Chapter 4

Controllability Analysis of Satellite Motion

Contents

This chapter examined the controllability of artificial satellites under the influence of zonal harmonic J_2 in cylindrical polar coordinate systems. The system is controlled when thrusters are applied in either r, θ, and z or θ and z directions. The equations that governs the satellite's motion have been linearized, and the Kalman controllability test is used to ensure that the system is controllable. We have also developed a controller u for the linearized system. The system's trajectory was plotted to demonstrate its controllability.

4.1 Introduction

Artificial satellites are integral to modern navigation, communication, and environmental monitoring [4, 42]. Researchers have extensively examined satellite motion through analytic, semi-analytic and numerical approaches. King-Hele [32] addressed the two-body problem of satellites analytically, factoring in Earth's oblateness. Raj [45] employed KS transformations [50] to regularize motion equations and solve them, accounting for atmospheric drag. Sehnal [48] investigated satellite motion considering perturbations from the upper terrestrial atmosphere. Knowles *et.al.* [33] analyzed sensor data to study the impact of geomagnetic storms on the upper atmosphere, affecting satellite orbits.

Yan and Kapila [55] formulated dynamical equations for satellite motion around an oblate Earth using a rotating frame, establishing conditions for a fixed osculating plane. Khalil [31] derived an analytical solution, incorporating atmospheric drag and

Earth's oblateness up to the 4th-order zonal harmonic. Bezdvěk and Vokrouhlicky' [7] presented a semi-analytical theory considering Earth's oblateness up to the 9th-order zonal harmonic and atmospheric drag, comparing predictions with real satellite data.

Hassan *et. al.* [25] regularized equations of perturbed motion due to Earth's oblateness using KS transformations and derived an algorithm to solve them using Picard's method. Chen and Jing [11] studied the relative motion of satellites under the effect of Earth's oblateness and atmospheric drag. Lee *et. al.* [35] simulated the rotational dynamics of satellites using Lie group variational approach. Reid and Misra [46] studied the formation flight of artificial satellites under the effect of aerodynamic forces. Xu and Chen [54] derived an analytical solution in terms of Keplerian angular elements of satellite orbit under the effect of atmospheric drag. Al-Bermani *et. al.* [3] investigated the effect of Earth's oblateness and atmospheric drag on the orbit of satellite Cosmos1484. Delhaise [16] derived an analytical solution of satellite motion by considering gravity and air drag using Lie transformations.

Sharma *et.al.* [49] explored satellite motion with different initial velocities, calculating orbital elements considering Earth's oblateness and atmospheric drag. They also predicted the time at which satellites would re-enter Earth's atmosphere for longer orbit stability. Hajovsky [24] used atmospheric drag as a controller for satellite trajectory, while B. Palancz [42, 43] employed pole placement for trajectory control. Lamba [34] discussed the controllability, observability and stability of artificial satellites using a state-space method, albeit with a two-dimensional model.

In this study, we consider satellite motion under the effect of the J_2 zonal harmonic in a cylindrical polar coordinate system and explore controllability using thrusters in various directions. Our observations suggest that satellite motion is controllable with thrusters placed in radial, azimuthal, and axial directions. We also investigated the trajectory controllability of satellites.

4.2 Preliminaries

In real life, most of the systems are nonlinear and this nonlinearity creates difficulty in finding solutions of the system. Hence it is required to approximate the nonlinear system by the appropriate linear system.

The motion of an artificial satellite under the effect of zonal harmonic J_2 is modeled in terms of system of nonlinear differential equations. Here, we introduce the concept of linear control theory followed by linearization of nonlinear control systems [9].

4.2.1 Linear Control Systems

Consider a linear control system,

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ x(t_0) &= x_0, \end{aligned} \tag{4.2.1}$$

where, $x_0, x(t) \in \mathbb{R}^n$ for all $t \in [t_0, t_1]$, $u \in L^2([t_0, t_1], \mathbb{R}^m)$. The matrices $A(t)$ and $B(t)$ are of order $n \times n$ and $n \times m$ respectively.
Let $\Phi(t, t_0)$ be the transition matrix of the homogeneous system $\dot{x}(t) = A(t)x(t)$ with initial condition $x(t_0) = x_0$ then solution of the system (4.2.1) is given by,

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)B(s)u(s)ds. \tag{4.2.2}$$

Definition 4.2.1. *[9] The system* (4.2.1) *is controllable over the interval* $[t_0, t_1]$*, if each pair of vectors* x_0 *and* x_1 *in* $\mathbb{R}^n$ *there is a control* $u \in L^2([t_0, t_1], \mathbb{R}^m)$ *such that the solution of* (4.2.1) *satisfies* $x(t_1) = x_1$*. This means there is a control* u *satisfying*

$$x_1 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, s)B(s)u(s)ds.$$

Theorem 4.2.1. *[9] The system* (4.2.1) *is controllable if and only if the controllability grammian of the system defined by* $W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, s)B(s)B^*(s)\Phi^*(t_1, s)ds$ *is invertible and control* u *of the system* (4.2.1) *is given by*

$$u(t) = B^*(t)\Phi^*(t_1, t)W^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)].$$

However, if the system is time-invariant, conditions reduce to Kalman condition which is given by,

Corollary 4.2.1. *[9] If matrices* A *and* B *are two time-invariant matrices of the system* (4.2.1) *then the system is controllable if and only if the rank of the controllability matrix* $Q = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n$*.*

4.2.2 Linearization of Differential Systems

Consider the nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \tag{4.2.3}$$

where the state x(t)is an n-dimensional vector, controller $u(t)$ is m-dimensional vector for all t, $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a non-linear function.

