Chapter 2

Mathematical Prelimarities of Orbital Mechanics

Contents

This chapter discusses the mathematical formulation of a two-body problem, various perturbing forces acting on the artificial satellites, the mathematical formulation of a two-body problem and various orbital elements.

2.1 Introduction

The study of the motion of the artificial satellite and its life span has been a topic of interest for many researchers over the past few decades. When the orbit of the satellite is low Earth orbit (LEO), the perturbation due to the oblateness of Earth and atmospheric drag plays a very important role. A variety of techniques, including analytic, semi-analytic, and numerical methods, have been employed to tackle the perturbed equations of motion in satellite dynamics. Raj[45] extensively investigated the movement of satellites influenced by Earth's oblateness and atmospheric drag, utilizing KS transformations for solving the equations of motion. King-Hele[32] provided an analytical solution for satellite motion considering Earth's oblateness. Sehnal[48] delved into the motion of satellites within the terrestrial upper atmosphere. Knowles *et. al.*[33] analyzed the impact of geomagnetic storms resulting from solar eruptions on the Earth's upper atmosphere and its effect on satellite motion. Yan and Kapila[55] developed dynamics for satellite motion around Earth's oblate shape using a rotating frame. Khalil[31] derived and solved Hamilton equations for satellite motion under Earth's oblateness and atmospheric drag using canonical transformations. Bezdvěk and Vokrouhlick'y[7] presented a semi-analytical theory for the long-term dynamics of low Earth orbit satellites, incorporating Earth's oblateness and atmospheric drag. Bhardwaj and Sethi[8] explored resonance in satellite motion affected by air drag. Hassan *et. al.*[25] attempted to solve equations governing satellite motion under Earth's oblateness using KS variables and Picard's iterative method. Chen and Jing[11] derived and solved differential equations governing relative satellite motion under Earth's oblateness and atmospheric drag, with applications in satellite attitude control and interplanetary mission orbital maneuvers. Lee *et. al.*[35] simulated satellite rotational dynamics using a Lie group variational integrator approach. Reid and Misra[46] studied the impact of aerodynamic forces on satellite formation flight. Xu and Chen[54] investigated the analytic solution of satellite orbit motion under atmospheric drag in terms of Keplerian angular elements. Al-Bermani *et. al.*[3] explored the effects of atmospheric drag and zonal harmonic J_2 on the near-Earth orbit satellite Cosmos1484. Delhaise[16] derived an analytic solution for satellite motion considering the combined effects of Earth's gravity and air drag using Lie transformations. Aghav and Gangal[2] designed and simplified an orbit determination algorithm for low Earth orbit navigation. This chapter discusses a mathematical model for the two-body problem, various perturbing forces acting on the artificial satellite and orbital elements.

2.2 Mathematical Formulation of Two-Body Problem

The two-body problem is a classic problem in physics and celestial mechanics that deals with the motion of two massive bodies under the influence of their gravitational attraction. In the context of artificial satellites, the two bodies typically refer to the satellite and the celestial body around which it orbits. This section aims to describe the mathematical formulation of the two-body problem for an artificial satellite [14]. Let:

- m_1 be the mass of the celestial body (e.g., Earth),
- m_2 be the mass of the artificial satellite,
- $\mathbf{r}_1$ be the position vector of the celestial body relative to an inertial reference frame,
- $\mathbf{r}_2$ be the position vector of the artificial satellite relative to the same reference frame,
- $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ be the relative position vector of the satellite concerning the celestial body,
- $\mathbf{F}$ be the gravitational force exerted by the celestial body on the satellite,
- G be the gravitational constant.

According to Newton's law of universal gravitation, the gravitational force $\mathbf{F}$ acting between the celestial body and the satellite is given by:

$$\mathbf{F} = -\frac{G m_1 m_2}{|\mathbf{r}|^3}\mathbf{r},$$

Newton's second law of motion states that the force acting on an object is equal to the mass of the object times its acceleration. Therefore, for the satellite:

$$m_2 \frac{d^2\mathbf{r}_2}{dt^2} = \mathbf{F}.$$

The two-body problem of an artificial satellite involves solving equations of motion to determine the trajectory of the satellite under the influence of gravitational attraction from the celestial body. This problem is essential for understanding orbital mechanics and predicting the motion of artificial satellites in space.

2.3 Perturbing Forces Acting on Artificial Satllites

Artificial satellites play a vital role in modern society, facilitating communication, navigation, weather forecasting, scientific research, and national security. The motion of an artificial satellite is influenced by various forces, including gravitational force, atmospheric drag, solar radiation pressure, and magnetic forces. Understanding these forces and their effects is essential for predicting and controlling the motion of satellites in space.

