Chapter 2

Eigenvalue Estimates for Fractional Sturm-Liouville Boundary Value Problem

2.1 Introduction

The Lyapunov inequality [85] has proved to be very useful in the study of spectral properties and oscillation theory of ordinary differential equations. This inequality can be stated as follows [23]:
The nontrivial solution to the boundary value problem
$u''(t) + q(t)u(t) = 0, a < t < b$, $u(a) = u(b) = 0$, exists, where $q : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)|ds > \frac{4}{b-a}.$$

The research on Lyapunov-Type Inequalities (LTIs) for Fractional Boundary Value Problems (FBVPs) has begun since 2013. In [45, 46, 47, 48, 63, 64, 99, 100, 101, 118], the authors have established LTIs for FBVPs of order α with different boundary conditions. In [100], Pathak obtained LTI for fractional boundary value problem with Hilfer derivative of order α, $1 < \alpha \leq 2$. Furthermore, the author applied LTI to obtain the lower bound for the smallest eigenvalue of corresponding eigenvalue problem. In addition, the Cauchy-Schwarz type inequality (CSI) is established to improve the lower bound estimation of the smallest eigenvalue and applied it to obtain intervals where certain Mittag-Leffler (M-L) function has no real zeros. The CSI provides better results than that of LTI. Motivated by the above work, we consider the following problem with Sturm-Liouville boundary conditions [64]:

$$ {}_a^C D_t^\alpha u(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha < 2 \tag{2.1}$$

$$pu(a) - ru'(a) = u(b) = 0, \tag{2.2}$$

where $p > 0$, $r \geq 0$ and $q : [a, b] \to \mathbb{R}$ is a continuous function. In [64], Jleli and Samet established a Lyapunov-type inequality for FBVP (2.1)-(2.2) as follows:

For $\dfrac{r}{p} > \dfrac{b-a}{\alpha-1}$

$$\int_a^b |q(s)|ds \geq \Big(1 + \frac{p}{r}(b-a)\Big)\frac{\Gamma\alpha}{(b-a)^{\alpha-1}} \tag{2.3}$$

and for $0 \leq \dfrac{r}{p} \leq \dfrac{b-a}{\alpha-1}$

$$\int_a^b |q(s)|ds \geq \frac{\Gamma\alpha}{max\{A(\alpha, \frac{r}{p}), B(\alpha, \frac{r}{p})\}}. \tag{2.4}$$

We establish CSI for FBVP (2.1)-(2.2). The outline of the chapter is as follows: first, we provide some preliminaries in section 2.2 which we will use in this paper. In section 2.3, we establish CSI for fractional Strum-Liouville boundary value problem containing Caputo derivative of order α, $1 < \alpha \leq 2$. We also give a comparison between the lower bound estimates of the smallest eigenvalue obtained from the LTI and CSI. In section 2.4, we use these inequalities to obtain an interval where a linear combination of certain Mittag- Leffler functions have no real zeros. Finally, a conclusion is given in section 2.5.

2.2 Preliminaries

In this section, we recall some basic definitions which are further used in this paper.

Definition 2.2.1. *The Caputo derivative of fractional order $\alpha > 0$ is defined by*

$${}_a^C D_t^\alpha\,(f(t)) = \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-s)^{m-\alpha-1} f^m(s)ds, t \in [a, b],$$

where m is the smallest integer greater of equal to α.

Definition 2.2.2. *The two-parameter M-L function is defined by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ (\alpha, \beta, z \in \mathbb{R}; \alpha, \beta > 0).$$

Definition 2.2.3. *The Pfaff Transformation is defined as*

$$ {}_2F_1(a,b;c;t) = (1-t)^{-a}{}_2F_1\Big(a, c-b; c; \frac{t}{t-1}\Big); |t| < \frac{1}{2}, $$

where, ${}_2F_1(a,b;c;t)$ *is a hypergeometric function.*

For more details, refer [79] and [106].

2.3 Main result

The main result is given in Theorem 2.3.1.