Let (x_0, u_0) be the reference point of the system (4.2.3) then Taylor series expansion of the the function at the reference point is given by:

$$f(x_0 + \delta x, u_0 + \delta u) = f(x_0, u_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)} \delta x + \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)} \delta u + \textit{higher order terms},$$

and therefore we have:

$$\dot{x}_0 + \delta\dot{x} \approx f(x_0, u_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)} \delta x + \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)} \delta u,$$

simplifying, we get

$$\delta x = \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)} \delta x + \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)} \delta u. \tag{4.2.4}$$

Define, $x = \delta x, u = \delta u, A = \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)}$ and $B = \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)}$ the system (4.2.4) becomes:

$$\dot{x} = Ax + Bu. \tag{4.2.5}$$

The equation (4.2.5) is a linear system corresponding to the system (4.2.3).

4.3 Controallabiliy Analysis of the Motion of Satellite

The equations of motion of satellite under the effect of the oblateness of the earth are given by

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \vec{a}_O, \tag{4.3.1}$$

where, $\mu = GM$, G is the gravitational constant and M is the mass of the earth, and $\vec{a}_O$ is the acceleration due to the oblateness of the earth, considering zonal harmonic J_2. The equations of motion in cylindrical coordinate systems represented by Humi[28],

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\mu r\left[\frac{1}{(r^2+z^2)^{\frac{3}{2}}} + \frac{3R^2 J_2(r^2-4z^2)}{2(r^2+z^2)^{\frac{7}{2}}}\right], \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0, \\ \ddot{z} &= -\mu z\left[\frac{1}{(r^2+z^2)^{\frac{3}{2}}} + \frac{3R^2 J_(3r^2-2z^2)}{2(r^2+z^2)^{\frac{7}{2}}}\right]. \end{aligned} \tag{4.3.2}$$

Under the effect of zonal harmonic J_2, the satellite will deviate from its desired orbit, hence its motion becomes uncontrollable. Eventually, it will hit on Earth. Hence, to control the motion of satellites we need to impose the controllers in the form of thrusters. Let $u_1,\ u_2$ and u_3 represents thrusters in the $r,\ \theta$ and z directions respectively. We analyzed seven different cases viz. applying thruster(s) in

1. only r direction,
2. only θ direction,
3. only z direction,
4. r and θ direction,
5. r and z direction,
6. θ and z direction,
7. r, θ and z direction.

and check the controllability of system in each case.

Further, we assume that the orbit of the satellite is circular with reference radius σ and the angle $\theta = \omega t$. Since we have a well-established theory of controllability for first order system, we apply the following transformation to the system (4.3.2) after adding controllers in various directions to reduce it to a system of first-order equations,

$$\begin{aligned} X_1 &= r - \sigma, \\ X_2 &= \dot{r}, \\ X_3 &= \sigma\left(\theta - \omega t\right), \\ X_4 &= \sigma\left(\dot{\theta} - \omega\right), \\ X_5 &= z, \\ X_6 &= \dot{z}. \end{aligned} \tag{4.3.3}$$

The study of controllability after applying thrusters in different directions is discussed below.

4.3.1 Adding the thruster $u_1(t)$ only in r direction

Adding thruster only in r direction, the system (4.3.2) becomes:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\mu r\left[\frac{1}{\left(r^2+z^2\right)^{3/2}} + \frac{3R^2 J_2\left(r^2 - 4z^2\right)}{2\left(r^2+z^2\right)^{7/2}}\right] + u_1(t), \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0, \\ \ddot{z} &= -\mu z\left[\frac{1}{\left(r^2+z^2\right)^{3/2}} + \frac{3R^2 J_2\left(3r^2 - 2z^2\right)}{2\left(r^2+z^2\right)^{7/2}}\right]. \end{aligned} \tag{4.3.4}$$

By transformation (4.3.3), system (4.3.4) takes the form:

$$
\begin{aligned}
\frac{dX_1}{dt} &= X_2, \\
\frac{dX_2}{dt} &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}}\right. \\
&\qquad\left. + \frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_1(t), \\
\frac{dX_3}{dt} &= X_4, \\
\frac{dX_4}{dt} &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)}, \\
\frac{dX_5}{dt} &= X_6, \\
\frac{dX_6}{dt} &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[3(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\}
\end{aligned}
\tag{4.3.5}
$$

Now we linearize the system (4.3.5) about origin, we take

$$
\begin{aligned}
f_1 &= X_2, \\
f_2 &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} \\
&\qquad + u_1(t), \\
f_3 &= X_4, \\
f_4 &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)}, \\
f_5 &= X_6, \\
f_6 &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[3(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\},
\end{aligned}
$$