2.3.1 Gravitational Force

The gravitational force is the primary force acting on the motion of an artificial satellite. According to Newton's law of universal gravitation, the gravitational force between two objects with masses m_1 and m_2 separated by a distance r is given by:

$$\mathbf{F}_g = -\frac{G m_1 m_2}{r^2}\hat{\mathbf{r}},$$

where G is the gravitational constant and $\hat{\mathbf{r}}$ is the unit vector pointing from the satellite to the center of the Earth. The gravitational force causes the satellite to orbit the Earth in an elliptical, circular or geostationary orbit.

2.3.2 Atmospheric Drag

Atmospheric drag is another significant force affecting the motion of an artificial satellite, especially in low Earth orbit (LEO). As the satellite moves through the Earth's atmosphere, it experiences resistance, which causes its velocity to decrease and its orbit to decay over time. The atmospheric drag force depends on factors such as the density of the atmosphere, the velocity of the satellite, and the cross-sectional area and shape of the satellite.

2.3.3 Solar Radiation Pressure

Solar radiation pressure is the force exerted by sunlight on the surface of the satellite. Photons from the Sun transfer momentum to the satellite's surface, causing it to experience a small but significant force. The solar radiation pressure force is directed away from the Sun and depends on factors such as the area and reflectivity of the satellite's surface.

2.3.4 Magnetic Forces

Magnetic forces may also influence the motion of an artificial satellite, especially in polar or high-altitude orbits. Interactions between the satellite's magnetic field and the Earth's magnetic field can cause perturbations in its trajectory, affecting its orientation, attitude, and orbit. Magnetic forces must be taken into account when designing and operating satellites in certain orbits.

2.3.5 Other Forces

In addition to gravitational force, atmospheric drag, solar radiation pressure, and magnetic forces, other forces may affect the motion of an artificial satellite. These forces include perturbations from other celestial bodies, such as the Moon and other planets, as well as non-gravitational forces such as thrust from onboard propulsion systems and tidal forces.

2.4 Mathematical Formulation of Motion under Perturbing Forces

The equation of motion of satellite without any additional perturbing force other than the gravitational force between Earth and satellite is given by

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}, \tag{2.4.1}$$

where $\mu = GM$, G is gravitational constant and M is mass of Earth. In the presence of perturbation, additional perturbing acceleration must be added on the right side of the equation (2.4.1). Since we are considering perturbation due to the oblateness of Earth and perturbation due to atmospheric drag, the equation of motion can be written as

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \vec{a}_O + \vec{a}_A, \tag{2.4.2}$$

where $\vec{a}_O$ is acceleration due to oblateness of Earth and $\vec{a}_A$ is acceleration due to atmospheric drag. The second-order equation (2.4.2) can be written as the following set of two first-order differential equations

$$\dot{\vec{r}} = \vec{v},$$
$$\dot{\vec{v}} = -\frac{\mu}{r^3}\vec{r} + \vec{a}_O + \vec{a}_A. \tag{2.4.3}$$

In the Cartesian coordinate system the system of equations (2.4.3) takes the form,

$$\begin{aligned}
\dot{x} &= v_x,\\
\dot{y} &= v_y,\\
\dot{z} &= v_z,\\
\dot{v_x} &= -\frac{\mu x}{r^3} + \vec{a}_{O_x} + \vec{a}_{A_x},\\
\dot{v_y} &= -\frac{\mu y}{r^3} + \vec{a}_{O_y} + \vec{a}_{A_y},\\
\dot{v_z} &= -\frac{\mu z}{r^3} + \vec{a}_{O_z} + \vec{a}_{A_z},
\end{aligned} \tag{2.4.4}$$

where $\vec{a}_{O_x}, \vec{a}_{O_y}$ and $\vec{a}_{O_z}$ are components of acceleration due to the oblateness of Earth in the direction x, y and z axis respectively and $\vec{a}_{A_x}, \vec{a}_{A_y}$ and $\vec{a}_{A_z}$ are components of acceleration due to atmospheric drag in x, y and z axis respectively.

