mathbb{V}(t), \\ {}_0^{\mathsf{C}} D_t^{\zeta} \mathbb{I}(t) =& b\mathbb{S}(t)\mathbb{V}(t) - a\mathbb{I}(t) - \epsilon\mathbb{I}(t)\mathbb{Z}(t), \\ {}_0^{\mathsf{C}} D_t^{\zeta} \mathbb{V}(t) =& m\mathbb{I}(t) - r\mathbb{V}(t) - q\mathbb{V}(t)\mathbb{B}(t), \\ {}_0^{\mathsf{C}} D_t^{\zeta} \mathbb{B}(t) =& \eta + c\mathbb{B}(t)\mathbb{V}(t) - \delta\mathbb{B}(t), \\ {}_0^{\mathsf{C}} D_t^{\zeta} \mathbb{Z}(t) =& \gamma + \omega\mathbb{I}(t)\mathbb{Z}(t) - \mu\mathbb{Z}(t), \end{aligned} \tag{4.6} $$

with initial conditions

$$ \mathbb{S}(0) = \mathbb{S}_0, \mathbb{I}(0) = \mathbb{I}_0, \mathbb{V}(0) = \mathbb{V}_0, \mathbb{B}(0) = \mathbb{B}_0, \mathbb{Z}(0) = \mathbb{Z}_0. $$

4.4.2 Existence and uniqueness of fractional Chikungunya virus model with adoptive immune response

In this section, we will discuss the existence and uniqueness of the considered model (4.6). Let us rewrite the considered model (4.6) in the following form:

$$\begin{aligned}
{}_0^{\mathsf{C}}D_t^{\zeta}\mathbb{S}(t) &= n_1(t,\mathbb{S},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}),\\
{}_0^{\mathsf{C}}D_t^{\zeta}\mathbb{I}(t) &= n_2(t,\mathbb{S},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}),\\
{}_0^{\mathsf{C}}D_t^{\zeta}\mathbb{V}(t) &= n_3(t,\mathbb{S},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}),\\
{}_0^{\mathsf{C}}D_t^{\zeta}\mathbb{B}(t) &= n_4(t,\mathbb{S},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}),\\
{}_0^{\mathsf{C}}D_t^{\zeta}\mathbb{Z}(t) &= n_5(t,\mathbb{S},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}),
\end{aligned} \tag{4.7}$$

where,

$$\begin{aligned}
n_1(t,\mathbb{S},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) &= \lambda - d\mathbb{S}(t) - b\mathbb{S}(t)\mathbb{V}(t),\\
n_2(t,\mathbb{S},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) &= b\mathbb{S}(t)\mathbb{V}(t) - a\mathbb{I}(t) - \epsilon\mathbb{I}(t)\mathbb{Z}(t),\\
n_3(t,\mathbb{S},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) &= m\mathbb{I}(t) - r\mathbb{V}(t) - q\mathbb{V}(t)\mathbb{B}(t),\\
n_4(t,\mathbb{S},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) &= \eta + c\mathbb{B}(t)\mathbb{V}(t) - \delta\mathbb{B}(t),\\
n_5(t,\mathbb{S},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) &= \gamma + \omega\mathbb{I}(t)\mathbb{Z}(t) - \mu\mathbb{Z}(t).
\end{aligned} \tag{4.8}$$

Let R be the vector's area dimensional space (one dimensional), if the partial derivative $\dfrac{dn}{dx_i}$, where, $i = 1,2,3,4,5$ are continuous in $\mathbb{R} = (x,t) : |t-t_0| \leq \mathbf{a}, |x-x_0| \leq \mathbf{b}$ then there constant $\delta \geq 0$ such that there exists a unique solution continuous vector solution $x(t) = [\mathbb{S}(t), \mathbb{I}(t), \mathbb{V}(t), \mathbb{B}(t), \mathbb{Z}(t)]$ in the interval $|t-t_0| \leq \delta$.

Consider, $n_1(t,\mathbb{S},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) = \lambda - d\mathbb{S}(t) - b\mathbb{S}(t)\mathbb{V}(t)$. Taking partial derivative of n_1 w.r.t. $\mathbb{S},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}$ respectively, we obtain the following:

$$\left|\frac{\partial n_1}{\partial \mathbb{S}}\right| = |-d - b\mathbb{V}|, \left|\frac{\partial n_1}{\partial \mathbb{I}}\right| = 0, \left|\frac{\partial n_1}{\partial \mathbb{V}}\right| = |-b\mathbb{S}|, \left|\frac{\partial n_1}{\partial \mathbb{B}}\right| = 0, \left|\frac{\partial n_1}{\partial \mathbb{Z}}\right| = 0.$$

Also, taking the partial derivative of n_2 to obtain the following:

$$\left|\frac{\partial n_2}{\partial \mathbb{S}}\right| = |bv|\,, \left|\frac{\partial n_2}{\partial \mathbb{I}}\right| = |-a-\epsilon\mathbb{Z}|\,, \left|\frac{\partial n_2}{\partial \mathbb{V}}\right| = |b\mathbb{S}|\,, \left|\frac{\partial n_2}{\partial \mathbb{B}}\right| = 0, \left|\frac{\partial n_2}{\partial \mathbb{Z}}\right| = |-\epsilon\mathbb{I}|\,.$$

Similarly, taking partial derivative of the functions n_3, n_4 and n_5 respectively. we get,

$$\left|\frac{\partial n_3}{\partial \mathbb{S}}\right| = 0, \left|\frac{\partial n_3}{\partial \mathbb{I}}\right| = |m|\,, \left|\frac{\partial n_3}{\partial \mathbb{V}}\right| = |-r-q\mathbb{B}|\,, \left|\frac{\partial n_3}{\partial \mathbb{B}}\right| = |-q\mathbb{V}|\,, \left|\frac{\partial n_3}{\partial \mathbb{Z}}\right| = 0,$$

$$\left|\frac{\partial n_4}{\partial \mathbb{S}}\right| = 0, \left|\frac{\partial n_4}{\partial \mathbb{I}}\right| = 0, \left|\frac{\partial n_4}{\partial \mathbb{V}}\right| = |c\mathbb{B}|\,, \left|\frac{\partial n_4}{\partial \mathbb{B}}\right| = |c\mathbb{V}-\delta|\,, \left|\frac{\partial n_4}{\partial \mathbb{Z}}\right| = 0,$$

$$\left|\frac{\partial n_5}{\partial \mathbb{S}}\right| = 0, \left|\frac{\partial n_5}{\partial \mathbb{I}}\right| = |\omega\mathbb{Z}|\,, \left|\frac{\partial n_5}{\partial \mathbb{V}}\right| = 0, \left|\frac{\partial n_5}{\partial \mathbb{B}}\right| = 0, \left|\frac{\partial n_5}{\partial \mathbb{Z}}\right| = |\omega\mathbb{I}-\mu|\,.$$

As the partial derivative for the functions n_1, n_2, n_3, n_4, n_5 are exists, it is continuous and bounded. Hence, the solution of fractional model (4.6) has exists and unique.

4.4.3 Stability analysis of fractional Chikungunya virus model with adoptive immune response

For the Chikungunya virus system (4.1), we discuss the stability though two unique equilibrium points of the system.

1. Disease free equilibrium point $\mathbb{E}^0$:

   At this stage the concentration of infected cells $\mathbb{I}$ and Chikungunya virus germs $\mathbb{V}$ are almost zero. Now, we equating right side of the system (4.1) with zero and with basic simplification, we get

$$\begin{aligned}\mathbb{E}^0 &= \left(\mathbb{S}^0, \mathbb{I}^0, \mathbb{V}^0, \mathbb{B}^0, \mathbb{Z}^0\right)\\ &= \left(\frac{\lambda}{d}, 0, 0, \frac{\eta}{\delta}, \frac{\gamma}{\mu}\right).\end{aligned} \tag{4.9}$$

2. Endemic equilibrium point $\mathbb{E}^{end}$:

   For $\mathbb{E}^{end}$, we equating right side of the system (4.1) with zero

$$\begin{aligned}
&\lambda - d\mathbb{S} - b\mathbb{S}\mathbb{V} = 0,\\
&b\mathbb{S}\mathbb{V} - a\mathbb{I} - \epsilon\mathbb{I}\mathbb{Z} = 0,\\
&m\mathbb{I} - r\mathbb{V} - q\mathbb{V}\mathbb{B} = 0,\\
&\eta + c\mathbb{B}\mathbb{V} - \delta\mathbb{B} = 0,\\
&\gamma + \omega\mathbb{I}\mathbb{Z} - \mu\mathbb{Z} = 0.
\end{aligned} \tag{4.10}$$

   With the basic simplification, the solution of system (4.10) will be

$$\begin{aligned}
&\mathbb{S} = \frac{(a+\epsilon\mathbb{Z})(r+q\mathbb{B}(t))}{bm}, \mathbb{I} = \frac{\mathbb{V}(r+q\mathbb{B})}{m},\\
&\mathbb{B} = \frac{\eta}{\delta - c\mathbb{V}}, \mathbb{Z} = \frac{m\gamma}{m\mu - \omega\mathbb{V}(r+q\mathbb{B})},
\end{aligned} \tag{4.11}$$

   substituting (4.11) into (4.10), we have

$$\mathbb{K}_1\mathbb{V}^4 + \mathbb{K}_2\mathbb{V}^3 + \mathbb{K}_3\mathbb{V}^2 + \mathbb{K}_4 V + \mathbb{K}_5 = 0, \tag{4.12}$$

   where,

$$\begin{aligned}
\mathbb{K}_1 &= abc^2r^2\omega,\\
\mathbb{K}_2 &= cr\left(-mbc(\omega\lambda+\epsilon\gamma) + a\left(-2b\omega(q\eta+r\delta) + c(rd\omega - mb\mu)\right)\right),\\
\mathbb{K}_3 &= a\omega(q\eta+r\delta)(rb\delta - 2cdr + bq\eta) + mac\mu(bq\eta + 2br\delta - cdr)\\
&\quad + mbc(q\eta+2r\delta)(\omega\lambda+\epsilon\gamma) + mc^2(mb\lambda\mu - dr\epsilon\gamma),\\
\mathbb{K}_4 &= ad\omega(q\eta+r\delta)^2 + ma\mu\left(-b\delta(q\eta+r\delta) + cd(q\eta+2r\delta)\right)\\
&\quad - m\left(-cd\epsilon\gamma(q\eta+2r\delta) + b\delta(q\eta+r\delta)(\omega\lambda+\epsilon\gamma) + 2mbc\delta\lambda\mu\right),\\
\mathbb{K}_5 &= md\delta(q\eta+r\delta)(\epsilon\gamma+a\mu)(\mathbb{R}_0 - 1).
\end{aligned}$$

Hence, the basic reproduction number $\mathbb{R}_0$ is

$$\mathbb{R}_0 = \frac{mb\mu\delta\lambda}{d(q\eta + r\delta)(\epsilon\gamma + a\mu)}. \tag{4.13}$$

For this $\mathbb{R}_0$, the Chikungunya virus system (4.1) having endemic equilibrium point $\mathbb{E}^{end}$ can be defined as

$$\mathbb{E}^{end} = (\mathbb{S}^*, \mathbb{I}^*, \mathbb{V}^*, \mathbb{B}^*, \mathbb{Z}^*), \tag{4.14}$$

where

$$\begin{aligned}
\mathbb{S}^* &= \frac{(r(\delta - c\mathbb{V}^*) + q\eta)\left(a\omega\mathbb{V}^*(rc\mathbb{V}^* - (r\delta + q\eta)) + m\mu(\delta - c\mathbb{V}^*)\right)}{bm(\delta - c\mathbb{V}^*)(\omega\mathbb{V}^*(rc\mathbb{V}^* - (r\delta + q\eta)) + m\mu(\delta - c\mathbb{V}^*))},\\
\mathbb{I}^* &= \frac{(r(\delta - c\mathbb{V}^*) + q\eta)\mathbb{V}^*}{m(\delta - c\mathbb{V}^*)},\\
\mathbb{B}^* &= \frac{\eta}{(\delta - c\mathbb{V}^*)},\\
\mathbb{Z}^* &= \frac{m\gamma(\delta - c\mathbb{V}^*)}{\omega\mathbb{V}^*(rc\mathbb{V}^* - (r\delta + q\eta)) + m\mu(\delta - c\mathbb{V}^*))}.
\end{aligned}$$

Remark 4.4.1. *To investigate the global stability of disease-free and endemic equilibrium of system* (4.1)*, we employ Lyapunov function [4, 51]. Let's define*

$$\mathbb{H}(x) = x - ln(x) - 1,$$

where $\mathbb{H}(x) \geq 0$ *for* $x > 0$ *and* $\mathbb{H}(1) = 0$.