therefore system (4.3.5) takes the form

$$\dot{X} = AX + BU,$$

where, $\dot{X} = \begin{bmatrix} \frac{dX_1}{dt} & \frac{dX_2}{dt} & \frac{dX_3}{dt} & \frac{dX_4}{dt} & \frac{dX_5}{dt} & \frac{dX_6}{dt} \end{bmatrix}'$, $A = \begin{bmatrix} \frac{\partial(f_1, f_2, f_3, f_4, f_5, f_6)}{\partial(X_1, X_2, X_3, X_4, X_5, X_6)} \end{bmatrix}$ at origin, $X = \begin{bmatrix} X_1 \; X_2 \; X_3 \; X_4 \; X_5 \; X_6 \end{bmatrix}'$, $B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \; \frac{\partial f_2}{\partial u_1} \; \frac{\partial f_3}{\partial u_1} \; \frac{\partial f_4}{\partial u_1} \; \frac{\partial f_5}{\partial u_1} \; \frac{\partial f_6}{\partial u_1} \end{bmatrix}'$ at origin and $u = \begin{bmatrix} u_1 \end{bmatrix}$. The values of A and B are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1.000002542612694 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -0.000294117647059 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -0.00000127311747 & 0 \end{bmatrix}, \tag{4.3.6}$$

and $B = \begin{bmatrix} 0 \; 1 \; 0 \; 0 \; 0 \; 0 \end{bmatrix}'$. The controllability matrix Q is given by

$$Q = \begin{bmatrix} B \; AB \; A^2B \; A^3B \; A^4B \; A^5B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -3 & 0 & 9 \\ 1 & 0 & -3 & 0 & 9 & 0 \\ 0 & 0 & -2 & 0 & 6 & 0 \\ 0 & -2 & 0 & 6 & 0 & -18 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of the matrix Q is 3, which is not equal to the dimensions of the state $X (= 6)$. By the Kalman's condition, the system is not controllable if we add the thruster only in radial direction r.

4.3.2 Adding the thruster $u_1(t)$ only in θ direction

Adding the thruster $u_1(t)$ only in θ direction, the system (4.3.2) becomes:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\mu r \left[\frac{1}{(r^2+z^2)^{3/2}} + \frac{3R^2 J_2 (r^2 - 4z^2)}{2(r^2+z^2)^{7/2}} \right], \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= u_1(t), \\ \ddot{z} &= -\mu z \left[\frac{1}{(r^2+z^2)^{3/2}} + \frac{3R^2 J_2 (3r^2 - 2z^2)}{2(r^2+z^2)^{7/2}} \right]. \end{aligned} \tag{4.3.7}$$

By transformation (4.3.3), system (4.3.7) takes the form:

$$\begin{aligned} \frac{dX_1}{dt} &= X_2, \\ \frac{dX_2}{dt} &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu(X_1+\sigma)\left\{ \frac{1}{\left[(X_1+\sigma)^2 + X_5^2\right]^{3/2}} + \frac{3R^2 J_2\left[(X_1+\sigma)^2 - 4X_5^2\right]}{2\left[(X_1+\sigma)^2 + X_5\right]^{7/2}} \right\}, \\ \frac{dX_3}{dt} &= X_4, \\ \frac{dX_4}{dt} &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)} + u_1(t), \\ \frac{dX_5}{dt} &= X_6, \\ \frac{dX_6}{dt} &= -\mu X_5 \left\{ \frac{1}{\left[(X_1+\sigma)^2 + X_5^2\right]^{3/2}} + \frac{3R^2 J_2\left[3(X_1+\sigma)^2 - 2X_5^2\right]}{2\left[(X_1+\sigma)^2 + X_5\right]^{7/2}} \right\}. \end{aligned} \tag{4.3.8}$$

We linearize the system (4.3.8) about origin by taking

$$\begin{aligned}
f_1 &= X_2,\\
f_2 &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2-\mu(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}}+\frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\},\\
f_3 &= X_4,\\
f_4 &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)}+u_1(t),\\
f_5 &= X_6,\\
f_6 &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}}+\frac{3R^2J_2\left[3(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\},
\end{aligned}$$

therefore the system (4.3.8) takes the form

$$\dot{X}=AX+BU,$$

where, $\dot{X}=\begin{bmatrix}\frac{dX_1}{dt} & \frac{dX_2}{dt} & \frac{dX_3}{dt} & \frac{dX_4}{dt} & \frac{dX_5}{dt} & \frac{dX_6}{dt}\end{bmatrix}'$, $A=\begin{bmatrix}\frac{\partial(f_1,f_2,f_3,f_4,f_5,f_6)}{\partial(X_1,X_2,X_3,X_4,X_5,X_6)}\end{bmatrix}$, at origin, $X=\begin{bmatrix}X_1\ X_2\ X_3\ X_4\ X_5\ X_6\end{bmatrix}'$, $B=\begin{bmatrix}\frac{\partial f_1}{\partial u_1} & \frac{\partial f_2}{\partial u_1} & \frac{\partial f_3}{\partial u_1} & \frac{\partial f_4}{\partial u_1} & \frac{\partial f_5}{\partial u_1} & \frac{\partial f_6}{\partial u_1}\end{bmatrix}'$ at origin, and $u=\begin{bmatrix}u_1\end{bmatrix}$. The matrix A is given by (4.3.6) and $B=\begin{bmatrix}0\ 0\ 0\ 1\ 0\ 0\end{bmatrix}'$. The controllability matrix Q is given by

$$Q=\begin{bmatrix}B\ AB\ A^2B\ A^3B\ A^4B\ A^5B\end{bmatrix}=\begin{bmatrix}0&0&2&0&-6&0\\0&2&0&-6&0&18\\0&1&0&-4&0&12\\1&0&-4&0&12&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},$$

and the rank of Q is 4, which is not equal to the dimensions of the state $X\,(=6)$. By the Kalman's condition, the system is not controllable if we add the thruster only in θ direction.