Lemma 2.3.1. *The FBVP* (2.1)-(2.2) *can be written in its equivalent integral form as [64]*

$$ u(t) = \int_a^b G(t,s)q(s)u(s)ds, \quad t \in [a,b], \tag{2.5} $$

where G is the Green's function given by

$$ G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{\Big(\dfrac{r}{p}+t-a\Big)(b-s)^{\alpha-1}}{\Big(\dfrac{r}{p}+b-a\Big)} - (t-s)^{\alpha-1}, a \le s \le t \le b, \\ \\ \dfrac{\Big(\dfrac{r}{p}+t-a\Big)(b-s)^{\alpha-1}}{\Big(\dfrac{r}{p}+b-a\Big)}, a \le t \le s \le b. \end{cases} \tag{2.6} $$

Lemma 2.3.2. *[100] Let* $u \in L^2[a,b]$*, then the Cauchy-Schwarz-type inequality of FBVP* (2.1)-(2.2) *is given by*

$$ 1 \le \Big\{ \int_a^b \int_a^b |G(t,s)q(s)|^2 dsdt \Big\}. \tag{2.7} $$

Proof. Taking the Cauchy-Scharz inequality in (2.5) we get,

$$|u(t)| \leq \left[\int_a^b |G(t,s)q(s)|^2 ds\right]^{\frac{1}{2}} \left[\int_a^b |u(s)|^2 ds\right]^{\frac{1}{2}}.$$

Squaring and integrating from a to b w.r.to. t gives

$$\int_a^b |u(t)|^2 dt \leq \int_a^b \left\{ \left[\int_a^b |G(t,s)q(s)|^2 ds\right] \left[\int_a^b |u(s)|^2 ds\right] \right\} dt$$

$$||u||_2 \leq \int_a^b \int_a^b |G(t,s)q(s)|^2 ds dt ||u||_2,$$

which proves the Lemma. □

Now, we consider Fractional Sturm-Liouville eigen value problem (FEP):

$$\begin{cases} {}_a^C D_t^\alpha \left(u(t)\right) + \lambda u(t) = 0, \; a < t < b, 1 < \alpha < 2 \\ pu(a) - ru'(a) = u(b) = 0. \end{cases} \tag{2.8}$$

We are ready to state and prove our main results.

Theorem 2.3.1. *If a nontrivial continuous solution of the problem* (2.8) *exists, then for FEP* (2.8) *the CSI is*

$$\lambda \geqslant \frac{1}{(\Gamma(\alpha))} \left\{ \frac{1}{(2\alpha-1)(\frac{r}{p}+b-a)^2} \left[\left(\frac{r}{p}-a\right)^2 (b-a) + \left(\frac{r}{p}-a\right)(b^2-a^2) + \frac{(b^3-a^3)}{3} \right] + \frac{(b-a)^{2\alpha}}{2\alpha(2\alpha-1)} - \frac{2\beta(1,\alpha)(b-a)^\alpha}{\left(\frac{r}{p}+b-a\right)} \int_a^b \left(\frac{r}{p}-a+t\right)(t-a)^\alpha \frac{{}_2F_1\left(1, 2\alpha, 1+\alpha, \frac{a-t}{b-t}\right)}{(b-t)} dt \right\}^{-\frac{1}{2}}, \tag{2.9}$$

where $\beta(m,n)$ is a Beta function.

Proof. Taking $q(t) = \lambda$ in (2.7) gives the inequality

$$\lambda \geq \left[\int_a^b \int_a^b |G(t,s)|^2 ds dt \right]^{-\frac{1}{2}}. \tag{2.10}$$

By substituting equation (2.6) in (2.10), after some simplifications we obtain (2.9), which concludes the proof. □

We consider following two cases.

