Chapter 3

Mathematical Control Theory

Contents

In this chapter, we lay the groundwork for understanding control theory by revisiting fundamental concepts and results from mathematical analysis, differential equations and linear algebra. These mathematical tools are essential for analyzing and designing control systems, particularly in assessing the controllability of both linear and nonlinear systems.

Control theory is a discipline focused on guiding the behavior of systems through external inputs to achieve specific goals. To effectively navigate this field, a solid grasp of mathematical principles is indispensable. Differential equations provide the language to describe the dynamic evolution of systems, while linear algebra offers techniques for representing and manipulating system dynamics.

Central to our discussion is the concept of controllability, which refers to the ability to steer a system from one state to another using control inputs. We explore criteria and methodologies for assessing controllability, drawing on mathematical insights to understand system behavior and control capabilities thoroughly.

Throughout this chapter, we integrate key findings from mathematical analysis, differential equations, and linear algebra to establish a robust theoretical framework for controllability analysis. By synthesizing these concepts, readers will gain the necessary tools to analyze and design control systems effectively, empowering them to address real-world challenges across various domains.

3.1 Control Theory

Kalman [30] introduced the concept of controllability for the linear system (3.1.1) and was subsequently extended to nonlinear systems, dominated by controllable linear parts, by Davison and Kunze [15], Mirza and Womack [38], Quinn and Carmichael [44] etc., by using the techniques of fixed point theory.

Our study found that the linear part's characteristics had a greater impact on the nonlinear system's controllability. Hence initially, we consider a finite-dimensional linear system represented by the differential equation

$$\left.\begin{aligned} &x'(t) = A(t)x(t) + B(t)u(t),\ 0 \leq t_0 < t \leq t_1 < \infty \\ &x(t_0) = x_0 \end{aligned}\right\}, \tag{3.1.1}$$

where, for each $t \in [t_0, t_1], x(t) \in \mathbb{R}^n$ is called the state of the system, $u(t) \in L^2([t_0, t_1], \mathbb{R}^m)$ is called the control vector; $A(t), B(t)$ are matrices of dimensions $n \times n$ and $n \times m$, respectively. Assume that the elements $a_{ik}(t)$ of $A(t)(i, k = 1, 2, ..., n)$ are absolutely integrable functions of $t \in [t_0, t_1]$ and elements $b_{il}(t)$ of

$B(t)(i = 1, 2, ..., n; l = 1, 2, ..., m)$ are piecewise continuous functions of $t \in [t_0, t_1]$.

Consider the homogeneous linear system

$$\left.\begin{array}{l} \frac{dx(t)}{dt} = A(t)x(t) \;\; t_0 \leq t \leq t_1 \\ x(t_0) = x_0 \end{array}\right\}, \tag{3.1.2}$$

where, the state $x(t) \in \mathbb{R}^n$, $A(t)$ is matrix of order $n \times n$. It follows easily that the equation (3.1.2) has a unique solution $x(\cdot)$ passing through the initial condition $x(t_0) = x_0$.

Proposition 3.1.1. *If $\phi_1(t), \phi_2(t), ..., \phi_n(t)$ are solutions of (3.1.2) with initial conditions $x_1, x_2, ..., x_n$ then their linear combination*

$$\phi(t) = \sum_{i=1}^{n} \alpha_i \phi_i(t), \qquad \alpha_i \in \mathbb{R},$$

is also a solution of (3.1.2) with initial condition

$$\phi(t_0) = \sum_{i=1}^{n} \alpha_i x_i.$$

Proposition 3.1.2. *Let $\phi_1(t), \phi_2(t), ..., \phi_m(t)$ be solution of (3.1.2) on $[t_0, t_1]$ and $s \in [t_0, t_1]$, then*

$$\{\phi_1(\cdot), \phi_2(\cdot), ..., \phi_m(\cdot), \}$$

are linearly independent solutions if and only if

$$\{\phi_1(s), \phi_2(s), ..., \phi_m(s)\},$$

is linearly independent set of vectors in $\mathbb{R}^n$.