Theorem 4.4.1. *Disease-free equilibrium* $\mathbb{E}^0$ *for Chikungunya virus system* (4.1) *is globally asymptotically stable if* $\mathbb{R}_0 \leq 1$.

Proof. Consider Lyapunov function $\mathbb{W}^0(\mathbb{S}, \mathbb{I}, \mathbb{V}, \mathbb{B}, \mathbb{Z})$ as:

$$\mathbb{W}^0 = \mathbb{S}^0\mathbb{H}\left(\frac{\mathbb{S}}{\mathbb{S}^0}\right) + \mathbb{I} + \frac{(\epsilon\gamma + a\mu)}{m\mu}\mathbb{V} + \frac{q\,(\epsilon\gamma + a\mu)\,\mathbb{B}^0}{mc\mu}\mathbb{H}\left(\frac{\mathbb{B}}{\mathbb{B}^0}\right) + \frac{\epsilon\mathbb{Z}^0}{\omega}\mathbb{H}\left(\frac{\mathbb{Z}}{\mathbb{Z}^0}\right), \tag{4.15}$$

where, $\mathbb{W}^0 \leq 0$ for each $\mathbb{S}, \mathbb{I}, \mathbb{V}, \mathbb{B}, \mathbb{Z} > 0$.

Calculate $\frac{d\mathbb{W}^0}{dt}$ to find the solution of system (4.1), we leads to,

$$\begin{aligned}
\frac{d\mathbb{W}^0}{dt} = & \quad (\lambda - dS - b\mathbb{S}\mathbb{V})\left(1 - \frac{\mathbb{S}^0}{\mathbb{S}}\right) + b\mathbb{S}\mathbb{V} - aI - \varepsilon\mathbb{I}\mathbb{Z} + \frac{q(\varepsilon\gamma + a\mu)(\eta - \delta\mathbb{B} + c\mathbb{B}\mathbb{V})}{mc\mu}\left(1 - \frac{\mathbb{B}^0}{\mathbb{B}}\right) \\
& + \frac{(\varepsilon\gamma + a\mu)}{m\mu}(m\mathbb{I} - r\mathbb{V} - q\mathbb{V}\mathbb{B}) + \frac{\varepsilon(\gamma - \mu\mathbb{Z} + \omega\mathbb{I}\mathbb{Z})}{\omega}\left(1 - \frac{\mathbb{Z}^0}{\mathbb{Z}}\right), \\
= & \quad (d\mathbb{S}^0 - d\mathbb{S})\left(1 - \frac{\mathbb{S}^0}{\mathbb{S}}\right) + b\mathbb{V}\mathbb{S}^0 - \frac{r(\varepsilon\gamma + a\mu)\mathbb{V}}{m\mu} - \frac{q(\varepsilon\gamma + a\mu)\mathbb{B}^0\mathbb{V}}{m\mu} \\
& + \frac{q(\varepsilon\gamma + a\mu)(\delta\mathbb{B}^0 - \delta B)}{mc\mu}\left(1 - \frac{\mathbb{B}^0}{\mathbb{B}}\right) + \frac{\varepsilon(\mu\mathbb{Z}^0 - \mu\mathbb{Z})}{\omega}\left(1 - \frac{\mathbb{Z}^0}{\mathbb{Z}}\right), \\
= & \quad - d\frac{(\mathbb{S} - \mathbb{S}^0)^2}{\mathbb{S}} - \frac{q\delta(\varepsilon\gamma + a\mu)}{mc\mu}\frac{(\mathbb{B} - \mathbb{B}^0)^2}{\mathbb{B}} - \frac{\varepsilon\mu}{\omega}\frac{(\mathbb{Z} - \mathbb{Z}^0)^2}{\mathbb{Z}} \\
& + \frac{(q\eta + r\delta)(\varepsilon\gamma + a\mu)}{m\delta\mu}(\mathbb{R}_0 - 1)\,\mathbb{V}. \qquad (4.16)
\end{aligned}$$

If $\mathbb{R}_0 \leq 1$, it leads to $\frac{d\mathbb{W}^0}{dt} \leq 0$. Therefore, $\frac{d\mathbb{W}^0}{dt} = 0 \Rightarrow \mathbb{S} = \mathbb{S}^0, \mathbb{B} = \mathbb{B}^0, \mathbb{Z} = \mathbb{Z}^0, \mathbb{V} = 0$.

Hence, By LaSalle's principle [56], disease-free equilibrium $\mathbb{E}_0$ for Chikungunya virus system (4.1) is globally asymptotically stable.

□

Theorem 4.4.2. *Endemic equilibrium $\mathbb{E}^{end}$ for Chikungunya virus system* (4.1) *is globally asymptotically stable if $\mathbb{R}_0 > 1$.*

Proof. We start with constructing Lyapunov function,

$$\begin{aligned}
\mathbb{W}^*(\mathbb{S}, \mathbb{I}, \mathbb{V}, \mathbb{B}, \mathbb{Z}) = \mathbb{S}^*\mathbb{H}\left(\frac{\mathbb{S}}{\mathbb{S}^*}\right) + \mathbb{I}^*\mathbb{H}\left(\frac{\mathbb{I}}{\mathbb{I}^*}\right) + \frac{(a + \varepsilon\mathbb{Z}^*)}{m}\mathbb{V}^*\mathbb{H}\left(\frac{\mathbb{V}}{\mathbb{V}^*}\right) + \\
\frac{q\,(a + \varepsilon\mathbb{Z}^*)}{m}\mathbb{B}^*\mathbb{H}\left(\frac{\mathbb{B}}{\mathbb{B}^*}\right) + \frac{\varepsilon}{\omega}\mathbb{Z}^*\mathbb{H}\left(\frac{\mathbb{Z}}{\mathbb{Z}^*}\right). \qquad (4.17)
\end{aligned}$$

We have $\mathbb{W}^* \geq 0$ for each $\mathbb{S}, \mathbb{I}, \mathbb{V}, \mathbb{B}, \mathbb{Z} > 0$. We now calculate $\frac{d\mathbb{W}^*}{dt}$ to find the solution of

system (4.1), which leads to,

$$\begin{aligned}\frac{d\mathbb{W}^*}{dt} = & \left(\lambda - d\mathbb{S} - b\mathbb{S}\mathbb{V}\right)\left(1 - \frac{\mathbb{S}^*}{\mathbb{S}}\right) + \left(b\mathbb{S}\mathbb{V} - a\mathbb{I} - \varepsilon\mathbb{I}\mathbb{Z}\right)\left(1 - \frac{\mathbb{I}^*}{\mathbb{I}}\right) \\ & + \frac{\left(a + \varepsilon\mathbb{Z}^*\right)\left(m\mathbb{I} - r\mathbb{V} - q\mathbb{V}\mathbb{B}\right)}{m}\left(1 - \frac{\mathbb{V}^*}{\mathbb{V}}\right) \\ & + \frac{q\left(a + \varepsilon\mathbb{Z}^*\right)\left(\eta - \delta\mathbb{V} + c\mathbb{B}\mathbb{V}\right)}{mc}\left(1 - \frac{\mathbb{B}^*}{\mathbb{B}}\right) + \frac{\varepsilon\left(\gamma - \mu\mathbb{Z} + \omega\mathbb{I}\mathbb{Z}\right)}{\omega}\left(1 - \frac{\mathbb{Z}^*}{\mathbb{Z}}\right). \end{aligned} \tag{4.18}$$

Considering the below mentioned condition's from (4.10) as

$$\begin{aligned} \lambda &= d\mathbb{S}^* - b\mathbb{S}^*\mathbb{V}^*, \\ b\mathbb{S}^*\mathbb{V}^* &= a\mathbb{I}^* + \varepsilon\mathbb{I}^*\mathbb{Z}^*, \\ m\mathbb{I}^* &= r\mathbb{V}^* + q\mathbb{B}^*\mathbb{V}^*, \\ \eta &= \delta\mathbb{B}^* - c\mathbb{B}^*\mathbb{V}^*, \\ \gamma &= \mu\mathbb{Z}^* - \omega\mathbb{I}^*\mathbb{Z}^*. \end{aligned} \tag{4.19}$$

Substituting (4.19) into (4.18), we get,

$$\begin{aligned}\frac{d\mathbb{W}^*}{dt} = & - d\frac{\left(\mathbb{S} - \mathbb{S}^*\right)^2}{\mathbb{S}} + 3b\mathbb{S}^*\mathbb{V}^* - b\mathbb{S}^*\mathbb{V}^*\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right) - b\mathbb{S}^*\mathbb{V}^*\frac{\mathbb{S}\mathbb{V}\mathbb{I}^*}{\mathbb{S}^*\mathbb{V}^*\mathbb{I}} - b\mathbb{S}^*\mathbb{V}^*\frac{\mathbb{V}^*\mathbb{I}}{\mathbb{V}\mathbb{I}^*} \\ & - \frac{2q\left(a + \varepsilon\mathbb{Z}^*\right)\mathbb{B}^*\mathbb{V}^*}{m} + \frac{q\left(a + \varepsilon\mathbb{Z}^*\right)\mathbb{B}\mathbb{V}^*}{m} + \frac{q\left(a + \varepsilon\mathbb{Z}^*\right)\mathbb{B}^*\mathbb{V}^*}{m}\left(\frac{\mathbb{B}^*}{\mathbb{B}}\right) \\ & - \frac{q\delta\left(a + \varepsilon\mathbb{Z}^*\right)}{mc}\frac{\left(\mathbb{B} - \mathbb{B}^*\right)^2}{\mathbb{B}} - 2\varepsilon\mathbb{I}^*\mathbb{Z}^* + \varepsilon\mathbb{I}^*\mathbb{Z} + \varepsilon\mathbb{I}^*\mathbb{Z}^*\left(\frac{\mathbb{Z}^*}{\mathbb{Z}}\right) - \frac{\varepsilon\mu}{\omega}\frac{\left(\mathbb{Z} - \mathbb{Z}^*\right)^2}{\mathbb{Z}} \\ = & - d\frac{\left(\mathbb{S} - \mathbb{S}^*\right)^2}{\mathbb{S}} + b\mathbb{S}^*\mathbb{V}^*\left(3 - \frac{\mathbb{S}^*}{\mathbb{S}} - \frac{\mathbb{S}\mathbb{V}\mathbb{I}^*}{\mathbb{S}^*\mathbb{V}^*\mathbb{I}} - \frac{\mathbb{V}^*\mathbb{I}}{\mathbb{V}\mathbb{I}^*}\right) + \frac{q\left(a + \varepsilon\mathbb{Z}^*\right)\mathbb{V}^*}{m}\frac{\left(\mathbb{B} - \mathbb{B}^*\right)^2}{\mathbb{B}} \\ & - \frac{q\delta\left(a + \varepsilon\mathbb{Z}^*\right)}{mc}\frac{\left(\mathbb{B} - \mathbb{B}^*\right)^2}{\mathbb{B}} + \varepsilon\mathbb{I}^*\frac{\left(\mathbb{Z} - \mathbb{Z}^*\right)^2}{\mathbb{Z}} - \frac{\varepsilon\mu}{\omega}\frac{\left(\mathbb{Z} - \mathbb{Z}^*\right)^2}{\mathbb{Z}} \\ = & - d\frac{\left(\mathbb{S} - \mathbb{S}^*\right)^2}{\mathbb{S}} + b\mathbb{S}^*\mathbb{V}^*\left(3 - \frac{\mathbb{S}^*}{\mathbb{S}} - \frac{\mathbb{S}\mathbb{V}\mathbb{I}^*}{\mathbb{S}^*\mathbb{V}^*\mathbb{I}} - \frac{\mathbb{V}^*\mathbb{I}}{\mathbb{V}\mathbb{I}^*}\right) - \frac{q\eta\left(a + \varepsilon\mathbb{Z}^*\right)}{mc\mathbb{B}^*}\frac{\left(\mathbb{B} - \mathbb{B}^*\right)^2}{\mathbb{B}} \\ & - \frac{\varepsilon\gamma}{\omega\mathbb{Z}^*}\frac{\left(\mathbb{Z} - \mathbb{Z}^*\right)^2}{\mathbb{Z}}. \end{aligned} \tag{4.20}$$