The Earth's gravitational potential can be modeled in terms of zonal harmonics Battin [6]. In the expression the value of J_2 zonal coefficient is 400 times higher than other J_n zonal coefficient, $n \geq 3$. Hence we consider only J_2 into account. If these higher order zonal coefficients are neglected and taking the gradient of scalar potential function then the components of acceleration due to oblateness of Earth in the direction of x, y and z direction respectively are,

$$
\begin{aligned}
\vec{a}_{O_x} &= -\frac{3\mu R^2 J_2 x(x^2+y^2-4z^2)}{2r^7}, \\
\vec{a}_{O_y} &= -\frac{3\mu R^2 J_2 y(x^2+y^2-4z^2)}{2r^7}, \\
\vec{a}_{O_z} &= -\frac{3\mu R^2 J_2 z(3x^2+3y^2-2z^2)}{2r^7},
\end{aligned} \tag{2.4.5}
$$

where $R = 6378.1363\ km$ is radius of Earth, $\mu = GM = 398600.436233\ km^3/sec^2$ and $J_2 = 1082.63 \times 10^{-6}$.

The acceleration due to atmospheric density is given by

$$\vec{a}_A = -\frac{1}{2}\rho\frac{C_D A}{m}|\vec{v_r}|\vec{v_r}, \tag{2.4.6}$$

where ρ is atmospheric density, C_D is drag coefficient, A is cross-sectional area of the satellite perpendicular to velocity vector, m is mass of satellite and $\vec{v_r}$ is satellite velocity vector relative to an atmosphere.

We take the simple exponential atmospheric model for which atmospheric density given by,

$$\rho = \rho_{pa} e^{\left[\frac{(r_{pa}-r)}{H}\right]}, \tag{2.4.7}$$

where ρ_{pa} is the density at initial perigee point, r_{pa} is the initial distance of satellite from Earth's surface, $r = |\vec{r}|$ and H is scale height. The ratio $B^* = \frac{C_D A}{m}$ is called the Ballistic coefficient.

We assume that the atmosphere rotates at the same angular speed as Earth. With this assumption the relative velocity vector is given by Wiesel[53]

$$\vec{v_r} = \vec{v} - \vec{\omega} \times \vec{r}, \tag{2.4.8}$$

where, $\vec{\omega}$ is the inertial rotation vector of the Earth given by

$$\vec{\omega} = \omega_e \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{2.4.9}$$

where, $\omega_e = 7.292115486 \times 10^{-5}\ rad/sec$. The cross product of the (2.4.8) and (2.4.9) gives three components of the relative velocity vector as

$$\vec{v_r} = \begin{bmatrix} v_x + \omega_e r_y \\ v_y - \omega_e r_x \\ v_z \end{bmatrix}. \tag{2.4.10}$$

Substituting (2.4.7), (2.4.10) and B^* in (2.4.6), we get the components of acceleration due to atmospheric drag in the direction of x, y and z axis respectively as

$$\begin{aligned} a_{A_x} &= -\frac{\rho_{pa} e^{[\frac{r_{pa}-r}{H}]} \sqrt{(v_x+\omega_e r_y)^2 + (vy - \omega_e r_x)^2 + v_z^2}\ (v_x + \omega_e r_y) B^*}{2}, \\ a_{A_y} &= -\frac{\rho_{pa} e^{[\frac{r_{pa}-r}{H}]} \sqrt{(v_x+\omega_e r_y)^2 + (vy - \omega_e r_x)^2 + v_z^2}\ (vy - \omega_e r_x) B^*}{2}, \\ a_{A_z} &= -\frac{\rho_{pa} e^{[\frac{r_{pa}-r}{H}]} \sqrt{(v_x+\omega_e r_y)^2 + (vy - \omega_e r_x)^2 + v_z^2}\ v_z B^*}{2}. \end{aligned} \tag{2.4.11}$$

Substituting (2.4.5) and (2.4.11) in (2.4.4), we get equations of motion of satellite under the oblateness of Earth and atmospheric drag as

$$
\begin{aligned}
\dot{x} &= v_x, \\
\dot{y} &= v_y, \\
\dot{z} &= v_z, \\
\dot{v_x} &= -\frac{\mu x}{r^3} - \frac{3\mu R^2 J_2 x(x^2+y^2-4z^2)}{2r^7} - \frac{\rho_{pa} e^{[\frac{r_{pa}-r}{H}]}\sqrt{(v_x+\omega_e r_y)^2+(vy-\omega_e r_x)^2+v_z^2}\,(v_x+\omega_e r_y)B^*}{2}, \\
\dot{v_y} &= -\frac{\mu y}{r^3} - \frac{3\mu R^2 J_2 y(x^2+y^2-4z^2)}{2r^7} - \frac{\rho_{pa} e^{[\frac{r_{pa}-r}{H}]}\sqrt{(v_x+\omega_e r_y)^2+(vy-\omega_e r_x)^2+v_z^2}\,(vy-\omega_e r_x)B^*}{2}, \\
\dot{v_z} &= -\frac{\mu z}{r^3} - \frac{3\mu R^2 J_2 z(3x^2+3y^2-2z^2)}{2r^7} - \frac{\rho_{pa} e^{[\frac{r_{pa}-r}{H}]}\sqrt{(v_x+\omega_e r_y)^2+(vy-\omega_e r_x)^2+v_z^2}\,v_z B^*}{2}.
\end{aligned}
\tag{2.4.12}
$$