An $n \times n$ matrix function $\Phi(t, t_0)$ is said to be transition matrix of homogeneous equation(3.1.2) if it satisfies the following:

$$\begin{aligned} \frac{d}{dt}\Phi(t,t_0) &= A(t)\Phi(t,t_0), \\ \Phi(t_0,t_0) &= I. \end{aligned}$$

Example:2.1.1 The matrix function given by

$$\Phi(t,t_0) = \begin{bmatrix} cos(t-t_0) & -sin(t-t_0) \\ sin(t-t_0) & cos(t-t_0) \end{bmatrix},$$

is the transition matrix for the system

$$\frac{d}{dt}x(t) = Ax(t),$$

$$x(t_0) = x_0,$$

where,

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

A is called the generator of the transition matrix $\Phi(t,t_0)$. The transition matrix has the following properties:

1. $\Phi(t,t) = I$, identity matrix of order n, for all $t \in [t_0, t_1]$
2. $\Phi(t,t_0) = \Phi(t,\tau)\Phi(\tau,t_0), \quad t \leq \tau \leq t_0$
3. $\frac{\partial}{\partial t}(\Phi(t,\tau)) = A(t)\Phi(t,\tau)$
4. If $\Phi_1(t), \Phi_2(t), \Phi_3(t)$ ...., $\Phi_n(t)$ are linearly independent solutions of the homogeneous system corresponding to (3.1.1) and $\Phi(t)$ is the matrix whose

columns are $\Phi_1(t)$, $\Phi_2(t)$, $\Phi_3(t)$, ..., $\Phi_n(t)$, then it can be shown easily that the transition matrix $\Phi(t,\tau)$ satisfies $\Phi(t,\tau) = \Phi(t)[\Phi(\tau)]^{-1}$.

Using the variation of the constant formula and Theorem 1 of Brockett [9], we have the following theorem concerning the solution of system (3.1.1).

Theorem 3.1.1. *The sequence of matrices M_k defined recursively by*

$$M_0 = I, \quad M_k = I + \int_{t_0}^{t} A(\tau)M_{k-1}(\tau)d\tau,$$

converges uniformly on $[t_0, t_1]$. Moreover, if Φ denotes the limit function of $M_k(t,t_0)$ then

$$\frac{d}{dt}\Phi(t,t_0) = A(t)\Phi(t,t_0) \ \textit{and} \ \ \Phi(t_0,t_0) = I,$$

and the solution of (3.1.1) which passes through x_0 at $t = t_0$ is given by

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau.$$

□

From Theorem 3.1.1, it follows that the explicit expression for $\Phi(t,t_0)$ is given by the Peano-Baker series (refer Brockett [9])

$$\Phi(t,t_0) = I + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2\, d\tau_1 + ..., \tag{3.1.3}$$

If A is a real constant $n \times n$ matrix, then the Peano-Baker series (3.1.3) becomes

$$\Phi(t,t_0) = I + A(t-t_0) + \frac{A^2(t-t_0)}{2!} + ... = e^{A(t-t_0)}.$$

A variety of definitions are available for controllability in the literature. We define controllability as follows (refer Russel [47]).

Definition 3.1.1. *The system (3.1.1) is said to be* ***controllable*** *over* $[t_0, t_1]$ *if for each pair of vectors* $x_0, x_1 \in \mathbb{R}^n$ *there exists a control* $u \in L^2([t_0, t_1], \mathbb{R}^m)$ *such that the solution of (3.1.1) with* $x(t_0) = x_0$ *also satisfies* $x(t_1) = x_1$*. That is,*

$$x_1 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau.$$

The above definition of controllability is referred to as a definition of complete controllability.

Remark 3.1.1. *The control* u *which steers* x_0 *to* x_1 *need not be unique and in general it depends on* x_0 *and* x_1*. The controllability is defined above as global controllability. If* x_0 *and* x_1 *are required only to belong to* $D \subset \mathbb{R}^n$*, then the resulting controllability is said to be local controllability.*

Let $\mathcal{C} : L^2(J, \mathbb{R}^m) \to \mathbb{R}^n$, J is the time interval of the form $[t_0, t_1]$ or $[0, T]$, be a **control operator** defined by

$$\mathcal{C}u = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau. \tag{3.1.4}$$

The following statements regarding the linear system (3.1.1) are equivalent:

1. The system (3.1.1) is controllable.
2. The control operator $\mathcal{C}$ is onto.
3. The adjoint operator $\mathcal{C}^*$ of $\mathcal{C}$ is one-one.