Utilizing the concept of arithmetic-geometric mean, we get

$$\begin{aligned} 3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\mathbb{V}\mathbb{I}^*}{\mathbb{S}^*\mathbb{V}^*\mathbb{I}}-\frac{\mathbb{V}^*\mathbb{I}}{\mathbb{V}\mathbb{I}^*} &\leq 0 \\ \frac{1}{3}\left(\frac{\mathbb{S}^*}{\mathbb{S}}+\frac{\mathbb{S}\mathbb{V}\mathbb{I}^*}{\mathbb{S}^*\mathbb{V}^*\mathbb{I}}+\frac{\mathbb{V}^*\mathbb{I}}{\mathbb{V}\mathbb{I}^*}\right) &\geq 1. \end{aligned} \tag{4.21}$$

Therefore, $\frac{d\mathbb{W}^*}{dt} \leq 0$ if $\mathbb{S}=\mathbb{S}^*, \mathbb{I}=\mathbb{I}^*, \mathbb{V}=\mathbb{V}^*, \mathbb{B}=\mathbb{B}^*$ and $\mathbb{Z}=\mathbb{Z}^*$.

Thus, By LaSalle's principle [56], endemic equilibrium $\mathbb{E}^{end}$ for Chikungunya virus system (4.1) is globally asymptotically stable. □

4.4.4 Fractional Chikungunya virus model with latently infected cells

The proposed Chikungunya virus model (4.2) with latently infected cells, which includes the well-known Caputo fractional derivative [24] is as follows:

$$\begin{aligned} {}^{\mathsf{C}}_{0}D_t^{\zeta}\mathbb{S}(t) =& \lambda - d\mathbb{S}(t) - b\mathbb{S}(t)\mathbb{V}(t), \\ {}^{\mathsf{C}}_{0}D_t^{\zeta}\mathbb{L}(t) =& (1-\rho)b\mathbb{S}(t)\mathbb{V}(t) - (\theta+\pi)\mathbb{L}(t), \\ {}^{\mathsf{C}}_{0}D_t^{\zeta}\mathbb{I}(t) =& \rho b\mathbb{S}(t)\mathbb{V}(t) - \pi\mathbb{L}(t) - a\mathbb{I}(t) - \epsilon\mathbb{I}(t)\mathbb{Z}(t), \\ {}^{\mathsf{C}}_{0}D_t^{\zeta}\mathbb{V}(t) =& m\mathbb{I}(t) - r\mathbb{V}(t) - q\mathbb{V}(t)\mathbb{B}(t), \\ {}^{\mathsf{C}}_{0}D_t^{\zeta}\mathbb{B}(t) =& \eta + c\mathbb{B}(t)\mathbb{V}(t) - \delta\mathbb{B}(t), \\ {}^{\mathsf{C}}_{0}D_t^{\zeta}\mathbb{Z}(t) =& \gamma + \omega\mathbb{I}(t)\mathbb{Z}(t) - \mu\mathbb{Z}(t), \end{aligned} \tag{4.22}$$

with initial conditions

$$\mathbb{S}(0)=\mathbb{S}_0, \mathbb{L}(0)=\mathbb{L}_0, \mathbb{I}(0)=\mathbb{I}_0, \mathbb{V}(0)=\mathbb{V}_0, \mathbb{B}(0)=\mathbb{B}_0, \mathbb{Z}(0)=\mathbb{Z}_0.$$

4.4.5 Existence and uniqueness of fractional Chikungunya virus model with latently infected cells

We will show in this section that the solution to the model (4.22) is under the consideration of existence and uniqueness using Banach's fixed point theorem. Additionally, we will use Schaefer's fixed point theorem to prove the solution's bounded ness and its existence.
Let us rewrite the considered model (4.22) in the following form:

$$\begin{aligned}
{}^{\mathsf{C}}_{0}D_t^{\zeta}\mathbb{S}(t) &= f_1(t,\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}),\\
{}^{\mathsf{C}}_{0}D_t^{\zeta}\mathbb{L}(t) &= f_2(t,\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}),\\
{}^{\mathsf{C}}_{0}D_t^{\zeta}\mathbb{I}(t) &= f_3(t,\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}),\\
{}^{\mathsf{C}}_{0}D_t^{\zeta}\mathbb{V}(t) &= f_4(t,\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}),\\
{}^{\mathsf{C}}_{0}D_t^{\zeta}\mathbb{B}(t) &= f_5(t,\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}),\\
{}^{\mathsf{C}}_{0}D_t^{\zeta}\mathbb{Z}(t) &= f_6(t,\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}),
\end{aligned} \tag{4.23}$$

where,

$$\begin{aligned}
f_1(t,\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) &= \lambda - d\mathbb{S}(t) - b\mathbb{S}(t)\mathbb{V}(t),\\
f_2(t,\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) &= (1-\rho)b\mathbb{S}(t)\mathbb{V}(t) - (\theta+\pi)\mathbb{L}(t),\\
f_3(t,\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) &= \rho b\mathbb{S}(t)\mathbb{V}(t) - \pi\mathbb{L}(t) - a\mathbb{I}(t) - \epsilon\mathbb{I}(t)\mathbb{Z}(t),\\
f_4(t,\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) &= m\mathbb{I}(t) - r\mathbb{V}(t) - q\mathbb{V}(t)\mathbb{B}(t),\\
f_5(t,\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) &= \eta + c\mathbb{B}(t)\mathbb{V}(t) - \delta\mathbb{B}(t),\\
f_6(t,\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) &= \gamma + \omega\mathbb{I}(t)\mathbb{Z}(t) - \mu\mathbb{Z}(t).
\end{aligned} \tag{4.24}$$

We will apply initial conditions and the fractional integral on (4.23). The integral equations that result from this process will be used to solve this fractional-order model.

$$\begin{aligned}
\mathbb{S}(t) &= \mathbb{S}(0) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-\delta)^{\zeta-1} f_1(t,\mathbb{S}(t))d\delta,\\
\mathbb{L}(t) &= \mathbb{L}(0) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-\delta)^{\zeta-1} f_2(t,\mathbb{L}(t))d\delta,\\
\mathbb{I}(t) &= \mathbb{I}(0) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-\delta)^{\zeta-1} f_3(t,\mathbb{I}(t))d\delta,\\
\mathbb{V}(t) &= \mathbb{V}(0) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-\delta)^{\zeta-1} f_4(t,\mathbb{V}(t))d\delta,\\
\mathbb{B}(t) &= \mathbb{B}(0) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-\delta)^{\zeta-1} f_5(t,\mathbb{B}(t))d\delta,\\
\mathbb{Z}(t) &= \mathbb{Z}(0) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-\delta)^{\zeta-1} f_6(t,\mathbb{Z}(t))d\delta.
\end{aligned} \tag{4.25}$$

Let $(\mathbb{D}, ||.||)$ be the Banach space and $\mathbb{H}^1([0,\mathbb{T}])$ be the set of functions $(f_1, f_2, f_3, f_4, f_5, f_6) : [0,\mathbb{T}] \times \mathbb{D} \to \mathbb{D}$ be assumed to be continuous having Chebyshev norm.

The continuous mappings $f_1, f_2, f_3, f_4, f_5, f_6$ satisfy the Lipschitz condition if

$$\sup_{0\le t\le T} \|\mathbb{S}\| \le \Psi_1, \qquad \sup_{0\le t\le T} \|\mathbb{L}\| \le \Psi_2, \qquad \sup_{0\le t\le T} \|\mathbb{I}\| \le \Psi_3,$$

$$\sup_{0\le t\le T} \|\mathbb{V}\| \le \Psi_4, \qquad \sup_{0\le t\le T} \|\mathbb{B}\| \le \Psi_5, \qquad \sup_{0\le t\le T} \|\mathbb{Z}\| \le \Psi_6.$$

Thus, firstly we have

$$\begin{aligned}
\|f_1(\mathbb{S}_1) - f_1(\mathbb{S}_2)\| &= \|(\lambda - d\mathbb{S}_1 - b\mathbb{S}_1\mathbb{V}) - (\lambda - d\mathbb{S}_2 - b\mathbb{S}_2\mathbb{V})\|\\
&= \|-d(\mathbb{S}_1 - \mathbb{S}_2) - b\mathbb{V}(\mathbb{S}_1 - \mathbb{S}_2)\|\\
&\le \left(d + b\sup_{0\le t\le T}\|\mathbb{V}\|\right)\|\mathbb{S}_1 - \mathbb{S}_2\|\\
&\le (d + b\Psi_4)\|\mathbb{S}_1 - \mathbb{S}_2\|\\
&= \mathbb{L}_{f_1}\|\mathbb{S}_1 - \mathbb{S}_2\|.
\end{aligned} \tag{4.26}$$

where $\mathbb{L}_{f_1} = (d + b\Psi_4) > 0$.

Similarly, we arrive at the following,

$$\begin{aligned}
\|f_2(\mathbb{L}_1) - f_2(\mathbb{L}_2)\| &= \mathbb{L}_{f_2} \|\mathbb{L}_1 - \mathbb{L}_2\| ; \mathbb{L}_{f_2} = \theta + \pi > 0, \\
\|f_3(\mathbb{I}_1) - f_3(\mathbb{I}_2)\| &= \mathbb{L}_{f_3} \|\mathbb{I}_1 - \mathbb{I}_2\| ; \mathbb{L}_{f_3} = a + \epsilon\Psi_6 > 0, \\
\|f_4(\mathbb{V}_1) - f_4(\mathbb{V}_2)\| &= \mathbb{L}_{f_4} \|\mathbb{V}_1 - \mathbb{V}_2\| ; \mathbb{L}_{f_4} = r + q\Psi_5 > 0, \\
\|f_5(\mathbb{B}_1) - f_5(\mathbb{B}_2)\| &= \mathbb{L}_{f_5} \|\mathbb{B}_1 - \mathbb{B}_2\| ; \mathbb{L}_{f_5} = c\Psi_4 - \delta > 0, \\
\|f_6(\mathbb{Z}_1) - f_6(\mathbb{Z}_2)\| &= \mathbb{L}_{f_6} \|\mathbb{Z}_1 - \mathbb{Z}_2\| ; \mathbb{L}_{f_6} = \omega\Psi_3 - \mu > 0.
\end{aligned} \tag{4.27}$$

Theorem 4.4.3. *Suppose* $(\mathbb{L}_{f_1}, \mathbb{L}_{f_2}, \mathbb{L}_{f_3}, \mathbb{L}_{f_4}, \mathbb{L}_{f_5}, \mathbb{L}_{f_6})\dfrac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)} < 1$ *and assuming that* $(f_1, f_2, f_3, f_4, f_5, f_6) : [0, \mathbb{T}] \times \mathbb{D} \to \mathbb{D}$ *are continuous and satisfies the Lipschitz criteria then the fractional model* (4.22) *has an unique solution.*

Proof. Considering the mapping $\Theta : \mathbb{H}^1([0, \mathbb{T}], \mathbb{D})$ where, Θ is defined in $(f_1, f_2, f_3, f_4, f_5, f_6) : [0, \mathbb{T}] \times \mathbb{D} \to \mathbb{D}$.

