Chapter 6

Shrinkage Testimation in Weibull Distribution

Chapter – 6

SHRINKAGE TESTIMATION IN WEIBULL DISTRIBUTION

6.1 Introduction

Weibull distribution is a continuous distribution. It is named after Swedish physicist Walodi Weibull (1939). He used this distribution to model data from problems dealing with yield strength of Bofor's Steel, fibre strength of Indian Cotton etc. In the context of life testing and reliability estimation this model fits well for the situations with changing failure rates i.e. when the failure rates increasing or decreasing. The Weibull distribution interpolates between Exponential distribution when $\beta = 1$ and a Rayleish distribution when $\beta = 2$. As 'β' converges to infinity the Weibull distribution converges to Dirac Delta distribution. This distribution is very widely used in Survival Analysis, Reliability and Engineering and Industrial Engineering. The Weibull distribution is alos useful in describing wear out, fatigue failure, vaccum tube failures, ball bearing failures etc.

In weather forecasting to describe wind speed distributions it is extensively used as the shape parameter of this distribution matches with natural distribution. Also, in general insurance to model the size of re-insurance claims, this model is appropriate.

In hydrology the Weibull distribution is applied to extreme events such as annual maximum one-day rainfalls and river discharges.

THE MODEL:

When we assume that some power (say) p^{th} of the failure time is distributed Exponentially, we get the Weibull distribution whose pdf is given by

$$f(x;\ \theta,\beta) = \begin{cases} \left(\frac{\beta}{\theta}\right)\left(\frac{x}{\theta}\right)^{\beta-1}\exp\left(-\left(\frac{x}{\theta}\right)^{\beta}\right), & x > 0,\ \theta,\beta > 0 \\ 0 & , o.w. \end{cases}$$

_____(6.1.1)

The parameters θ and β are called the life and shape parameter respectively. This distribution is also useful in describing the wear out or fatigue failures. Cohen (1965), Harter and Moore (1965) have derived maximum likelihood estimators in Weibull distribution based on Complete and Censored samples. Bain and Antle (1967) have given estimators which are practical and based on Monte - Carlo methods. Mann and Fertig (1975) have derived simplified efficient point and interval estimator for Weibull distribution parameters, which are either maximum likelihood or best linear invariant.

Singh and Bhatkulikar (1978) have studied shrunken estimators in Weibull distribution, Pandey and Singh (1984) considered estimating the shape parameter of this distribution by shrinkage towards an interval.

Due to considerable handling of manufactured items or past information as in the case of wind speed data and in many other situations in life testing and reliability estimation where one may have an initial estimate of the shape parameter β in the form of either a guess β_0 or an interval (β_1, β_2) $(\beta_1 < \beta_2)$ in which β lies. We have proposed shrinkage testimators for the shape parameter both using point and interval guess. We have studied the risk properties of the proposed estimator using an asymmetric loss function.

In section 6.2 two shrinkage testimators for the shape parameter using point guess have been proposed and their risks are derived in section 6.3. In section 6.4 we have derived the relative risks of the proposed testimators with respect to $\hat{\beta}$ the best available estimator in the absence of any other information. Section 6.5 is devoted to the recommendations and conclusions of $\hat{\beta}_{ST_1}$ and $\hat{\beta}_{ST_2}$.

In section 6.6 deals with the shrinkage testimation of 'β' by shrinkage towards an interval. Section 6.7 is devoted to the derivation of the risk of the proposed testimator and its relative risk is derived in section 6.8. Section 6.9 gives the recommendations and conclusions of the proposed testimator.

6.2 SHRINKAGE TESTIMATOR USING POINT GUESS

Let x have the distribution

$$f(x;\ \theta,\beta) = \left(\frac{\beta}{\theta}\right)\left(\frac{x}{\theta}\right)^{\beta-1} \exp\left(-\left(\frac{x}{\theta}\right)^{\beta}\right), \qquad x > 0,\ \ \theta,\beta > 0$$

Suppose that a guess of β (say) β_0 is given and a random sample of size 'n' ($x_1, x_2 \ldots\ldots, x_n$) is available from this distribution and we are interested in constructing an estimator of β using the sample information and hopefully the guess value β_0. Let $x_1 \leq x_2 \ldots\ldots \leq x_r$ denote the r smallest ordered observations in a sample of size n from the Weibull distribution.

Then, the shrinkage Testimator of β (say) $\hat{\beta}_{ST_1}$ can be proposed as follows:

1. First test with a sample of size 'n' the null hypothesis $H_0 : \beta = \beta_0$ against the alternative $H_1 : \beta \neq \beta_0$ where β_0 is the point guess value of β.
2. If H_0 is accepted at α % level of significance i.e. $\chi_1^2 < \frac{2\,\mathrm{Tr}}{\beta_0} < \chi_2^2$, where $Tr = -\sum(x_{(i)} - x_{(r)})$; $N = n\, k_{r,n}$ and χ_1^2 , χ_2^2 refer to critical points of

unbiased portioning of χ^2 distribution with 2N degrees of freedom, N = $n\,k_{r,n}$. Then use the conventional shrinkage estimator $\hat{\beta}_s = k\hat{\beta} + (1-k)\beta_0$ with shrinkage factor 'k' otherwise ignore β_0 and use an unbiased estimator $\hat{\beta}$ of β.