4. The matrix $\mathcal{C}\mathcal{C}^*$ is positive definite.

The operator $\mathcal{C}\mathcal{C}^*$ is known as **controllability Grammian** and is denoted by $W(t_0, t_1)$. Thus we have

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B^*(\tau)\Phi^*(t_1, \tau)d\tau. \tag{3.1.5}$$

By Definition 3.1.1, the system (3.1.1) is globally controllable if $(x_1 - \Phi(t_1, t_0)x_0)$ lies in the Range of $\mathcal{C}$ for all $x_0, x_1 \in \mathbb{R}^n$. But $(x_1 - \Phi(t_1, t_0)x_0) \in$ Range $(\mathcal{C})$ if and only if $(x_1 - \Phi(t_1, t_0)x_0) \in$ Range $(\mathcal{C}\mathcal{C}^*)$= Range $(W(t_0, t_1))$.

When $W(t_0, t_1)$ is invertible, the control function defined by

$$u(t) = B^*(t)\Phi^*(t_1, t)W^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0], \tag{3.1.6}$$

steers the system (3.1.1) from $x(t_0) = x_0$ to $x(t_1) = x_1$.

Hence, we have the following characterization for controllability.

Theorem 3.1.2. *The linear system (3.1.1) is (globally) controllable if and only if the controllability Grammian $W(t_1, t_0)$ defined in (3.1.5) is nonsingular. That is,*

$$det \;\; W(t_0, t_1) \neq 0.$$

□

Definition 3.1.2. *A bounded linear operator $P : \mathbb{R}^n \to L^2(J, \mathbb{R}^m)$ is called a **steering operator** for (3.1.1) if for any $\alpha \in \mathbb{R}^n, u = P\alpha$ steers 0 to α.*

An $m \times n$ matrix function $P(t)$ is called a steering function, if the operator P defined by $(P\alpha)(t) = P(t)\alpha$ is a steering operator.

We observe that

1. A bounded linear operator $P : \mathbb{R}^n \to L^2(J, \mathbb{R}^m)$ is a steering operator if and only if $\mathcal{C}P = I$ and

2. An $m \times n$ matrix function $P(t)$ is a steering function if and only if

$$\int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)P(\tau)d\tau = I. \tag{3.1.7}$$

We have the following characterization regarding the controllability of (3.1.1) in terms of steering operator and steering function.

Theorem 3.1.3. *The following are equivalent:*

1. *The system (3.1.1) is controllable.*

2. *There exists a steering function for (3.1.1).*

3. *There exists a steering operator for (3.1.1).*

□

Remark 3.1.2. *If $\mathcal{C}\mathcal{C}^*$ is invertible (that is, the controllability Grammian is non-singular) then*

$$P = \mathcal{C}^*(\mathcal{C}\mathcal{C}^*)^{-1},$$

(the More-Penrose inverse of $\mathcal{C}$) is a steering operator. In this case

$$P(t) = B^*(t)\Phi^*(t_1, t)W^{-1}(t_0, t_1),$$

is a steering function. Further, $P(t)$ is an optimal steering function, (refer Russel [47]).

Now we consider the finite-dimensional nonlinear time-varying system, with control, represented by the equation

$$\left.\begin{array}{l} \frac{dx}{dt} = A(t)x + B(t)u + f(t,x), \quad 0 \le t_0 < t \le t_1 < \infty \\ x(t_0) = x_0 \end{array}\right\}, \tag{3.1.8}$$

where $A(t)$ and $B(t)$ are as in (3.1.1) and $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function satisfying Caratheodory conditions, that is, f is measurable with respect to t for all x and continuous with respect to x for almost all $t \in [t_0, t_1]$ (Joshi and Bose [29]). All quantities in (3.1.8) are assumed to be real.

A solution of (3.1.8) is an absolutely continuous function in $L^2([t_0, t_1], \mathbb{R}^n)$ which satisfies (3.1.8) almost everywhere. A solution $x(t)$ exists for (3.1.8) if and only if $x(t)$ satisfies the integral equation

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau + \int_{t_0}^{t} \Phi(t, \tau)f(\tau, x(\tau))d\tau,$$

where $\Phi(t, \tau)$ is the transition matrix of the homogeneous linear part. We shall be interested in the global controllability of (3.1.8).