Using (4.26)- (4.27) and for all $((\mathbb{S}_1, \mathbb{S}_2), (\mathbb{L}_1, \mathbb{L}_2), (\mathbb{I}_1, \mathbb{I}_2), (\mathbb{V}_1, \mathbb{V}_2), (\mathbb{B}_1, \mathbb{B}_2), (\mathbb{Z}_1, \mathbb{Z}_2)) \in \mathbb{H}^1([0, \mathbb{T}], \mathbb{D})$ and $0 \le t \le \mathbb{T}$, we get

$$\begin{aligned}
\|\Theta(\mathbb{S}_1(t)) - \Theta(\mathbb{S}_2(t))\| &= \left\| \frac{1}{\Gamma(\zeta)} \int_0^t (t-\delta)^{\zeta-1} f_1(t, \mathbb{S}_1(t)) d\delta \right. \\
&\quad \left. - \left(\frac{1}{\Gamma(\zeta)} \int_0^t (t-\delta)^{\zeta-1} f_1(t, \mathbb{S}_2(t)) d\delta \right) \right\| \\
&\le \frac{1}{\Gamma(\zeta)} \int_0^t (t-\delta)^{\zeta-1} \|f_1(t, \mathbb{S}_1(t)) - f_1(t, \mathbb{S}_2(t))\| d\delta \\
&\le \frac{\mathbb{L}_{f_1}}{\Gamma(\zeta)} \int_0^t (t-\delta)^{\zeta-1} \|\mathbb{S}_1(t) - \mathbb{S}_2(t)\| d\delta \\
&\le \mathbb{L}_{f_1} \left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)} \right) \|\mathbb{S}_1 - \mathbb{S}_2\| .
\end{aligned} \tag{4.28}$$

In a same manner,

$$\|\Theta(\mathbb{L}_1(t)) - \Theta(\mathbb{L}_2(t))\| \leq \mathbb{L}_{f_2}\left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)}\right)\|\mathbb{L}_1 - \mathbb{L}_2\|, \tag{4.29}$$

$$\|\Theta(\mathbb{I}_1(t)) - \Theta(\mathbb{I}_2(t))\| \leq \mathbb{L}_{f_3}\left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)}\right)\|\mathbb{I}_1 - \mathbb{I}_2\|, \tag{4.30}$$

$$\|\Theta(\mathbb{V}_1(t)) - \Theta(\mathbb{V}_2(t))\| \leq \mathbb{L}_{f_4}\left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)}\right)\|\mathbb{V}_1 - \mathbb{V}_2\|, \tag{4.31}$$

$$\|\Theta(\mathbb{B}_1(t)) - \Theta(\mathbb{B}_2(t))\| \leq \mathbb{L}_{f_5}\left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)}\right)\|\mathbb{B}_1 - \mathbb{B}_2\|, \tag{4.32}$$

$$\|\Theta(\mathbb{Z}_1(t)) - \Theta(\mathbb{Z}_2(t))\| \leq \mathbb{L}_{f_6}\left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)}\right)\|\mathbb{Z}_1 - \mathbb{Z}_2\|. \tag{4.33}$$

It is evident from the condition that $(\mathbb{L}_{f_1}, \mathbb{L}_{f_2}, \mathbb{L}_{f_3}, \mathbb{L}_{f_4}, \mathbb{L}_{f_5}, \mathbb{L}_{f_6})\dfrac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)} < 1$. As a contraction mapping, the operator Θ has a distinct fixed point in $0 \leq t \leq \mathbb{T}$, according to the application of the Banach contraction mapping principle. □

Using Schaefer's fixed point theorem, we examine whether the fractional model (4.22) has solution.

Theorem 4.4.4. *Given that* $(f_1, f_2, f_3, f_4, f_5, f_6) : [0, \mathbb{T}] \times \mathbb{D} \to \mathbb{D}$ *are continuous and there exist constants* $(\mathbb{L}_{f_1}, \mathbb{L}_{f_2}, \mathbb{L}_{f_3}, \mathbb{L}_{f_4}, \mathbb{L}_{f_5}, \mathbb{L}_{f_6}) > 0$ *such that,*

$$\|f_1(t, \mathbb{S})| \leq \mathbb{L}_{f_1}(k + \|\mathbb{S}\|),\ \|f_2(t, \mathbb{L})| \leq \mathbb{L}_{f_2}(k + \|\mathbb{L}\|),\ \|f_3(t, \mathbb{I})| \leq \mathbb{L}_{f_3}(k + \|\mathbb{I}\|),$$

$$\|f_4(t, \mathbb{V})| \leq \mathbb{L}_{f_4}(k + \|\mathbb{V}\|),\ \|f_5(t, \mathbb{B})| \leq \mathbb{L}_{f_5}(k + \|\mathbb{B}\|),\ \|f_6(t, \mathbb{Z})| \leq \mathbb{L}_{f_6}(k + \|\mathbb{Z}\|),$$

where, $0 < k \leq 1$ *is an arbitrary number, then model* (4.22) *has at least one solution.*

Proof. Let $\{\mathbb{S}^{m+1}\}_\infty, \{\mathbb{L}^{m+1}\}_\infty, \{\mathbb{I}^{m+1}\}_\infty, \{\mathbb{V}^{m+1}\}_\infty, \{\mathbb{B}^{m+1}\}_\infty, \{\mathbb{Z}^{m+1}\}_\infty$ be sequences such that $\mathbb{S}^{m+1} \to \mathbb{S}^m, \mathbb{L}^{m+1} \to \mathbb{L}^m, \mathbb{I}^{m+1} \to \mathbb{I}^m, \mathbb{V}^{m+1} \to \mathbb{V}^m, \mathbb{B}^{m+1} \to \mathbb{B}^m, \mathbb{Z}^{m+1} \to \mathbb{Z}^m$ in

$\mathbb{H}^1([0,\mathbb{T}],\mathbb{D})$. For each $0 \leq t \leq T$, we have,

$$\begin{aligned}
&\left\|\Theta\mathbb{S}^{m+1}(t) - \Theta\mathbb{S}^{m}(t)\right\| \\
&= \left\| \frac{1}{\Gamma(\zeta)} \int_0^t (t-\delta)^{\zeta-1} f_1(t, \mathbb{S}^{m+1}(t)) d\delta - \frac{1}{\Gamma(\zeta)} \int_0^t (t-\delta)^{\zeta-1} f_1(t, \mathbb{S}^{m}(t)) d\delta \right\| \\
&\leq \frac{1}{\Gamma(\zeta)} \int_0^t (t-\delta)^{\zeta-1} \left\| f_1(t, \mathbb{S}^{m+1}(t)) - f_1(t, \mathbb{S}^{m}(t)) \right\| d\delta \\
&\leq \mathbb{L}_{f_1} \left(\frac{\mathbb{T}^{\zeta}}{\Gamma(\zeta+1)} \right) \left\| \mathbb{S}^{m+1} - \mathbb{S}^{m} \right\|,
\end{aligned} \tag{4.34}$$

where, $\|\mathbb{S}^{m+1} - \mathbb{S}^{m}\| \to 0$ as $m \to \infty$. Similarly we also obtain

$$\left\|\Theta\mathbb{L}^{m+1}(t) - \Theta\mathbb{L}^{m}(t)\right\| \leq \mathbb{L}_{f_2} \left(\frac{\mathbb{T}^{\zeta}}{\Gamma(\zeta+1)} \right) \left\| \mathbb{L}^{m+1} - \mathbb{L}^{m} \right\|, \tag{4.35}$$

$$\left\|\Theta\mathbb{I}^{m+1}(t) - \Theta\mathbb{I}^{m}(t)\right\| \leq \mathbb{L}_{f_3} \left(\frac{\mathbb{T}^{\zeta}}{\Gamma(\zeta+1)} \right) \left\| \mathbb{I}^{m+1} - \mathbb{I}^{m} \right\|, \tag{4.36}$$

$$\left\|\Theta\mathbb{V}^{m+1}(t) - \Theta\mathbb{V}^{m}(t)\right\| \leq \mathbb{L}_{f_4} \left(\frac{\mathbb{T}^{\zeta}}{\Gamma(\zeta+1)} \right) \left\| \mathbb{V}^{m+1} - \mathbb{V}^{m} \right\|, \tag{4.37}$$

$$\left\|\Theta\mathbb{B}^{m+1}(t) - \Theta\mathbb{B}^{m}(t)\right\| \leq \mathbb{L}_{f_2} \left(\frac{\mathbb{T}^{\zeta}}{\Gamma(\zeta+1)} \right) \left\| \mathbb{B}^{m+1} - \mathbb{B}^{m} \right\|, \tag{4.38}$$

$$\left\|\Theta\mathbb{Z}^{m+1}(t) - \Theta\mathbb{Z}^{m}(t)\right\| \leq \mathbb{L}_{f_2} \left(\frac{\mathbb{T}^{\zeta}}{\Gamma(\zeta+1)} \right) \left\| \mathbb{Z}^{m+1} - \mathbb{Z}^{m} \right\|, \tag{4.39}$$

where, $\|\mathbb{L}^{m+1} - \mathbb{L}^{m}\| \to 0$, $\|\mathbb{I}^{m+1} - \mathbb{I}^{m}\| \to 0$, $\|\mathbb{V}^{m+1} - \mathbb{V}^{m}\| \to 0$,
$\|\mathbb{B}^{m+1} - \mathbb{B}^{m}\| \to 0$, $\|\mathbb{Z}^{m+1} - \mathbb{Z}^{m}\| \to 0$ as $m \to \infty$. Thus the operator Θ is continuous.