4.3.3 Adding the thruster $u_1(t)$ only in z direction

The system (4.3.2) is written as

$$\begin{aligned}
\ddot{r} - r\dot{\theta}^2 &= -\mu r\left[\frac{1}{(r^2+z^2)^{3/2}} + \frac{3R^2J_2\left(r^2-4z^2\right)}{2\left(r^2+z^2\right)^{7/2}}\right],\\
r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0,\\
\ddot{z} &= -\mu z\left[\frac{1}{(r^2+z^2)^{3/2}} + \frac{3R^2J_2\left(3r^2-2z^2\right)}{2\left(r^2+z^2\right)^{7/2}}\right] + u_1(t).
\end{aligned} \tag{4.3.9}$$

By transformation (4.3.3), system (4.3.9) takes the form:

$$\begin{aligned}
\frac{dX_1}{dt} &= X_2,\\
\frac{dX_2}{dt} &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu\left(X_1+\sigma\right)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\},\\
\frac{dX_3}{dt} &= X_4,\\
\frac{dX_4}{dt} &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)},\\
\frac{dX_5}{dt} &= X_6,\\
\frac{dX_6}{dt} &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[3\left(X_1+\sigma\right)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_1\left(t\right).
\end{aligned} \tag{4.3.10}$$

Linearizing the system (4.3.10) about origin by taking

$$\begin{aligned}
f_1 &= X_2,\\
f_2 &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2-\mu\,(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}}+\frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\},\\
f_3 &= X_4,\\
f_4 &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)},\\
f_5 &= X_6,\\
f_6 &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}}+\frac{3R^2J_2\left[3\,(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\}+u_1\,(t)\,,
\end{aligned}$$

and the system (4.3.10) takes the form

$$\dot{X}=AX+BU,$$

where, $\dot{X}=\begin{bmatrix}\frac{dX_1}{dt} & \frac{dX_2}{dt} & \frac{dX_3}{dt} & \frac{dX_4}{dt} & \frac{dX_5}{dt} & \frac{dX_6}{dt}\end{bmatrix}'$, $A=\begin{bmatrix}\frac{\partial(f_1,f_2,f_3,f_4,f_5,f_6)}{\partial(X_1,X_2,X_3,X_4,X_5,X_6)}\end{bmatrix}$ at origin, $\begin{bmatrix}X=X_1\ X_2\ X_3\ X_4\ X_5\ X_6\end{bmatrix}'$, $B=\begin{bmatrix}\frac{\partial f_1}{\partial u_1} & \frac{\partial f_2}{\partial u_1} & \frac{\partial f_3}{\partial u_1} & \frac{\partial f_4}{\partial u_1} & \frac{\partial f_5}{\partial u_1} & \frac{\partial f_6}{\partial u_1}\end{bmatrix}'$ at origin and $U=\begin{bmatrix}u_1\end{bmatrix}$. The values of A as (4.3.6) and $B=\begin{bmatrix}0\,0\,0\,0\,0\,1\end{bmatrix}'$. The controllability matrix Q is given by

$$Q=\begin{bmatrix}B\ AB\ A^2B\ A^3B\ A^4B\ A^5B\end{bmatrix}=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\\1&0&0&0&0&0\end{bmatrix},$$

Therefore, rank of the matrix $Q=2$ which is not equal to the dimensions of the state $X\,(=6)$. By the Kalman's condition, the system is not controllable if we add the thruster only in z direction.

4.3.4 Adding thrusters $u_1(t)$ and $u_2(t)$ in r and θ direction

The system (4.3.2) becomes:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\mu r \left[\frac{1}{\left(r^2+z^2\right)^{3/2}} + \frac{3R^2 J_2 \left(r^2 - 4z^2\right)}{2\left(r^2+z^2\right)^{7/2}} \right] + u_1(t), \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= u_2(t), \\ \ddot{z} &= -\mu z \left[\frac{1}{\left(r^2+z^2\right)^{3/2}} + \frac{3R^2 J_2 \left(3r^2 - 2z^2\right)}{2\left(r^2+z^2\right)^{7/2}} \right]. \end{aligned} \tag{4.3.11}$$

By transformation (4.3.3), system (4.3.11) takes the form

$$\begin{aligned} \frac{dX_1}{dt} &= X_2, \\ \frac{dX_2}{dt} &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu\,(X_1+\sigma)\left\{ \frac{1}{\left[(X_1+\sigma)^2 + X_5^2\right]^{3/2}} \right. \\ &\qquad \left. + \frac{3R^2 J_2\left[(X_1+\sigma)^2 - 4X_5^2\right]}{2\left[(X_1+\sigma)^2 + X_5\right]^{7/2}} \right\} + u_1(t), \\ \frac{dX_3}{dt} &= X_4, \\ \frac{dX_4}{dt} &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)} + u_2(t), \\ \frac{dX_5}{dt} &= X_6, \\ \frac{dX_6}{dt} &= -\mu X_5 \left\{ \frac{1}{\left[(X_1+\sigma)^2 + X_5^2\right]^{3/2}} + \frac{3R^2 J_2\left[3\,(X_1+\sigma)^2 - 2X_5^2\right]}{2\left[(X_1+\sigma)^2 + X_5\right]^{7/2}} \right\}. \end{aligned} \tag{4.3.12}$$