There exists a control u which steers the initial x_0 at time $t = t_0$ to the given final state x_1 at time $t = t_1$ if and only if there exists a solution x of (3.1.8) satisfying

$$x_1 = x(t_1) = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau + \int_{t_0}^{t_1} \Phi(t_1, \tau)f(\tau, x(\tau))d\tau, \tag{3.1.9}$$

That is

$$\left[x_1 - \Phi(t_1, t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1, \tau)f(\tau, x(\tau))d\tau\right] = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau.$$

Suppose the linear part of (3.1.8) is controllable. Thus by Theorem 3.1.3, there exists a steering function $P(t)$ for the linear part of the system (3.1.8). If there exists x satisfying (3.1.9) then the steering control for (3.1.8) is given by (using Definition 3.1.2)

$$u(t) = P(t)\Big[x_1 - \Phi(t_1,t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1,\tau)f(\tau,x(\tau))d\tau\Big]. \qquad (3.1.10)$$

Thus, the state of the system (3.1.8) is given by

$$\begin{aligned} x(t) &= \Phi(t,t_0)x_0 + \int_{t_0}^{t} \phi(t,\tau)f(\tau,x(\tau))d\tau + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)P(\tau) \\ &\quad \Big[x_1 - \Phi(t_1,t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1,s)f(s,x(s))ds.\Big]d\tau \qquad (3.1.11) \end{aligned}$$

Suppose that (3.1.11) is solvable, then we have $x(t_0) = x_0$ and $x(t_1) = x_1$. This implies that the system (3.1.8) is controllable with the control defined by (3.1.10).

Hence, the controllability of the nonlinear system (3.1.8) is equivalent to the solvability of the coupled equations:

$$\begin{aligned} x(t) &= \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + \int_{t_0}^{t} \Phi(t,\tau)f(\tau,x(\tau))d\tau, \\ u(t) &= P(t)\Big[x_1 - \Phi(t_1,t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1,\tau)F(\tau,x(\tau))d\tau\Big]. \end{aligned}$$

Let $X_1 = L^2(J,\mathbb{R}^m), X_2 = L^2(J,\mathbb{R}^n)$. Define operators $K, N : X_2 \to X_2, H : X_1 \to$

X_2 and $L : X_2 \to X_1$ as follows

$$\begin{aligned} (Kx)(t) &= \int_{t_0}^{t} \Phi(t,\tau)x(\tau)d\tau; & (Nx)(t) &= f(t,x(t)); \\ (Hu)(t) &= \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau; & (Lx)(t) &= P(t)\int_{t_0}^{t_1} \Phi(t_1,\tau)x(\tau)d\tau. \end{aligned}$$

Clearly K, H and L are continuous linear operators and N is a nonlinear operator, called the Nemytskii operator (refer to Joshi and Bose [29]). Using these definitions, the coupled equations can be written as a coupled operator equations

$$x = Hu + KNx + w_1,$$

$$u = u_1 - LNx,$$

where, $w_1 = \Phi(t, t_0)x_0$ and $u_1 = P(t)[x_1 - \Phi(t_1, t_0)x_0]$. Thus, the controllability of the nonlinear system (3.1.8) is equivalent to the solvability of the above operator equation. Now we consider the linear infinite dimensional system in Banach space described by the equation

$$\left.\begin{aligned} &x'(t) = Ax(t) + B(t)u(t), \;\; 0 \le t_0 < t \le t_1 < \infty \\ &x(0) = x_0 \end{aligned}\right\}, \tag{3.1.12}$$

where, the state $x(t)$ takes values in a Banach space X for each $t \in J = [t_0, t_1]$, control function $u(.)$ is given in $L^2(J, U)$, a Banach space of admissible control functions, with U as a Banach space. Here A is the infinitesimal generator of a strongly continuous semigroup $S(t), t \ge 0$ in the Banach space X.

If X is a finite-dimensional space and A is a matrix then $S(t - s) = e^{A(t-s)}$ reduces to the transition matrix.