Next, we demonstrate that, on the set of $\mathbb{H}^1([0,\mathbb{T}],\mathbb{D})$, the operator Θ is a bounded function with one-to-one correspondence. For each $\mathbb{S} \in \mathbb{H}_{\mathbb{S}}, \mathbb{L} \in \mathbb{H}_{\mathbb{L}}, \mathbb{I} \in \mathbb{H}_{\mathbb{I}}, \mathbb{V} \in \mathbb{H}_{\mathbb{V}}, \mathbb{B} \in \mathbb{H}_{\mathbb{B}}, \mathbb{Z} \in \mathbb{H}_{\mathbb{Z}}$ and for $e > 0$, there correspond a value $g > 0$. Where $\|\Theta\mathbb{S}\| \leq g$, $\|\Theta\mathbb{L}\| \leq g$, $\|\Theta\mathbb{I}\| \leq g$, $\|\Theta\mathbb{V}\| \leq g$, $\|\Theta\mathbb{B}\| \leq g$, $\|\Theta\mathbb{Z}\| \leq g$, and for $0 \leq t \leq \mathbb{T}$, the subset of Banach space having all

continuous functions, can be defined as

$$\mathbb{H}_{\mathbb{S}} = \left\{\mathbb{S} \in \mathbb{H}^1([0,\mathbb{T}],\mathbb{D}) : \|\mathbb{S}\| \leq e\right\}, \mathbb{H}_{\mathbb{L}} = \left\{\mathbb{L} \in \mathbb{H}^1([0,\mathbb{T}],\mathbb{D}) : \|\mathbb{L}\| \leq e\right\},$$

$$\mathbb{H}_{\mathbb{I}} = \left\{\mathbb{I} \in \mathbb{H}^1([0,\mathbb{T}],\mathbb{D}) : \|\mathbb{I}\| \leq e\right\}, \mathbb{H}_{\mathbb{V}} = \left\{\mathbb{V} \in \mathbb{H}^1([0,\mathbb{T}],\mathbb{D}) : \|\mathbb{V}\| \leq e\right\},$$

$$\mathbb{H}_{\mathbb{B}} = \left\{\mathbb{B} \in \mathbb{H}^1([0,\mathbb{T}],\mathbb{D}) : \|\mathbb{B}\| \leq e\right\}, \mathbb{H}_{\mathbb{Z}} = \left\{\mathbb{Z} \in \mathbb{H}^1([0,\mathbb{T}],\mathbb{D}) : \|\mathbb{Z}\| \leq e\right\}.$$

So for any $0 \leq t \leq \mathbb{T}$,

$$\begin{aligned}
\|\Theta S\| &\leq \|\mathbb{S}(0)\| + \frac{1}{\Gamma(\zeta)} \int_0^t (t-\delta)^{\zeta-1} \|f_1(t,\mathbb{S}(t))\| \, d\delta \\
&\leq \|\mathbb{S}(0)\| + \frac{\|f_1(t,\mathbb{S}(t))\|}{\Gamma(\zeta)} \int_0^t (t-\delta)^{\zeta-1} d\delta \\
&\leq \|\mathbb{S}(0)\| + \mathbb{L}_{f_1}(k + \|S\|) \left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)}\right) \\
&\leq \|\mathbb{S}(0)\| + \mathbb{L}_{f_1}(k + e) \left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)}\right).
\end{aligned}$$

Using the same strategy, we have

$$\begin{aligned}
\|\Theta\mathbb{L}\| &\leq \|\mathbb{L}(0)\| + \mathbb{L}_{f_2}(k + e) \left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)}\right), \\
\|\Theta\mathbb{I}\| &\leq \|\mathbb{I}(0)\| + \mathbb{L}_{f_3}(k + e) \left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)}\right), \\
\|\Theta\mathbb{V}\| &\leq \|\mathbb{V}(0)\| + \mathbb{L}_{f_4}(k + e) \left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)}\right), \\
\|\Theta\mathbb{B}\| &\leq \|\mathbb{B}(0)\| + \mathbb{L}_{f_5}(k + e) \left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)}\right), \\
\|\Theta\mathbb{Z}\| &\leq \|\mathbb{Z}(0)\| + \mathbb{L}_{f_6}(k + e) \left(\frac{\mathbb{T}^\zeta}{\Gamma(\zeta+1)}\right).
\end{aligned}$$

Conversely, suppose Φ maps a bounded and equaicontinuous set in $\mathbb{H}^1([0,\mathbb{T}],D)$.

If $0 \leq t_1 \leq t_2 \leq T$, then

$$\begin{aligned}
&\|\Theta(\mathbb{S}(t_1)) - \Theta(\mathbb{S}(t_2))\| \\
&= \frac{1}{\Gamma(\zeta)} \left\| \int_0^{t_1} (t_1-\delta)^{\zeta-1} f_1(t,\mathbb{S}(t))d\delta - \int_0^{t_2} (t_2-\delta)^{\zeta-1} f_1(t,\mathbb{S}(t))d\delta \right\| \\
&\leq \frac{1}{\Gamma(\zeta)} \left\| \int_0^{t_1} \left((t_1-\delta)^{\zeta-1} - (t_2-\delta)^{\zeta-1}\right) f_1(t,\mathbb{S}(t))d\delta \right\| + \frac{1}{\Gamma(\zeta)} \left\| \int_{t_1}^{t_2} (t_2-\delta)^{\zeta-1} f_1(t,\mathbb{S}(t))d\delta \right\| \\
&\leq \mathbb{L}_{f_1} \frac{(k+e)}{\Gamma(\zeta)} \left\| \int_0^{t_1} \left((t_1-\delta)^{\zeta-1} - (t_2-\delta)^{\zeta-1}\right) d\delta + \int_{t_1}^{t_2} (t_2-\delta)^{\zeta-1} d\delta \right\| \\
&\leq \left(\frac{\mathbb{L}_{f_1}(k+e)\mathbb{T}^\zeta}{\Gamma(\zeta+1)} \right) \left(t_1^\zeta - t_2^\zeta + 2(t_2-t_1)^\zeta \right).
\end{aligned}$$

Similarly, we also obtain

$$\begin{aligned}
\|\Theta(\mathbb{L}(t_1)) - \Theta(\mathbb{L}(t_2))\| &\leq \left(\frac{\mathbb{L}_{f_2}(k+e)\mathbb{T}^\zeta}{\Gamma(\zeta+1)} \right) \left(t_1^\zeta - t_2^\zeta + 2(t_2-t_1)^\zeta \right), \\
\|\Theta(\mathbb{I}(t_1)) - \Theta(\mathbb{I}(t_2))\| &\leq \left(\frac{\mathbb{L}_{f_3}(k+e)\mathbb{T}^\zeta}{\Gamma(\zeta+1)} \right) \left(t_1^\zeta - t_2^\zeta + 2(t_2-t_1)^\zeta \right), \\
\|\Theta(\mathbb{V}(t_1)) - \Theta(\mathbb{V}(t_2))\| &\leq \left(\frac{\mathbb{L}_{f_4}(k+e)\mathbb{T}^\zeta}{\Gamma(\zeta+1)} \right) \left(t_1^\zeta - t_2^\zeta + 2(t_2-t_1)^\zeta \right), \\
\|\Theta(\mathbb{B}(t_1)) - \Theta(\mathbb{B}(t_2))\| &\leq \left(\frac{\mathbb{L}_{f_5}(k+e)\mathbb{T}^\zeta}{\Gamma(\zeta+1)} \right) \left(t_1^\zeta - t_2^\zeta + 2(t_2-t_1)^\zeta \right), \\
\|\Theta(\mathbb{Z}(t_1)) - \Theta(\mathbb{Z}(t_2))\| &\leq \left(\frac{\mathbb{L}_{f_6}(k+e)\mathbb{T}^\zeta}{\Gamma(\zeta+1)} \right) \left(t_1^\zeta - t_2^\zeta + 2(t_2-t_1)^\zeta \right).
\end{aligned}$$

The expression tends to zero as $t_1 \to t_2$ on the right side of the inequality. By Arzela-Ascoli theorem, Θ is a continuous function.

Now, to show that

$$N(\Theta) = \left\{ (\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) \in \mathbb{H}^1([0,\mathbb{T}],\mathbb{D}) : (\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) = \lambda(\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) \right\}$$

is bounded for any $\lambda \in (0,1)$.

Assume $(\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) \in N(\Theta)$, such that $(\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z}) = \lambda\Theta(\mathbb{S},\mathbb{L},\mathbb{I},\mathbb{V},\mathbb{B},\mathbb{Z})$.

For each $t \in [0,\mathbb{T}]$ which gives

$$\begin{aligned}\|\mathbb{S}(t)\| &\leq \mathbb{S}(0) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-\delta)^{\zeta-1}\,\|f_1(t,\mathbb{S}(t))\|\,d\delta \\ &\leq \mathbb{S}(0) + \frac{\mathbb{L}_{f_1}(k+e)}{\Gamma(\zeta)}\int_0^t (t-\delta)^{\zeta-1} d\delta \\ &\leq \mathbb{S}(0) + \frac{\mathbb{L}_{f_1}(k+e)\mathbb{T}^{\zeta}}{\Gamma(\zeta+1)} < \infty.\end{aligned}$$

In a same manner, $\|\mathbb{L}(t)\| < \infty$, $\|\mathbb{I}(t)\| < \infty$, $\|\mathbb{V}(t)\| < \infty$, $\|\mathbb{B}(t)\| < \infty$, $\|\mathbb{Z}(t)\| < \infty$. As we have prove that $N(\Theta)$ is bounded and we demonstrated, Θ has a fixed point, which is provided by Schaefer's fixed point theorem. Hence the model has a solution. □

4.4.6 Stability analysis of fractional Chikungunya virus model with latently infected cells

For the Chikungunya virus system (4.2), we discuss the stability though two unique equilibrium points of the system.

1. Disease free equilibrium point $\mathbb{E}^0_{\mathbb{L}}$:

   At this stage the concentration of actively infected cells $\mathbb{I}$, latently infected cells $\mathbb{L}$ and Chikungunya virus germs $\mathbb{V}$ are almost zero. Now, we equating right side of the system (4.2) with zero and with basic simplification, we get

$$\begin{aligned}\mathbb{E}^0_{\mathbb{L}} &= \left(\mathbb{S}^0,\mathbb{L}^0,\mathbb{I}^0,\mathbb{V}^0,\mathbb{B}^0,\mathbb{Z}^0\right) \\ &= \left(\frac{\lambda}{d},0,0,\frac{\eta}{\delta},\frac{\gamma}{\mu}\right).\end{aligned} \tag{4.40}$$

2. Endemic equilibrium point $\mathbb{E}_{\mathbb{L}}^{end}$:

   For $\mathbb{E}_{\mathbb{L}}^{end}$, we equating right side of the system (4.2) with zero

$$\begin{aligned}
&\lambda - d\mathbb{S} - b\mathbb{S}\mathbb{V} = 0,\\
&(1-\rho)b\mathbb{S}\mathbb{V} - (\theta+\pi)\mathbb{L} = 0,\\
&b\rho\mathbb{S}\mathbb{V} + \pi\mathbb{L} - a\mathbb{I} - \epsilon\mathbb{I}\mathbb{Z} = 0,\\
&m\mathbb{I} - r\mathbb{V} - q\mathbb{V}\mathbb{B} = 0,\\
&\eta + c\mathbb{B}\mathbb{V} - \delta\mathbb{B} = 0,\\
&\gamma + \omega\mathbb{I}\mathbb{Z} - \mu\mathbb{Z} = 0,
\end{aligned} \tag{4.41}$$

   with the basic simplification, the solution of system (4.41) will be

$$\mathbb{S} = \frac{\lambda}{d + b\mathbb{V}},\ \mathbb{L} = \frac{b(1-\rho)\mathbb{S}\mathbb{V}}{(\theta+\pi)},\ \mathbb{I} = \frac{\mathbb{V}(r+qB)}{m},\ \mathbb{B} = \frac{\eta}{\delta - c\mathbb{V}},\ \mathbb{Z} = \frac{m\gamma}{m\mu - \omega\mathbb{V}(r+q\mathbb{B})}. \tag{4.42}$$

   Substituting (4.42) into (4.41), we have

$$\mathbb{K}_6\mathbb{V}^4 + \mathbb{K}_7\mathbb{V}^3 + \mathbb{K}_8\mathbb{V}^2 + \mathbb{K}_9\mathbb{V} + \mathbb{K}_{10} = 0, \tag{4.43}$$

   where,

$$\begin{aligned}
\mathbb{K}_6 =\ & abc^2r^2\omega(\theta-\pi),\\
\mathbb{K}_7 =\ & cr\left(-mbc(\theta+\pi)(\omega\lambda+\epsilon\gamma) + a(\theta+\pi)\left(-2b\omega(q\eta+r\delta) + c(rd\omega - mb\mu)\right)\right),\\
\mathbb{K}_8 =\ & a\omega(\theta+\pi)(q\eta+r\delta)(rb\delta - 2cdr + bq\eta) + mac\mu(\theta+\pi)(bq\eta + 2br\delta - cdr)\\
& + mbc(\theta+\pi)(q\eta+2r\delta)(\omega\lambda+\epsilon\gamma) + mc^2(\theta+\pi)(mb\lambda\mu - dr\epsilon\gamma),\\
\mathbb{K}_9 =\ & ad\omega(\theta+\pi)(q\eta+r\delta)^2 + ma\mu(\theta+\pi)\left(-b\delta(q\eta+r\delta) + cd(q\eta+2r\delta)\right)\\
& - m(\theta+\pi)\left(-cd\epsilon\gamma(q\eta+2r\delta) + b\delta(q\eta+r\delta)(\omega\lambda+\epsilon\gamma) + 2mbc\delta\lambda\mu\right),\\
\mathbb{K}_{10} =\ & md\delta(q\eta+r\delta)(\epsilon\gamma+a\mu)(\theta+\pi)(\mathbb{R}_0^{\mathbb{L}} - 1).
\end{aligned}$$