Case 1: Taking $a = 0, b = 1, p = 1$ and $r = 2$ in (2.8), we get the following FEP:

$${}_0^C D_t^{\alpha} (u(t)) + \lambda u(t) = 0, 0 < t < 1, 1 < \alpha < 2 \tag{2.11}$$

$$u(0) - 2u'(0) = u(1) = 0. \tag{2.12}$$

Case 2: Taking $a = 0, b = 1, p = 2$ and $r = 1$ in (2.8), gives the eigenvalue problem:

$${}_0^C D_t^{\alpha} (u(t)) + \lambda u(t) = 0, 0 < t < 1, 1 < \alpha < 2 \tag{2.13}$$

$$2u(0) - u'(0) = u(1) = 0. \tag{2.14}$$

Next, we give three methods to estimate the lower bound for the smallest eigenvalue of problems (2.11)-(2.12) and (2.13)-(2.14) by using the following definitions given in [100].

Definition 2.3.1. *A Lyapunov-Type Inequality Lower Bound (LTILB) is defined as a lower bound estimate for the smallest eigenvalue obtained from Lyapunov-type inequalities given by* (2.3) *and* (2.4).

We obtain a lower bound for the smallest eigenvalue of problem (2.11) with boundary conditions (2.12) is:

$$\lambda \geqslant \frac{3}{2}\Gamma(\alpha), \tag{2.15}$$

and for the problem (2.13)-(2.14) it is:

$$\lambda \geq \frac{\Gamma\alpha}{max\{A(\alpha, \frac{1}{2}), B(\alpha, \frac{1}{2})\}}. \tag{2.16}$$

Definition 2.3.2. *A Cauchy-Schwarz Inequality Lower Bound (CSILB) is defined as an estimate of the lower bound for the smallest eigenvalue obtained from the Cauchy-Schwarz inequality of type given in equation* (2.9)*.*

We obtain the CSIs of problems (2.11)-(2.12) and (2.13) -(2.14), after some simplifications and using Pfaff transformation in (2.9) respectively as follows :

$$\lambda \geqslant \frac{1}{\Gamma(\alpha)}\left\{\frac{1}{(2\alpha-1)}\left[\frac{19}{27}+\frac{1}{2\alpha}\right]-\frac{2}{3}\int_0^1 (2+t)t^{\alpha}\beta(1,\alpha)\ {}_2F_1(1-\alpha,1;\alpha+1,t)dt\right\}^{-\frac{1}{2}}, \tag{2.17}$$

$$\lambda \geqslant \frac{1}{\Gamma(\alpha)}\left\{\frac{1}{(2\alpha-1)}\left[\frac{13}{27}+\frac{1}{2\alpha}\right]-\frac{4}{3}\int_0^1 \left(\frac{1}{2}+t\right)t^{\alpha}\beta(1,\alpha)\ {}_2F_1(1-\alpha,1;\alpha+1,t)dt\right\}^{-\frac{1}{2}}, \tag{2.18}$$

provided $\alpha > \frac{1}{2}$.

In [38], eigenvalues $\lambda \in \mathbb{R}$ of problems (2.11)-(2.12) and (2.13)-(2.14) are the solutions of the linear combination of certain M-L functions are respectively as follows:

$$2E_{\alpha,1}(-\lambda) + E_{\alpha,2}(-\lambda) = 0, \tag{2.19}$$

$$E_{\alpha,1}(-\lambda) + 2E_{\alpha,2}(-\lambda) = 0. \tag{2.20}$$

Now, comparing the non-zero solutions of equations (2.19)-(2.20) for $1.5 < \alpha \leq 2$ with CSILB given by equations (2.17)-(2.18) and LTILB given by the equations (2.15)-(2.16) respectively, we get the following comparison figures.