Definition 3.1.3. *A strongly continuous family* $\{S(t)\}_{t\ge 0}$ *of bounded operators in* *a Banach space X is called a* ***semigroup*** *generated by A if*

(i) $S(t+s)x = S(t)S(s)x,\ x \in X$ *and* $t, s \geq 0,$

(ii) $S(0)x = x,\ x \in X,$

(iii)$t \mapsto S(t)x$ *is continuous for* $t \geq 0, x \in X,$

(iv) $Ax = \lim_{t \to 0^+} \frac{S(t)x - x}{t} = \frac{d^+ S(t)x}{dt}|_{t=0},\ \ x \in D(A),$ *where*

$D(A) = \{x \in X : \lim_{t \to 0^+} \frac{S(t)x - x}{t} \text{exists}\}.$

Remark 3.1.3. *In Definition 3.1.3, the condition (iv) gives the generator of the semigroup in terms of an operator A (refer Yang [56]).*

Example:2.1.2 Consider the one-dimensional heat equation on $\Omega = (0,1)$

$$\left.\begin{aligned} &\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \text{ in } \Omega \times (0,T) \\ &y(x,0) = y_0(x) \text{ in } \Omega \\ &y(0,t) = 0 = y(1,t) \text{ in } (0,T). \end{aligned}\right\},$$

The above system can be associated with the evolution equation

$$\frac{dy}{dt} = Ay \ on \ H = L^2(0,1),$$

where $A : H \to H$ such that $Ay = y''$

$$\mathcal{D}(A) = \{y \in H \colon y, y' \ are \ absolutely \ continuous,\ y(0) = y(1) = 0\}$$

and H is a Hilbert space. It is easy to show that A generates semigroup $S(t), t \geq 0$

in $L^2(0,1)$ given by

$$S(t)y = \sum_{n=1}^{\infty} 2\ \exp(-n^2\pi^2 t)\sin n\pi x \int_0^1 \sin n\pi y(\tau)d\tau, \quad y \in L^2(0,1).$$

3.2 Some Results from Analysis

This section deals with some basic results from mathematical analysis and differential equations, including some definitions, lemmas and theorems which will be of frequent use in the subsequent chapters.

Let X be a real Banach space and X^* be the dual of X. Let Y be another real Banach space and F is an operator with domain in X and range in Y. Then we have the following definition:

Definition 3.2.1. *A sequence $\{x_n\}$ in X* ***converges*** *to $x_0 \in X$, if $\lim_{n\to\infty} ||x_n - x_0|| = 0$. It is denoted by $x_n \to x_0$.*

Definition 3.2.2. *A sequence $\{x_n\}$ in X* ***converges weakly*** *to $x_0 \in X$, if $f(x_n)$ converges to $f(x_0)$ in $\mathbb{R}$ for every linear functional $f \in X^*$. It is denoted by $x_n \rightharpoonup x_0$.*

Definition 3.2.3. *X is said to be* ***compact*** *or more precisely,* ***sequentially compact*** *if every sequence in X has a convergent subsequence.*

Definition 3.2.4. *A* ***relative compact*** *subspace Z of X is a subset whose closure is compact.*

Definition 3.2.5. *The operator $F : X \to Y$ is called* ***continuous*** *at x_0 if $x_n \to x_0 \Rightarrow Fx_n \to Fx_0$*

Definition 3.2.6. *F is called* ***bounded*** *if it maps every bounded sequence $\{x_n\}$ in X into bounded sequence $\{Fx_n\}$ in Y and F is called* ***compact*** *if for any bounded sequence $\{x_n\}$ in X, the sequence $\{Fx_n\}$ has a convergent subsequence in Y.*

Definition 3.2.7. *F is called **weakly compact** if for any bounded sequence $\{x_n\}$ in X, the sequence $\{Fx_n\}$ has a weakly convergent subsequence in Y.*

Definition 3.2.8. *Let Lip be the set of all operators $L : X \to X$ such that there exists a constant $\alpha > 0$ satisfying $\|Lx - Ly\| \leq \alpha\|x - y\| \; \forall x, y \in X$. For $L \in$ Lip , we define*

$$\|L\|^* = \sup_{x,y \in X; x \neq y} \frac{\|Lx - Ly\|}{\|x - y\|}$$

*If $L \in$ Lip with $\|L\|^* = \alpha$, we say that L is **Lipschitz continuous** with constant α. If $\alpha < 1$ then we say that L is a contraction.*

We note that :

1. $L, L_1 \in \text{Lip} \Rightarrow \|LL_1\|^* \leq \|L\|^*\|L_1\|^*$
2. $L \in BL(X) \Rightarrow \|L\|^* = \|L\|$

For more details refer to Dolezal [18].