For linearizing the system (4.3.12) about origin, we take

$$\begin{aligned}
f_1 &= X_2,\\
f_2 &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu\,(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}}\right.\\
&\qquad\left. + \frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_1(t),\\
f_3 &= X_4,\\
f_4 &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)} + u_2(t),\\
f_5 &= X_6,\\
f_6 &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[3\,(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\}
\end{aligned} \tag{4.3.13}$$

Therefore the system (4.3.12) take the form

$$\dot{X} = AX + BU,$$

where, $\dot{X} = \begin{bmatrix}\frac{dX_1}{dt} & \frac{dX_2}{dt} & \frac{dX_3}{dt} & \frac{dX_4}{dt} & \frac{dX_5}{dt} & \frac{dX_6}{dt}\end{bmatrix}'$, $A = \begin{bmatrix}\frac{\partial(f_1, f_2, f_3, f_4, f_5, f_6)}{\partial(X_1, X_2, X_3, X_4, X_5, X_6)}\end{bmatrix}$ at origin, $X = \begin{bmatrix}X_1\ X_2\ X_3\ X_4\ X_5\ X_6\end{bmatrix}'$, $B = \begin{bmatrix}\frac{\partial f_1}{\partial u_1} & \frac{\partial f_2}{\partial u_1} & \frac{\partial f_3}{\partial u_1} & \frac{\partial f_4}{\partial u_1} & \frac{\partial f_5}{\partial u_1} & \frac{\partial f_6}{\partial u_1}\\ \frac{\partial f_1}{\partial u_2} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_4}{\partial u_2} & \frac{\partial f_5}{\partial u_2} & \frac{\partial f_6}{\partial u_2}\end{bmatrix}'$ at origin and $u = \begin{bmatrix}u_1\ u_2\end{bmatrix}'$. We obtain the values of A as (4.3.6)and $B = \begin{bmatrix}0\ 1\ 0\ 0\ 0\ 0\\ 0\ 0\ 0\ 1\ 0\ 0\end{bmatrix}'$. The controllability matrix Q is given by

$$Q = \begin{bmatrix} B \; AB \; A^2B \; A^3B \; A^4B \; A^5B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2 & -3 & 0 & 0 & -6 & 9 & 0 \\ 1 & 0 & 0 & 2 & -3 & 0 & 0 & -6 & 9 & 0 & 0 & 18 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & -4 & 6 & 0 & 0 & 12 \\ 0 & 1 & -2 & 0 & 0 & -4 & 6 & 0 & 0 & 12 & -18 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

The rank of the matrix $Q = 4$ which is not equal to the dimensions of the state $X\,(= 6)$. By the Kalman's condition, we conclude that the system is not controllable if we add the thrusters in r and θ direction.

4.3.5 Adding thrusters $u_1(t)$ and $u_2(t)$ in r and z direction

The system (4.3.2) becomes:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\mu r \left[\frac{1}{(r^2+z^2)^{3/2}} + \frac{3R^2 J_2 (r^2 - 4z^2)}{2(r^2+z^2)^{7/2}} \right] + u_1(t), \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0, \\ \ddot{z} &= -\mu z \left[\frac{1}{(r^2+z^2)^{3/2}} + \frac{3R^2 J_2 (3r^2 - 2z^2)}{2(r^2+z^2)^{7/2}} \right] + u_2(t). \end{aligned} \tag{4.3.14}$$

By transformation (4.3.3), system (4.3.14) takes the form

$$\begin{aligned}
\frac{dX_1}{dt} &= X_2, \\
\frac{dX_2}{dt} &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu\,(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}}\right. \\
&\qquad\qquad \left. + \frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_1(t), \\
\frac{dX_3}{dt} &= X_4, \\
\frac{dX_4}{dt} &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)}, \\
\frac{dX_5}{dt} &= X_6, \\
\frac{dX_6}{dt} &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[3\,(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_2(t).
\end{aligned} \tag{4.3.15}$$

For linearizing the system (4.3.15) about origin, we take

$$\begin{aligned}
f_1 &= X_2, \\
f_2 &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu\,(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}}\right. \\
&\qquad\qquad \left. + \frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_1(t), \\
f_3 &= X_4, \\
f_4 &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)}, \\
f_5 &= X_6, \\
f_6 &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[3\,(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_2(t),
\end{aligned} \tag{4.3.16}$$

Therefore the system (4.3.15) takes the form

$$\dot{X} = AX + BU,$$

where, $\dot{X} = \begin{bmatrix} \frac{dX_1}{dt} & \frac{dX_2}{dt} & \frac{dX_3}{dt} & \frac{dX_4}{dt} & \frac{dX_5}{dt} & \frac{dX_6}{dt} \end{bmatrix}'$, $A = \begin{bmatrix} \frac{\partial(f_1, f_2, f_3, f_4, f_5, f_6)}{\partial(X_1, X_2, X_3, X_4, X_5, X_6)} \end{bmatrix}$ at origin, $X = \begin{bmatrix} X_1 \; X_2 \; X_3 \; X_4 \; X_5 \; X_6 \end{bmatrix}'$, $B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_2}{\partial u_1} & \frac{\partial f_3}{\partial u_1} & \frac{\partial f_4}{\partial u_1} & \frac{\partial f_5}{\partial u_1} & \frac{\partial f_6}{\partial u_1} \\ \frac{\partial f_1}{\partial u_2} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_4}{\partial u_2} & \frac{\partial f_5}{\partial u_2} & \frac{\partial f_6}{\partial u_2} \end{bmatrix}'$ at origin and $u = \begin{bmatrix} u_1 \; u_2 \end{bmatrix}'$, The matrix A is given in (4.3.6) and $B = \begin{bmatrix} 0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1 \end{bmatrix}'$. The controllability matrix Q is given by

$$Q = \begin{bmatrix} B \; AB \; A^2B \; A^3B \; A^4B \; A^5B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 9 & 0 \\ 1 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & -18 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

The rank of the matrix Q is 5, which is not equal to the dimensions of the state $X\,(= 6)$. By Kalman's condition, we conclude that the system is not controllable if we add the thrusters in r and z direction.