Hence, the basic reproduction number $\mathbb{R}_0^{\mathbb{L}}$ is

$$\mathbb{R}_0^{\mathbb{L}} = \frac{mb\mu\delta\lambda(\theta+\rho\pi)}{d(\theta+\pi)(q\eta+r\delta)(\epsilon\gamma+a\mu)}. \tag{4.44}$$

For this $\mathbb{R}_0^{\mathbb{L}}$, the Chikungunya virus system (4.2) having endemic equilibrium point $\mathbb{E}_{\mathbb{L}}^{end}$ can be defined as

$$\mathbb{E}_{\mathbb{L}}^{end} = (\mathbb{S}^*, \mathbb{I}^*, \mathbb{V}^*, \mathbb{B}^*, \mathbb{Z}^*), \tag{4.45}$$

where,

$$\mathbb{S}^* = \frac{\lambda}{d+b\mathbb{V}^*}, \qquad \mathbb{L}^* = \frac{b\lambda(1-\rho)\mathbb{V}^*}{(\theta+\pi)(d+b\mathbb{V}^*)}, \qquad \mathbb{I}^* = \frac{(r(\delta-c\mathbb{V}^*)+q\eta)\mathbb{V}^*}{m(\delta-c\mathbb{V}^*)},$$
$$\mathbb{B}^* = \frac{\eta}{\delta-c\mathbb{V}^*}, \qquad \mathbb{Z}^* = \frac{m\gamma(\delta-c\mathbb{V}^*)}{m\mu(\delta-c\mathbb{V}^*)+\omega(rc\mathbb{V}^*-(r\delta+q\eta))\mathbb{V}^*}.$$

Theorem 4.4.5. *Disease-free equilibrium $\mathbb{E}_{\mathbb{L}}^0$ for Chikungunya virus system (4.2) is globally asymptotically stable if*

$$\mathbb{R}_0^{\mathbb{L}} \leq 1.$$

Proof. By following the exactly same process of Theorem 4.4.1, we can easily prove that the Chikungunya virus system (4.2) with latently infected cells are globally stable for $\mathbb{R}_0^{\mathbb{L}} \leq 1$. □

Theorem 4.4.6. *Disease-free equilibrium $\mathbb{E}_{\mathbb{L}}^{end}$ for Chikungunya virus system (4.2) is globally asymptotically stable if*

$$\mathbb{R}_0^{\mathbb{L}} > 1.$$

Proof. We can proceed the exactly same process of Theorem 4.4.2 and we can achieve that the Chikungunya virus system (4.2) with latently infected cells are globally stable for $\mathbb{R}_0^{\mathbb{L}} > 1$. □

4.5 Methodology

In this section, we utilized a recently proposed novel numerical scheme by Abdon Atangana and Kolade M. Owolabi [15]. To understand the methodology, let us consider the FIVP:

$$ {}^{C}{}_{0}D_{t}^{\zeta}\chi(t) = \kappa(t, \chi(t)), \quad t \in [0, \mathbb{T}], \zeta > 0, \tag{4.46} $$

with initial condition:

$$ \chi(0) = \chi_0. \tag{4.47} $$

Now, we discuss the numerical methods to solve (4.46).

In [15], authors derived a numerical scheme based on the fundamental theorem of calculus and calculating the difference between two times t_{s+1} and t_s. Employing calculus's fundamental theorem on (4.46),we get

$$ \chi(t) - \chi_0 = \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} \kappa(\tau, \chi(\tau)) d\tau. \tag{4.48} $$

Thus, at $t = t_{s+1}, s = 0, 1, 2, \ldots$

$$ \chi(t_{s+1}) - \chi(0) = \frac{1}{\Gamma(\zeta)} \int_0^{t_{s+1}} (t_{s+1}-\tau)^{\zeta-1} \kappa(t, \chi(t)) dt, \tag{4.49} $$

and at $t = t_s$, we obtain

$$ \chi(t_s) - \chi(0) = \frac{1}{\Gamma(\zeta)} \int_0^{t_s} (t_s-\tau)^{\zeta-1} \kappa(t, \chi(t)) dt. \tag{4.50} $$

Subtracting (4.49)-(4.50), it leads to

$$ \begin{aligned} \chi(t_{s+1}) =& \chi(t_s) + \frac{1}{\Gamma(\zeta)} \int_0^{t_{s+1}} (t_{s+1}-\tau)^{\zeta-1} \kappa(t, \chi(t)) dt \\ & - \frac{1}{\Gamma(\zeta)} \int_0^{t_s} (t_s-\tau)^{\zeta-1} \kappa(t, \chi(t)) dt. \end{aligned} \tag{4.51} $$

Now, (4.51) can be written as

$$\chi(t_{s+1}) = \chi(t_s) + \Theta_{\zeta,1} - \Theta_{\zeta,2}, \tag{4.52}$$

where $\Theta_{\zeta,1} = \frac{1}{\Gamma(\zeta)} \int_0^{t_{s+1}} (t_{s+1} - \tau)^{\zeta-1} \kappa(t, \chi(t)) dt$ and $\Theta_{\zeta,2} = \frac{1}{\Gamma(\zeta)} \int_0^{t_s} (t_s - \tau)^{\zeta-1} \kappa(t, \chi(t)) dt$. Using the Lagrange approximation for the function $\kappa(t, \chi(t))$, we have

$$\begin{aligned} \kappa(t, \chi(t)) &\simeq \frac{t - t_{s-1}}{t_s - t_{s-1}} \kappa(t_s, \chi_s) + \frac{t - t_s}{t_{s-1} - t_s} \kappa(t_{s-1}, \chi_{s-1}) \\ &= \frac{t - t_{s-1}}{\Delta} \kappa(t_s, \chi_s) - \frac{t - t_s}{\Delta} \kappa(t_{s-1}, \chi_{s-1}). \end{aligned} \tag{4.53}$$

The use of the above expression leads to

$$\Theta_{\zeta,1} = \frac{1}{\Gamma(\zeta)} \int_0^{t_{s+1}} (t_{s+1} - t)^{\zeta-1} \left\{ \frac{t - t_{s-1}}{\Delta} \kappa(t_s, \chi_s) - \frac{t - t_s}{\Delta} \kappa(t_{s-1}, \chi_{s-1}) \right\} dt$$

On integrating, we obtain

$$\Theta_{\zeta,1} = \frac{\kappa(t_s, \chi_s)}{\Delta\Gamma(\zeta)} \left\{ \frac{2\Delta t_{s+1}^{\zeta}}{\zeta} - \frac{t_{s+1}^{\zeta+1}}{\zeta+1} \right\} - \frac{\kappa(t_{s-1}, \chi_{s-1})}{\Delta\Gamma(\zeta)} \left\{ \frac{\Delta t_{s+1}^{\zeta}}{\zeta} - \frac{t_{s+1}^{\zeta+1}}{\zeta+1} \right\}. \tag{4.54}$$

Similarly, we get

$$\Theta_{\zeta,2} = \frac{\kappa(t_s, \chi_s)}{\Delta\Gamma(\zeta)} \left\{ \frac{\Delta t_s^{\zeta}}{\zeta} - \frac{t_s^{\zeta+1}}{\zeta+1} \right\} + \frac{\kappa(t_{s-1}, \chi_{s-1})}{\Delta\Gamma(\zeta)} \left\{ \frac{t_s^{\zeta+1}}{\zeta+1} \right\}. \tag{4.55}$$

We have the final approximate solution of IVP (4.46) - (4.47) by substituting (4.54) and (4.55) in (4.52), we get

$$\begin{aligned} \chi(t_{s+1}) = &\chi(t_s) + \frac{\kappa(t_s, \chi_s)}{\Delta\Gamma(\zeta)} \left[\frac{2\Delta t_{s+1}^{\zeta}}{\zeta} - \frac{t_{s+1}^{\zeta+1}}{\zeta+1} + \frac{\Delta t_s^{\zeta}}{\zeta} - \frac{t_s^{\zeta+1}}{\zeta+1} \right] \\ &+ \frac{\kappa(t_{s-1}, \chi_{s-1})}{\Delta\Gamma(\zeta)} \left[-\frac{\Delta t_{s+1}^{\zeta}}{\zeta} + \frac{t_{s+1}^{\zeta+1}}{\zeta+1} + \frac{t_s^{\zeta+1}}{\zeta+1} \right]. \end{aligned} \tag{4.56}$$

4.5.1 Methodology for fractional Chikungunya virus model with adoptive immune response

Here, we utilize (4.56) for numerical simulation of (4.6).

$$\begin{aligned}
\mathbb{S}\left(t_{s+1}\right) =& \mathbb{S}\left(t_s\right) + \frac{f_1\left(t_s, \mathbb{S}_s\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_1 + \frac{f_1\left(t_{s-1}, \mathbb{S}_{s-1}\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_2,\\
\mathbb{I}\left(t_{s+1}\right) =& \mathbb{I}\left(t_s\right) + \frac{f_2\left(t_s, \mathbb{I}_s\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_1 + \frac{f_2\left(t_{s-1}, \mathbb{I}_{s-1}\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_2,\\
\mathbb{V}\left(t_{s+1}\right) =& \mathbb{V}\left(t_s\right) + \frac{f_3\left(t_s, \mathbb{V}_s\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_1 + \frac{f_3\left(t_{s-1}, \mathbb{V}_{s-1}\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_2,\\
\mathbb{B}\left(t_{s+1}\right) =& \mathbb{B}\left(t_s\right) + \frac{f_4\left(t_s, \mathbb{B}_s\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_1 + \frac{f_4\left(t_{s-1}, \mathbb{B}_{s-1}\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_2,\\
\mathbb{Z}\left(t_{s+1}\right) =& \mathbb{Z}\left(t_s\right) + \frac{f_5\left(t_s, \mathbb{Z}_s\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_1 + \frac{f_5\left(t_{s-1}, \mathbb{Z}_{s-1}\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_2,
\end{aligned} \tag{4.57}$$

where $\mathbb{K}_1 = \left[\frac{2\Delta t_{s+1}^{\zeta}}{\zeta} - \frac{t_{s+1}^{\zeta+1}}{\zeta+1} + \frac{\Delta t_s^{\zeta}}{\zeta} - \frac{t_s^{\zeta+1}}{\zeta+1}\right]$ and $\mathbb{K}_2 = \left[-\frac{\Delta t_{s+1}^{\zeta}}{\zeta} + \frac{t_{s+1}^{\zeta+1}}{\zeta+1} + \frac{t_s^{\zeta+1}}{\zeta+1}\right]$.