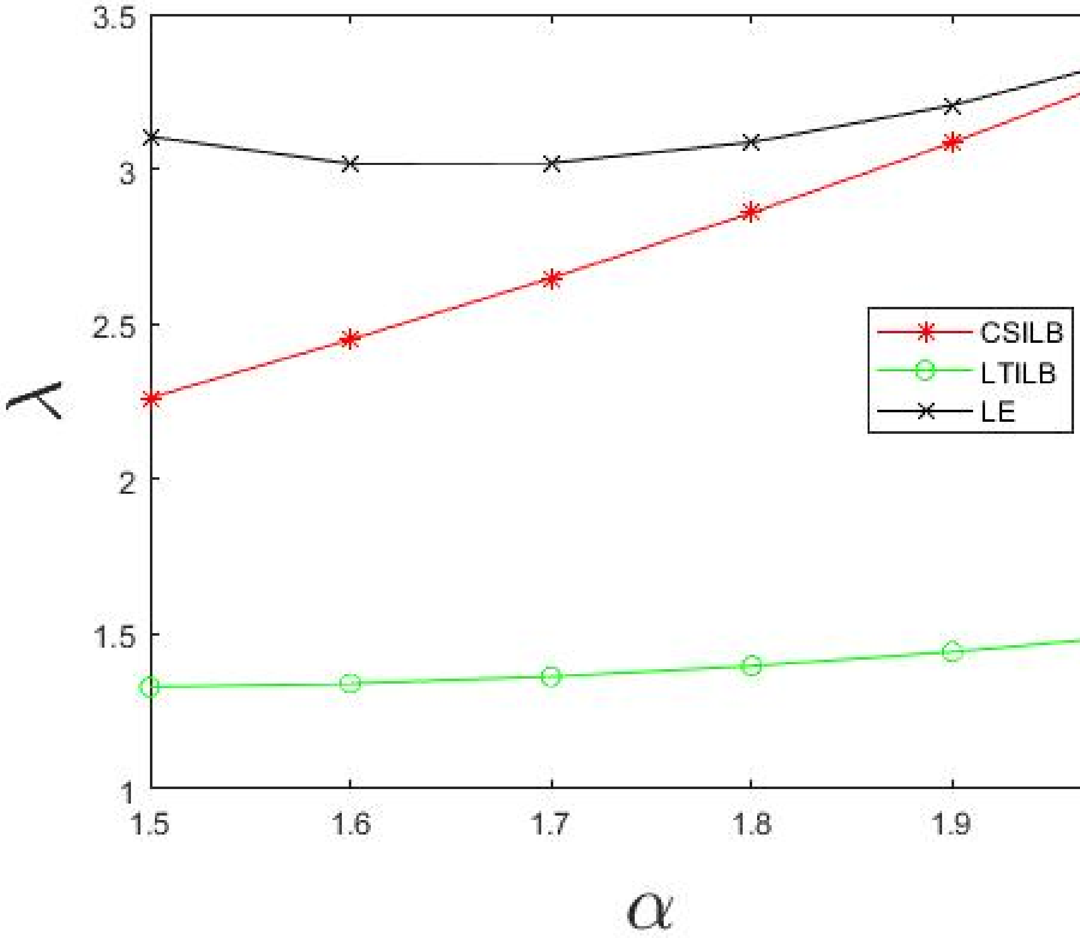

Figure 2.1: Comparison of the lower bounds for λ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue for (2.11)-(2.12). ($- \circ -$: LTILB; $- * -$: CSILB; $- \times -$:LE - the Lowest Eigenvalue λ))

These figures clearly demonstrates that between the two estimates considered here, the LTILB provides the worse estimate and the CSILB provide better estimate for the smallest eigenvalues of (2.11)-(2.12) as in figure 2.1 and (2.13)-(2.14) as in figure 2.2.

We consider the integer order case, i.e. $\alpha = 2$. For this case, the LTILB and CSILB for the smallest λ of (2.11)-(2.12) are given as 1.5 and 3.3310 and for (2.13)-(2.14), 2.6667 and 5.1117 respectively. (See equations (2.15), (2.16), (2.17) and (2.18)). For $\alpha = 2$, the problems (2.11)-(2.12) and (2.13)-(2.14) can be solved in closed form using the tools from integer order