4.3.6 Adding thrusters $u_1(t)$ and $u_2(t)$ in θ and z direction

The system (4.3.2) becomes:

$$\begin{aligned}\ddot{r} - r\dot{\theta}^2 &= -\mu r \left[\frac{1}{(r^2+z^2)^{3/2}} + \frac{3R^2 J_2 (r^2 - 4z^2)}{2(r^2+z^2)^{7/2}} \right], \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= u_1(t), \\ \ddot{z} &= -\mu z \left[\frac{1}{(r^2+z^2)^{3/2}} + \frac{3R^2 J_2 (3r^2 - 2z^2)}{2(r^2+z^2)^{7/2}} \right] + u_2(t).\end{aligned} \tag{4.3.17}$$

By transformation (4.3.3), system (4.3.17) takes the form

$$\begin{aligned}\frac{dX_1}{dt} &= X_2, \\ \frac{dX_2}{dt} &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu(X_1+\sigma)\left\{ \frac{1}{\left[(X_1+\sigma)^2 + X_5^2\right]^{3/2}} + \frac{3R^2 J_2 \left[(X_1+\sigma)^2 - 4X_5^2\right]}{2\left[(X_1+\sigma)^2 + X_5\right]^{7/2}} \right\}, \\ \frac{dX_3}{dt} &= X_4, \\ \frac{dX_4}{dt} &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)} + u_1(t), \\ \frac{dX_5}{dt} &= X_6, \\ \frac{dX_6}{dt} &= -\mu X_5 \left\{ \frac{1}{\left[(X_1+\sigma)^2 + X_5^2\right]^{3/2}} + \frac{3R^2 J_2 \left[3(X_1+\sigma)^2 - 2X_5^2\right]}{2\left[(X_1+\sigma)^2 + X_5\right]^{7/2}} \right\} + u_2(t).\end{aligned} \tag{4.3.18}$$

For linearizing the system (4.3.18) about origin, we take

$$\begin{aligned}
f_1 &= X_2,\\
f_2 &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu\,(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\},\\
f_3 &= X_4,\\
f_4 &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)} + u_1(t),\\
f_5 &= X_6,\\
f_6 &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[3\,(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_2(t),
\end{aligned}$$

Therefore the system (4.3.18) takes the form:

$$\dot{X} = AX + BU,$$

where, $\dot{X} = \begin{bmatrix} \frac{dX_1}{dt} & \frac{dX_2}{dt} & \frac{dX_3}{dt} & \frac{dX_4}{dt} & \frac{dX_5}{dt} & \frac{dX_6}{dt} \end{bmatrix}'$, $A = \begin{bmatrix} \frac{\partial(f_1, f_2, f_3, f_4, f_5, f_6)}{\partial(X_1, X_2, X_3, X_4, X_5, X_6)} \end{bmatrix}$ at origin, $X = \begin{bmatrix} X_1\ X_2\ X_3\ X_4\ X_5\ X_6 \end{bmatrix}'$, $B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_2}{\partial u_1} & \frac{\partial f_3}{\partial u_1} & \frac{\partial f_4}{\partial u_1} & \frac{\partial f_5}{\partial u_1} & \frac{\partial f_6}{\partial u_1} \\ \frac{\partial f_1}{\partial u_2} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_4}{\partial u_2} & \frac{\partial f_5}{\partial u_2} & \frac{\partial f_6}{\partial u_2} \end{bmatrix}'$ at origin and $u = \begin{bmatrix} u_1\ u_2 \end{bmatrix}'$. The matrix A is given in (4.3.6) and $B = \begin{bmatrix} 0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1 \end{bmatrix}'$. The controllability matrix Q is given by

$$Q = \begin{bmatrix} B\ AB\ A^2B\ A^3B\ A^4B\ A^5B \end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -6 & 0 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 18 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 12 & 0\\
1 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 12 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

The rank of Q is 6, which is equal to the dimensions of the state $X\,(=6)$. Hence by the Kalman's condition, we conclude that the system is controllable if we add thrusters $u_1(t)$ and $u_2(t)$ in θ and z directions. The figure-1 shows, that the system is steered from the initial point $[1\ 2\ 3\ 4\ 5\ 6]'$ to the final point $[6\ 5\ 4\ 3\ 2\ 1]'$ during the time interval $[0, 10]$, by applying the controllers, i.e. thrusters $u_1(t)$ and $u_2(t)$ in θ and z direction.