4.5.2 Methodology for fractional Chikungunya virus model with latently infected cells

Here, we utilize (4.56) for numerical simulation of (4.22).

$$\begin{aligned}
\mathbb{S}\left(t_{s+1}\right) =& \mathbb{S}\left(t_s\right) + \frac{f_1\left(t_s, \mathbb{S}_s\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_1 + \frac{f_1\left(t_{s-1}, \mathbb{S}_{s-1}\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_2,\\
\mathbb{L}\left(t_{s+1}\right) =& \mathbb{L}\left(t_s\right) + \frac{f_2\left(t_s, \mathbb{L}_s\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_1 + \frac{f_2\left(t_{s-1}, \mathbb{L}_{s-1}\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_2,\\
\mathbb{I}\left(t_{s+1}\right) =& \mathbb{I}\left(t_s\right) + \frac{f_3\left(t_s, \mathbb{I}_s\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_1 + \frac{f_3\left(t_{s-1}, \mathbb{I}_{s-1}\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_2,\\
\mathbb{V}\left(t_{s+1}\right) =& \mathbb{V}\left(t_s\right) + \frac{f_4\left(t_s, \mathbb{V}_s\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_1 + \frac{f_4\left(t_{s-1}, \mathbb{V}_{s-1}\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_2,\\
\mathbb{B}\left(t_{s+1}\right) =& \mathbb{B}\left(t_s\right) + \frac{f_5\left(t_s, \mathbb{B}_s\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_1 + \frac{f_5\left(t_{s-1}, \mathbb{B}_{s-1}\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_2,\\
\mathbb{Z}\left(t_{s+1}\right) =& \mathbb{Z}\left(t_s\right) + \frac{f_6\left(t_s, \mathbb{Z}_s\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_1 + \frac{f_6\left(t_{s-1}, \mathbb{Z}_{s-1}\right)}{\Delta\Gamma(\zeta)}\mathbb{K}_2,
\end{aligned} \tag{4.58}$$

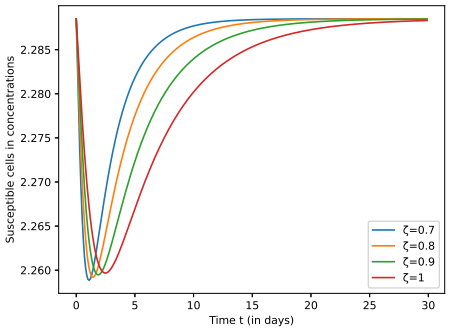
where $\mathbb{K}_1 = \left[\frac{2\Delta t_{s+1}^{\zeta}}{\zeta} - \frac{t_{s+1}^{\zeta+1}}{\zeta+1} + \frac{\Delta t_s^{\zeta}}{\zeta} - \frac{t_s^{\zeta+1}}{\zeta+1}\right]$ and $\mathbb{K}_2 = \left[-\frac{\Delta t_{s+1}^{\zeta}}{\zeta} + \frac{t_{s+1}^{\zeta+1}}{\zeta+1} + \frac{t_s^{\zeta+1}}{\zeta+1}\right]$.

4.6 Result and Discussion

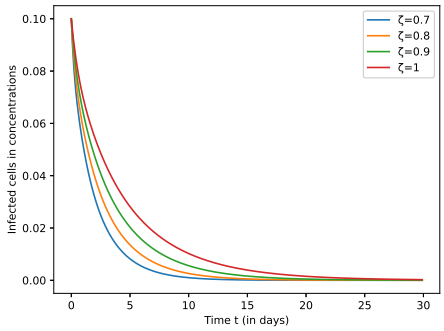
This section addresses the numerical solution of the fractional Chikungunya virus model with adaptive immune response as (4.6) and with latently infected cells as (4.22), which is obtained using Atangana and Owolabi's recently proposed numerical technique with the specific parameter values $\lambda = 1.826, \mu = 1.2, c = 1.2129, d = 0.7979, \gamma = 0.5, \omega = 0.5, \epsilon = 0.4441, m = 2.02, q = 0.5964, r = 0.4418, \eta = 1.402, \delta = 1.251, a = 0.5$.

The solution of the system using Caputo fractional derivative are depicted through a several graphs for fractional orders $\zeta = 0.7, 0.8, 0.9$ and 1.0, as shown in figures – (4.3a) to (4.6f). Particularly, figures – (4.3a) to (4.3e) show the behavior of the each compartment of the fractional Chikungunya virus system (4.6) with adaptive immune response for 30 days, when Chikungunya virus transmission rate is 0.1 and figures – (4.4a) to (4.4e) are the showcase for the same when Chikungunya virus transmission rate is 0.9. Similarly, figures – (4.5a) to (4.5f) display the behavior of the each compartment of the fractional Chikungunya virus system (4.22) with latenty infected cells response for 30 days, when Chikungunya virus transmission rate is 0.1 and figures – (4.6a) to (4.6f) are for the same when Chikungunya virus transmission rate is 0.9.

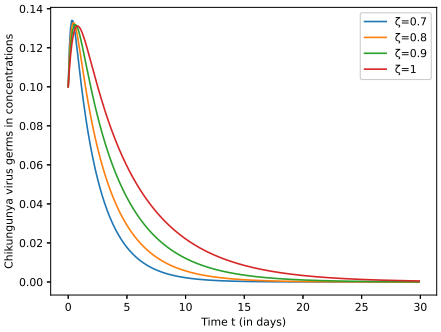
Figures – (4.7) and (4.8) represent the behavior of the system when susceptible cells get infected due to the Chikungunya virus with different transmission rate. Figure – (4.9) is a plot of the system when antibodies respond against Chikungunya virus and figure – (4.10) is a graph of the system when CTL cells act as a defence system against Chikungunya virus.



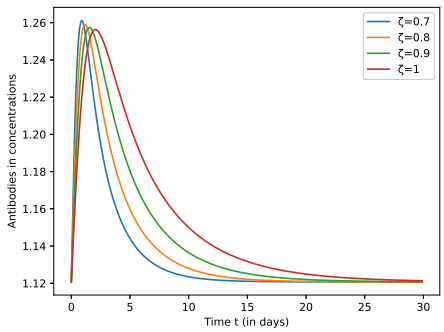
(a) Simulation for concentrations of susceptible cells at some fractional-order ζ.



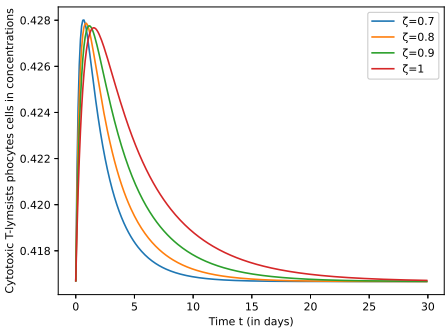
(b) Simulation for concentrations of infected cells at some fractional-order ζ.



(c) Simulation for concentrations of chikungunya virus germs at some fractional-order ζ.

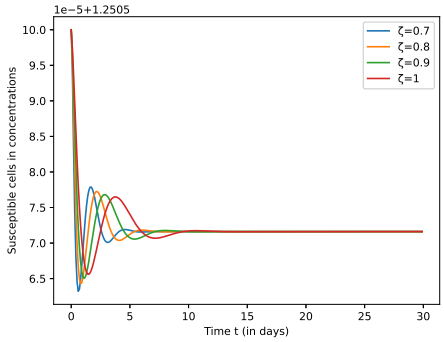


(d) Simulation for concentrations of antibodies at some fractional-order ζ.

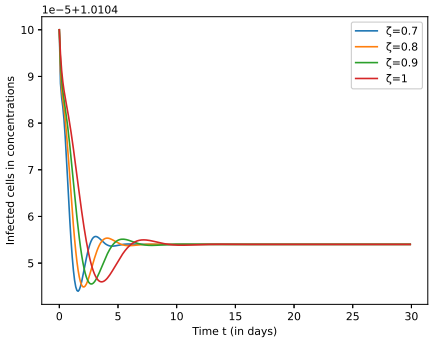


(e) Simulation for concentrations of Cytotoxic T-lymsists phocytes cells at some fractional-order ζ.

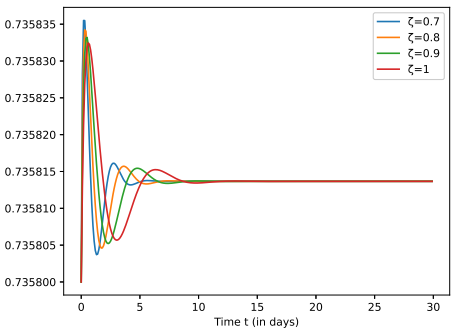
Figure 4.3: Behavior of within-host fractional chikungunya virus model with adaptive immune response when $b = 0.1$.



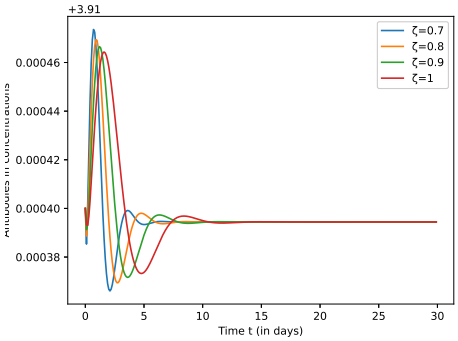
(a) Simulation for concentrations of susceptible cells at some fractional-order ζ.



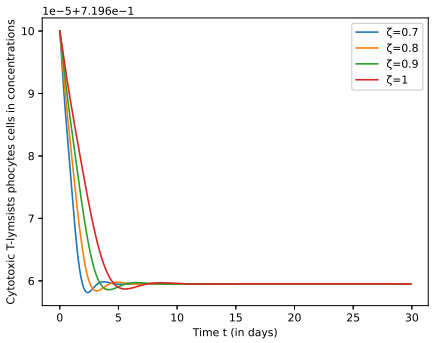
(b) Simulation for concentrations of infected cells at some fractional-order ζ.



(c) Simulation for concentrations of chikungunya virus germs at some fractional-order ζ.

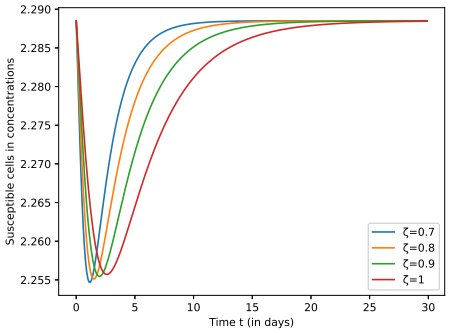


(d) Simulation for concentrations of antibodies at some fractional-order ζ.

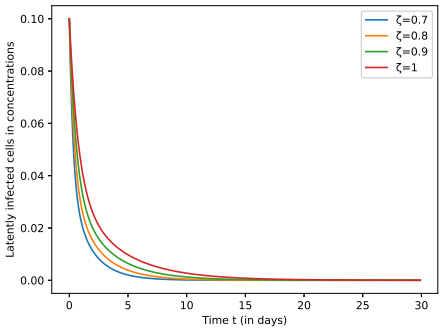


(e) Simulation for concentrations of Cytotoxic T-lymsists phocytes cells at some fractional-order ζ.

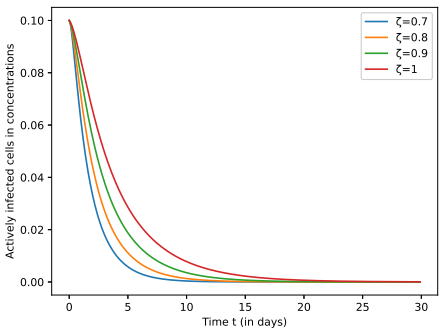
Figure 4.4: Behavior of within-host fractional chikungunya virus model with adaptive immune response for when $b = 0.9$.



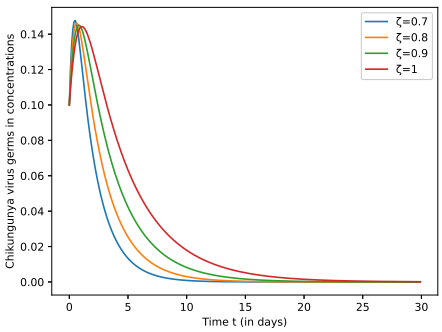
(a) Simulation for concentrations of susceptible cells at some fractional-order ζ.