The following Fixed point theorem will be used to prove the controllability results of different nonlinear systems.

Theorem 3.2.1. ***Banach contraction Principle*** *(refer Limaye [36])*
Let X be a Banach space and $F : X \to X$ be a contraction on X. Then F has precisely one fixed point, and the fixed point can be computed by the iterative scheme $x_{n+1} = Fx_n$, x_0 being any arbitrary initial guess.

Let us consider a metric d on a set $\mathcal{T}$, let $B(\mathcal{T})$ denote the set of all real-valued bounded functions on $\mathcal{T}$, that is,

$$B(\mathcal{T}) = \{x : \mathcal{T} \to \mathbb{R}, \sup_{t \in \mathcal{T}} |x(t)| < \infty\}.$$

For $x, y \in B(\mathcal{T})$, let

$$d_\infty(x, y) = \sup_{t \in \mathcal{T}} |x(t) - y(t)|.$$

Then d_∞ is a metric on $B(\mathcal{T})$, known as the sup metric. Let d be a metric on a set X. For $x \in X$ and $r > 0$, the set

$$U_d = \{y \in X : d(x, y) < r\}$$

is called the open ball about x of radius r.

Definition 3.2.9. *A subset E of X is said to be* ***bounded*** *if $E \subset U(x, r)$ for some $x \in X$ and $r > 0$.*

Definition 3.2.10. *A subset E of a metric space X is said to be* ***totally bounded*** *if for every $\epsilon > 0$, there are $x_1, x_2, ..., x_n$ in X such that*

$$E \subset U(x_1, \epsilon) \cup U(x_2, \epsilon) \cup ... \cup U(x_n, \epsilon).$$

Now, let

$$C(\mathcal{T}) = \{x \in B(\mathcal{T}) : \ x \text{ continuous on } \mathcal{T}\}$$

then we have following definitions

Definition 3.2.11. *A subset E of $C(\mathcal{T})$ is* ***uniformly bounded*** *in the sup metric if and only if there is some $\alpha > 0$ such that $|x(t)| \leq \alpha$ for all $x \in E$ and all $t \in \mathcal{T}$.*

Definition 3.2.12. *A subset E of $C(\mathcal{T})$ is said to be* ***equicontinuous*** *at $t \in T$ if for every $\epsilon > 0$, there is some $\delta > 0$ such that $|x(t) - x(s)| < \epsilon$ for all $x \in E$ and $s \in \mathcal{T}$ with $d(s, t) < \delta$, where δ may depend on t, but not $x \in E$.*

Theorem 3.2.2. ***Arzela-Ascoli*** *(refer Limaye [36])*
Let $\mathcal{T}$ be a compact metric space and $E \subset C(\mathcal{T})$. Suppose that E is bounded as well as equicontinuous at each $t \in \mathcal{T}$. Then

(a) ***Ascoli-1883***

E is uniformly bounded on $\mathcal{T}$. In fact E is totally bounded in the sup metric on $C(\mathcal{T})$.

(b) ***Arzela - 1889***

Every sequence in E contains a uniformly convergent subsequence.

More detail of the above analysis is available in (Limaye [36]).

Theorem 3.2.3. ***Schauder Fixed Point Theorem*** *(refer Choudhari [12])*
Let E be a nonempty convex closed subset of a normed linear space X and let F be a relatively compact subset of E. Then every continuous mapping of E into F has a fixed point. □

Theorem 3.2.4. *Let A be a real symmetric matrix and $\lambda_1, \lambda_2, ..., \lambda_k$ be distinct eigenvalues of A. Let $u_i \in \mathbb{R}^n$ be non-zero vector such that*

$$Au_i = \lambda_i u_i, \ \ 1 \leq i \leq k,$$

then $\{u_1, u_2, ..., u_k\}$ forms an orthonormal set. □

Theorem 3.2.5. *Let A be an $n \times n$ real symmetric matrix such that all its eigenvalues are distinct. Then there exists an orthogonal matrix P such that*

$$P^{-1}AP = D,$$

where D is a diagonal matrix with diagonal entries being the eigen values of A.