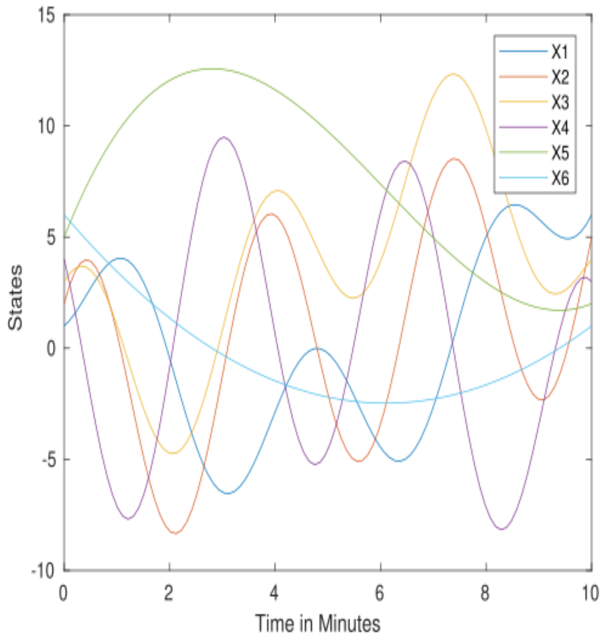

Figure 4.1: State Control of the System under the effect of zonal harmonic J_2

The graph of the controllers i.e. thrusters $u_1(t)$ and $u_2(t)$ in θ are shown in the figure-2:

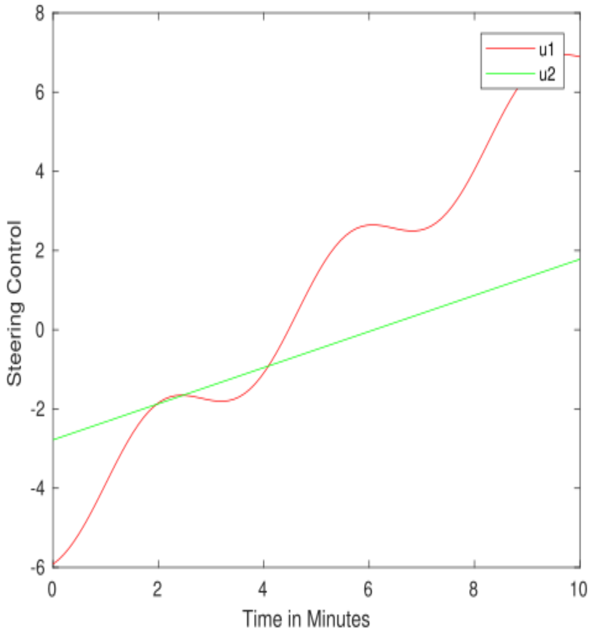

Figure 4.2: Steering Control of the System under the effect of zonal harmonic J_2

4.3.7 If we add the thrusters, in all the three directions i.e. r, θ and z directions

The system (4.3.2) is written as

$$\begin{aligned}
\ddot{r} - r\dot{\theta}^2 &= -\mu r \left[\frac{1}{(r^2+z^2)^{3/2}} + \frac{3R^2 J_2 (r^2 - 4z^2)}{2(r^2+z^2)^{7/2}} \right] + u_1(t), \\
r\ddot{\theta} + 2\dot{r}\dot{\theta} &= u_2(t), \\
\ddot{z} &= -\mu z \left[\frac{1}{(r^2+z^2)^{3/2}} + \frac{3R^2 J_2 (3r^2 - 2z^2)}{2(r^2+z^2)^{7/2}} \right] + u_3(t).
\end{aligned} \tag{4.3.19}$$

By transformation (4.3.3), the system (4.3.19) takes the form

$$
\begin{aligned}
\frac{dX_1}{dt} &= X_2,\\
\frac{dX_2}{dt} &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu\,(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}}\right.\\
&\qquad\left. + \frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_1(t),\\
\frac{dX_3}{dt} &= X_4,\\
\frac{dX_4}{dt} &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)} + u_2(t),\\
\frac{dX_5}{dt} &= X_6,\\
\frac{dX_6}{dt} &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[3\,(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_3(t).
\end{aligned}
\tag{4.3.20}
$$

Now we linearize the system (4.3.20) about origin, we take

$$
\begin{aligned}
f_1 &= X_2,\\
f_2 &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu\,(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}}\right.\\
&\qquad\left. + \frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_1(t),\\
f_3 &= X_4\\
f_4 &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)} + u_2(t),\\
f_5 &= X_6,\\
f_6 &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[3\,(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_3(t),
\end{aligned}
$$

and write the system (4.3.20) in the form

$$\dot{X} = AX + BU,$$

where $\dot{X} = \begin{bmatrix} \frac{dX_1}{dt} & \frac{dX_2}{dt} & \frac{dX_3}{dt} & \frac{dX_4}{dt} & \frac{dX_5}{dt} & \frac{dX_6}{dt} \end{bmatrix}'$, $A = \begin{bmatrix} \frac{\partial(f_1, f_2, f_3, f_4, f_5, f_6)}{\partial(X_1, X_2, X_3, X_4, X_5, X_6)} \end{bmatrix}$, at origin, $X = \begin{bmatrix} X_1 \; X_2 \; X_3 \; X_4 \; X_5 \; X_6 \end{bmatrix}'$, $B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_2}{\partial u_1} & \frac{\partial f_3}{\partial u_1} & \frac{\partial f_4}{\partial u_1} & \frac{\partial f_5}{\partial u_1} & \frac{\partial f_6}{\partial u_1} \\ \frac{\partial f_1}{\partial u_2} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_4}{\partial u_2} & \frac{\partial f_5}{\partial u_2} & \frac{\partial f_6}{\partial u_2} \\ \frac{\partial f_1}{\partial u_3} & \frac{\partial f_2}{\partial u_3} & \frac{\partial f_3}{\partial u_3} & \frac{\partial f_4}{\partial u_3} & \frac{\partial f_5}{\partial u_3} & \frac{\partial f_6}{\partial u_3} \end{bmatrix}'$, at origin and $u = \begin{bmatrix} u_1 \; u_2 \; u_3 \end{bmatrix}'$. The matrix A is given by (4.3.6) and $B = \begin{bmatrix} 0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1 \end{bmatrix}'$. The controllability matrix $Q = \begin{bmatrix} B \; AB \; A^2B \; A^3B \; A^4B \; A^5B \end{bmatrix}$ is given by