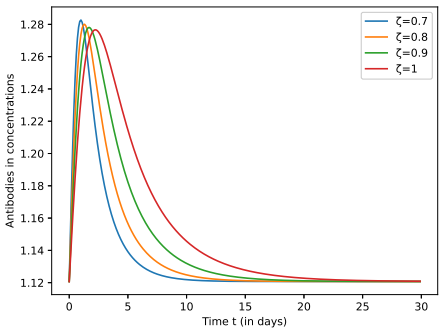
(b) Simulation for concentrations of latently infected cells at some fractional-order ζ.



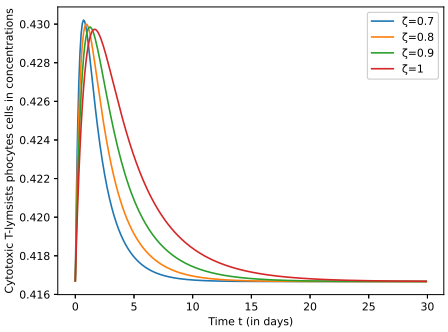
(c) Simulation for concentrations of actively infected cells at some fractional-order ζ.



(d) Simulation for concentrations of chikungunya virus germs at some fractional-order ζ.

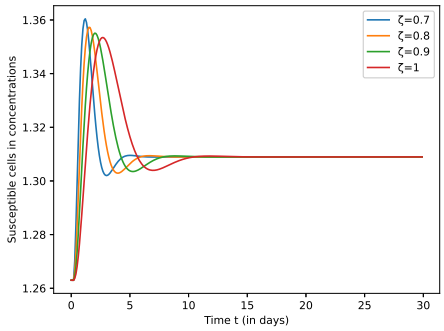


(e) Simulation for concentrations of antibodies at some fractional-order ζ.

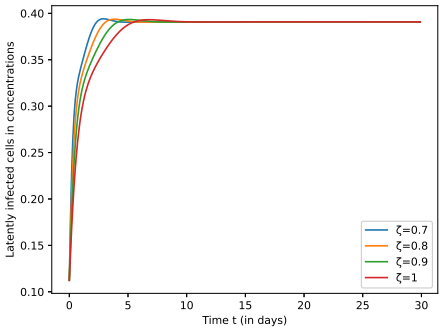


(f) Simulation for concentrations of Cytotoxic T-lymsists phocytes cells at some fractional-order ζ.

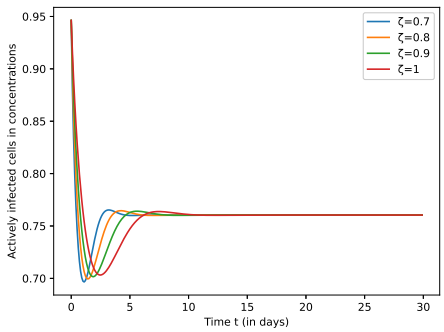
Figure 4.5: Behavior of within-host fractional chikungunya virus model with latency when $b = 0.1$.



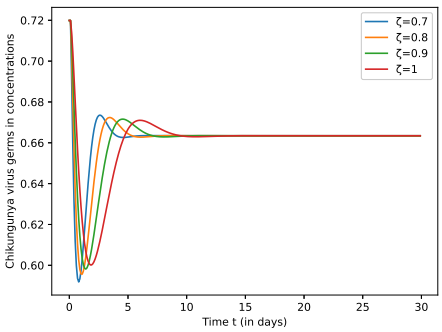
(a) Simulation for concentrations of susceptible cells at some fractional-order ζ.



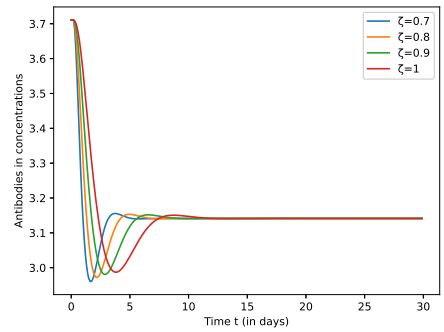
(b) Simulation for concentrations of latently infected cells at some fractional-order ζ.



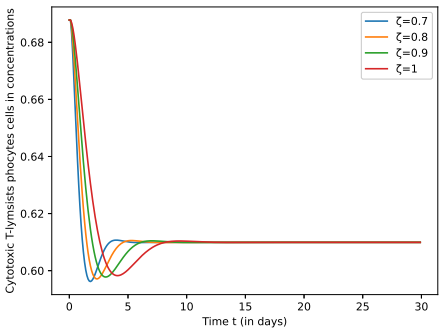
(c) Simulation for concentrations of actively infected cells at some fractional-order ζ.



(d) Simulation for concentrations of chikungunya virus germs at some fractional-order ζ.



(e) Simulation for concentrations of antibodies at some fractional-order ζ.



(f) Simulation for concentrations of Cytotoxic T-lymsists phocytes cells at some fractional-order ζ.

Figure 4.6: Behavior of within-host fractional chikungunya virus model with latency when $b = 0.9$.

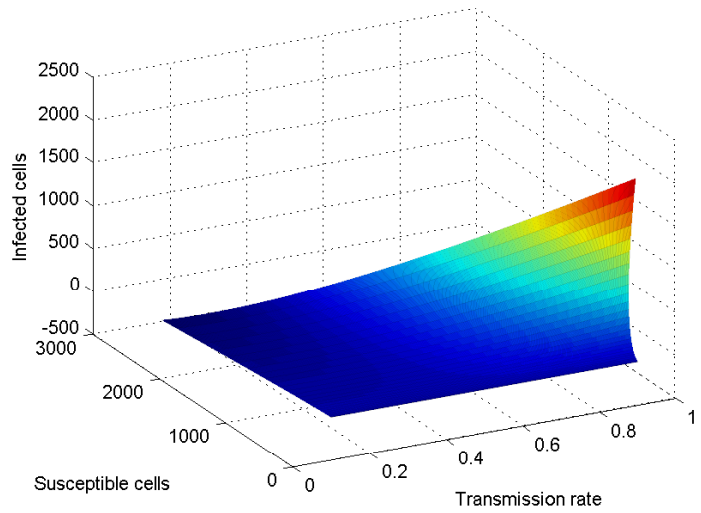


Figure 4.7: When the Chikungunya virus transmission rate b is between 0.1 and 1, susceptible cells get infected in the concentrations.

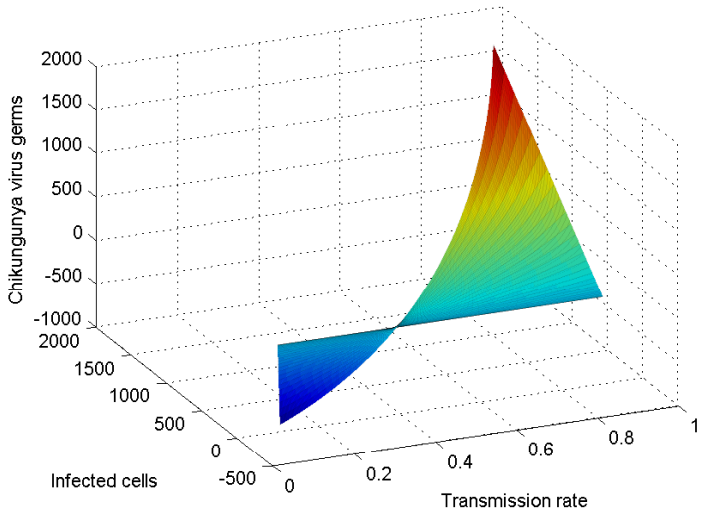


Figure 4.8: When the Chikungunya virus transmission rate b is between 0.1 and 1, Uninfected target cells get infected due to virus in the concentrations.

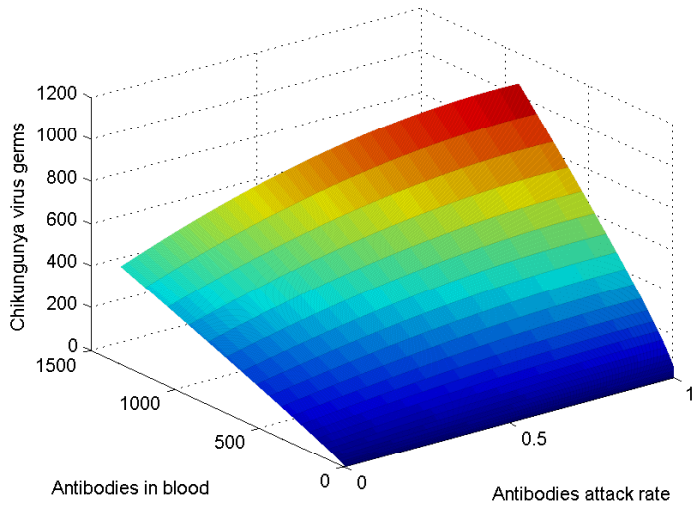


Figure 4.9: Antibodies in human body respond against Chikungunya virus when the antibodies attack rate q is between 0 to 1.

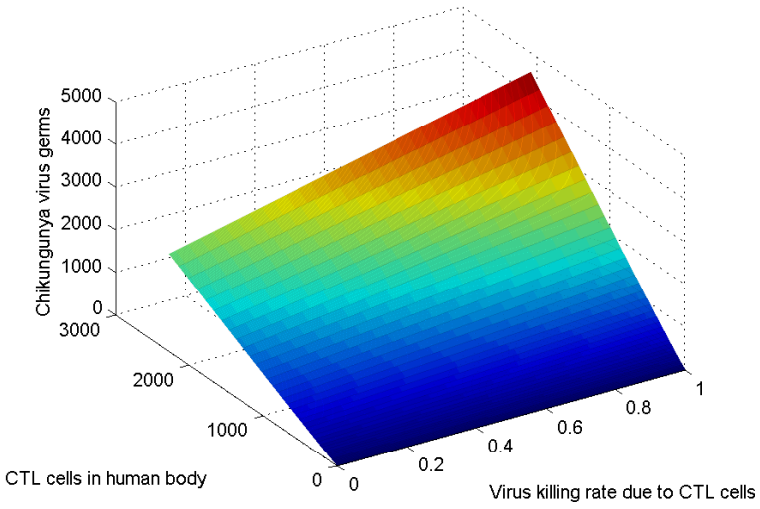


Figure 4.10: Cytotoxic T-lymphocytes (CTL) cells act as a defence system against Chikungunya virus when the virus killing rate due to CTL cells ϵ is between 0 to 1.

4.7 Conclusion

We introduced a fractional-order Chikungunya virus model with adaptive immune response and with latently infected cells in human body. Atangana and Owolabi's numerical scheme has simple and straightforward approach to develop python code which provides the numerical and graphical computations of presented fractional model. We demonstrated the existence and uniqueness of both fractional Chikugunya virus models using the fixed point theorems of Banach and Schaefer. We estimated disease-free and endemic equilibrium point for the suggested system and analyzed global stability in the view of the basic reproduction number and Lyapunov function. Each of these figures demonstrates the smooth convergence of the fraction-order $\zeta = 0.7$ to 1 which indicates the stable evolution of the system. It is observed that the number of susceptible cells in concentrations are very high initially while the number of infected cells reduce within the first 10 days of Chikungunya virus infection. The trajectories in figure (4.3d) illustrate a gradual increase in the recovery process of infected humans with Chikungunya germs. The growth of the Chikungunya virus is initially grow rapidly for the first 10 days, but then it is in control rapidly, as evidenced in figure (4.3e). Consequently, the contamination in blood rises while the transmission rate of infection decreases significantly after 10 days, as illustrated in figures (4.4e), (4.5f) and (4.6f). Hence by understanding the key drivers of Chikungunya virus transmission, policy-makers can develop targeted strategies to mitigate the impact of the virus on global health.