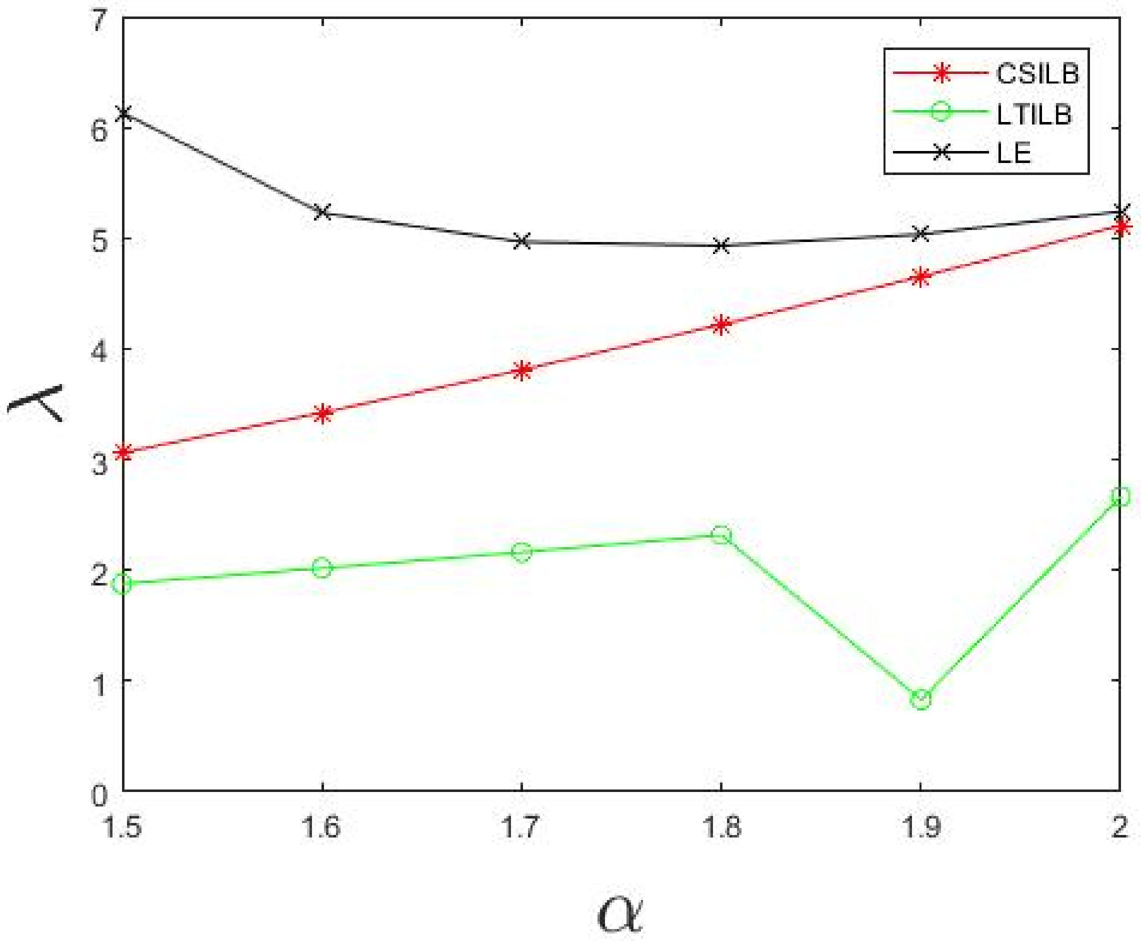

Figure 2.2: Comparison of the lower bounds for λ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue for (2.13)-(2.14). ($-\circ-$: LTILB; $-*-$: CSILB; $-\times-$:LE - the Lowest Eigenvalue λ))

calculus. Results show, the smallest eigenvalues of (2.11)-(2.12) and (2.13)-(2.14) are the roots of equations (2.19) and (2.20) respectively, which give the smallest eigenvalues as 3.3731 and 5.2392. Comparing these λ with its estimate above, it is clear that between LTILB and CSILB for the integer α the CSILB provides the best estimate for the smallest eigenvalue.