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & -3 & 0 & 0 & 0 & -6 & 0 & 9 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & -3 & 0 & 0 & 0 & -6 & 0 & 9 & 0 & 0 & 0 & 18 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & -4 & 0 & 6 & 0 & 0 & 0 & 12 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 & -4 & 0 & 6 & 0 & 0 & 0 & 12 & 0 & -18 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of matrix Q is 6, which is equal to the dimensions of the state X $(= 6)$. Hence by Kalman's condition, the system is controllable if we add the thrusters in r, θ, and z direction. Figure-3 shows the trajectories of states of the system (4.3.19) with initial state $[1\ 2\ 3\ 4\ 5\ 6]'$ and desired final state $[6\ 5\ 4\ 3\ 2\ 1]'$ respectively.

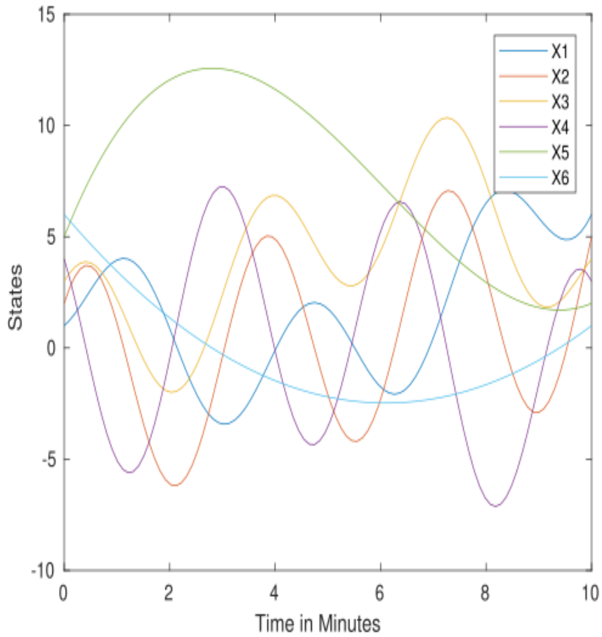

Figure 4.3: State Control of the System under the effect of zonal harmonic J_2

We can see from Figure 3 that the initial state is steered to the final state during the time interval $[0, 10]$. The graph of the controllers i.e. thrusters in all three directions r, θ, and z are shown in figure-4.

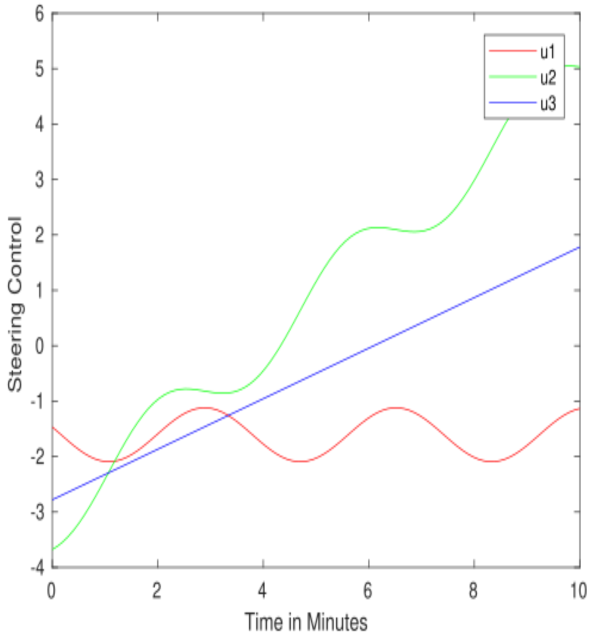

Figure 4.4: Steering Control of the System under the effect of zonal harmonic J_2

4.4 Conclusion

We have studied controllability analysis for seven different cases by applying controllers in (1) r- direction, (2) θ- direction, (3) z- direction, (4) r and θ directions, (5) r and z directions, (6) θ and z directions and (7) r, θ and z directions. Applying Kalman's rank condition we found that, the system (4.3.2) is uncontrollable if we apply thrusters i.e controllers in (1) r- direction, (2) θ- direction, (3) z- direction, (4) r and θ directions, (5) r and z directions, and it is controllable if thrusters are applied in (6) θ and z directions and (7) r, θ and z directions.

From this study, we found that to control the motion of the satellite under the effect of zonal harmonic J_2 we need to plug the controllers in the form of thrusters in all three directions. If the thruster in r direction fails then also the motion of the satellite is controllable, but if the thruster in any other direction(s) fails then the motion of the satellite will become uncontrollable and it may hit the Earth's surface.