2.5 Orbital Elements

The study of celestial mechanics has long been a cornerstone of human understanding of the motion of celestial bodies. One crucial aspect of celestial mechanics is the description of the motion of celestial bodies in space, such as planets, moons, asteroids and artificial satellites. Orbital elements serve as fundamental parameters used to characterize and describe the trajectory of an object in space relative to a reference frame. In this section, we delve into the introduction and historical background of orbital elements, exploring their significance, development, and applications throughout history and in modern space exploration. The origins of orbital elements can be traced back to ancient astronomers who observed the motion of celestial bodies and sought to understand and predict their paths in the sky. Ancient civilizations such as the Babylonians, Greeks, Egyptians and Chinese made significant contributions to the field of astronomy, developing mathematical models and techniques to describe the apparent motion of celestial objects relative to Earth.

In the Western world, the ancient Greeks, notably Claudius Ptolemy, developed geocentric models of the universe, wherein the Earth was considered the center of the cosmos and celestial bodies moved in circular or epicyclic orbits around it. While these models were influential for centuries, they were eventually supplanted by heliocentric models proposed by Nicolaus Copernicus, Johannes Kepler and Galileo Galilei during the Renaissance and Scientific Revolution. One of the most significant advancements in the understanding of celestial motion came with Johannes Kepler's formulation of his laws of planetary motion in the early 17th century. Kepler's laws, derived from meticulous observations of the planets made by Tycho Brahe, described the motion of planets around the Sun in elliptical orbits with the Sun at one focus. Kepler introduced three key parameters, known as orbital elements, to characterize these orbits:

1. **Semimajor Axis** (a): The average distance from the center of the ellipse to its farthest point.
2. **Eccentricity** (e): A measure of the deviation of the orbit from a perfect circle.
3. **Inclination** (i): The angle between the plane of the orbit and a reference plane, such as the ecliptic plane for solar system objects.

In addition to the parameters introduced by Kepler, modern orbital mechanics defines several additional elements that further characterize an orbit:

- **Longitude of Ascending Node** (Ω): The angle between the reference direction and the line where the orbit crosses the reference plane from south to north.
- **Argument of Periapsis** (ω): The angle between the ascending node and the periapsis, measured in the orbital plane.

- **True Anomaly (ν)**: The angle between the periapsis and the position of the orbiting body, measured at the focus of the ellipse.
- **Mean Anomaly (M)**: The fraction of the orbital period that has elapsed since the body passed the periapsis, expressed as an angle.

These additional elements provide a more complete description of the orientation and position of an orbiting body within its orbit.

In the modern era, advancements in observational technology, computational methods, and space exploration have led to further refinements in our understanding of orbital mechanics and the development of precise techniques for determining orbital elements. With the advent of space-based telescopes, radar tracking systems and spacecraft missions, scientists have been able to gather vast amounts of data on the motion of celestial bodies and artificial satellites.

Organizations such as NASA, the European Space Agency (ESA) and other space agencies routinely compute and disseminate orbital elements for thousands of satellites orbiting the Earth and other celestial bodies. These orbital elements enable precise prediction and tracking of satellite orbits, essential for space missions, satellite communications, navigation and space situational awareness.

Orbital elements represent a fundamental tool for characterizing the motion of celestial bodies in space. From their origins in ancient astronomical observations to their modern applications in space exploration, orbital elements have played a central role in advancing our understanding of the cosmos and enabling the exploration and utilization of space. As we continue to push the boundaries of space exploration and scientific discovery, orbital elements will remain indispensable for navigating the complexities of the universe and unlocking its mysteries.

2.6 Conclusion

Sharma *et.al*. [49] analyzed the orbit of the satellite under various initial conditions and they found that the motion of an artificial satellite is influenced by various forces, including oblateness of the earth, and atmospheric drag. Understanding these forces and their effects is crucial for predicting and controlling the motion of satellites in space, optimizing their orbits and ensuring the success of satellite-based missions.