2.4 Applications

We now consider an application of the lower bounds for the smallest eigenvalues of FEPs (2.11)-(2.12) and (2.13)-(2.14) found in equations (2.15)-(2.20).

Theorem 2.4.1. *Let* $1.5 < \alpha \leq 2$. *The linear combination of certain Mittag-Leffler functions* $2E_{\alpha,1}(-z) + E_{\alpha,2}(-z)$ *have no real zeros in the following domains:*
LTILB:

$$z \in \Big(-\frac{3}{2}\Gamma(\alpha), 0\Big] \tag{2.21}$$

CSILB:

$$z \in \Bigg(-\frac{1}{\Gamma(\alpha)}\Big\{\frac{1}{(2\alpha-1)}\Big[\frac{19}{27}+\frac{1}{2\alpha}\Big]-\frac{2}{3}C_1(\alpha)\Big\}^{-\frac{1}{2}}, 0\Bigg], \tag{2.22}$$

where, $C_1(\alpha) = \int_0^1 (2+t)t^{\alpha}\beta(1,\alpha)\ {}_2F_1(1-\alpha, 1; \alpha+1, t)dt.$

Proof. Let λ be the smallest eigenvalue of the equation (2.12), then $z = \lambda$ is the smallest value for which $2E_{\alpha,1}(-z) + E_{\alpha,2}(-z) = 0$. If there is another z smaller than λ for which $2E_{\alpha,1}(-z)+E_{\alpha,2}(-z) = 0$,then it will contradict that λ is the smallest eigenvalue. Therefore, $2E_{\alpha,1}(-z) + E_{\alpha,2}(-z)$ have no real zeros for $z \in (-\lambda, 0]$. Thus, $2E_{\alpha,1}(-z) + E_{\alpha,2}(-z)$ have no real zeros for

$$z \in \Bigg(-\frac{1}{\Gamma(\alpha)}\Big\{\frac{1}{(2\alpha-1)}\Big[\frac{19}{27}+\frac{1}{2\alpha}\Big]-\frac{2}{3}C_1(\alpha)\Big\}^{-\frac{1}{2}}, 0\Bigg].$$

This proves (2.22). Proof of (2.21) is given in [64]. □

Theorem 2.4.2. *Let* $1.5 < \alpha \leq 2$. *The linear combination of certain Mittag- Leffler functions* $E_{\alpha,1}(-z) + 2E_{\alpha,2}(-z)$ *have no real zeros in the following domains:*
LTILB:

$$z \in \Bigg(-\frac{\Gamma\alpha}{max\{A(\alpha,\frac{1}{2}), B(\alpha,\frac{1}{2})\}}, 0\Bigg]. \tag{2.23}$$

CSILB:

$$z \in \left(-\frac{1}{\Gamma(\alpha)}\left\{\frac{1}{(2\alpha-1)}\left[\frac{13}{27}+\frac{1}{2\alpha}\right]-\frac{4}{3}C_2(\alpha)\right\}^{-\frac{1}{2}}, 0\right], \tag{2.24}$$

where, $C_2(\alpha) = \int_0^1 \left(\frac{1}{2}+t\right)t^{\alpha}\beta(1,\alpha)\, {}_2F_1(1-\alpha, 1; \alpha+1, t)dt.$

Proof. The proof is similar to the proof of Theorem 2.4.1 . □

2.5 Conclusion

We established Cauchy-Schwarz-type inequality for fractional Strum-Liouville boundary value problem containing Caputo derivative of order α, $1 < \alpha \leq 2$ to determine a lower bound for the smallest eigenvalues. We give a comparison between the smallest eigenvalues and its lower bounds obtained from the Lyapunov-type and Cauchy-Schwarz-type inequalities.The results indicate that the Cauchy-Schwarz-type inequality gives better lower bound estimates for the smallest eigenvalues than the Lyapunov-type inequality. We then used these inequalities to obtain an interval where a linear combination of certain Mittag- Leffler functions have no real zeros.