Estimators of this type with 'k' arbitrary and lying between '0' and '1' have been proposed by Singh and Bhatkulikar (1978) and have calculated relative efficiencies with respect to $\hat{\beta}$. In the present study we have studied the risk properties of shrinkage testimators of β using an asymmetric loss function.

We have proposed another testimator $\hat{\beta}_{ST_2}$ which removes the arbitrariness in the choice of a shrinkage factor for a given 'α'. Also, in all such studies we have considered a two sided alternative. i.e. $H_1 : \beta \neq \beta_0$ which appears more appropriate for shrinkage problems. It is to be noted that a UMP test of $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$ does not exist. However, a UMPU test of an equivalent hypothesis $H_0 : b = 1$ against $H_1 : b \neq 1$ exist and is to reject H_0 whenever 2Tr $< \chi_1^2$ and 2Tr $> \chi_2^2$. So, we define

$$\hat{\beta}_{ST_1} = \begin{cases} k\left(\frac{N-1}{Tr} - 1\right) + 1\,, & if\ H_0\ is\ accepted \\ \frac{N-1}{Tr}\,, & otherwise \end{cases} \qquad \text{\_\_\_\_(6.2.1)}$$

Where $0 \leq k \leq 1$. Again, we take $k = \frac{2\,Tr}{\chi^2}$ and define another shrinkage testimator $\hat{\beta}_{ST_2}$ of β as follows:

$$\hat{\beta}_{ST_2} = \begin{cases} \frac{2\,Tr}{\chi^2}\left(\frac{N-1}{Tr} - 1\right) + 1\,, & if\ H_0\ is\ accepted \\ \frac{N-1}{Tr}\,, & otherwise \end{cases} \qquad \text{\_\_\_\_(6.2.2)}$$

6.3 Risk of Testimators

In this section we derive the risk of these two testimators which are defined in the previous section.

6.3.1 Risk of $\hat{\beta}_{ST_1}$

The risk of $\hat{\beta}_{ST_1}$ under L(Δ) is defined by

$$R(\hat{\beta}_{ST_1}) = E[\hat{\beta}_{ST_1} \mid L\,(\Delta)]$$

$$= E\left[\left(k\left(\tfrac{N-1}{Tr}-1\right)+1\right)/\tfrac{\chi_1^2}{2} < Tr < \tfrac{\chi_2^2}{2}\right]\cdot P\left[\tfrac{\chi_1^2}{2} < Tr < \tfrac{\chi_2^2}{2}\right]$$

$$+ E\left[\left(\tfrac{N-1}{Tr}\right)/Tr < \tfrac{\chi_1^2}{2} \cup Tr > \tfrac{\chi_2^2}{2}\right]\cdot P\left[Tr < \tfrac{\chi_1^2}{2} \cup Tr > \tfrac{\chi_2^2}{2}\right]$$

(6.3.1.1)

$$= e^{-a}\int_{\frac{\chi_1^2}{2}}^{\frac{\chi_2^2}{2}} e^{a\left(\frac{k\left(\frac{N-1}{Tr}-1\right)+1}{\beta}\right)} f(Tr)dTr - a\int_{\frac{\chi_1^2}{2}}^{\frac{\chi_2^2}{2}}\left[\frac{k\left(\frac{N-1}{Tr}-1\right)+1}{\beta}-1\right] f(Tr)dTr$$

$$-\int_{\frac{\chi_1^2}{2}}^{\frac{\chi_2^2}{2}} f(Tr)dTr + e^{-a}\int_0^{\frac{\chi_1^2}{2}} e^{\left(\frac{N-1}{Tr}\right)} f(Tr)dTr + e^{-a}\int_{\frac{\chi_2^2}{2}}^{\infty} e^{\left(\frac{N-1}{Tr}\right)} f(Tr)dTr$$

$$-a\int_0^{\frac{\chi_1^2}{2}}\left(\frac{N-1}{Tr}-1\right) f(Tr)dTr - a\int_{\frac{\chi_2^2}{2}}^{\infty}\left(\frac{N-1}{Tr}-1\right) f(Tr)dTr - \int_0^{\frac{\chi_1^2}{2}} f(Tr)dTr - \int_{\frac{\chi_2^2}{2}}^{\infty} f(Tr)dTr$$

(6.3.1.2)

$$f(Tr) = \frac{1}{b^N \Gamma(N)} e^{\frac{-Tr}{b}} (Tr)^{N-1} dTr \quad ; Tr > 0,\ b > 0$$

Straight forward integration of (6.3.1.2) gives

$$R(\hat{\beta}_{ST_1}) = \left[\begin{array}{c} I_1 - \left\{ I\left(\frac{\chi_2^2}{2b}, N-1\right) - I\left(\frac{\chi_1^2}{2b}, N-1\right) \right\} ak \\ + \left\{ I\left(\frac{\chi_2^2}{2b}, N\right) - I\left(\frac{\chi_1^2}{2b}, N\right) \right\} a(kb - b + 1) \\ - \left\{ I\left(\frac{\chi_2^2}{2b}, N\right) - I\left(\frac{\chi_1^2}{2b}, N\right) \right\} + I_2 + I_3 - \\ a \left\{ I\left(\frac{\chi_1^2}{2b}, N-1\right) - I\left(\frac{\chi_2^2}{2b}, N-1\right) + 1 \right\} \\ + a \left\{ I\left(\frac{\chi_1^2}{2b}, N\right) - I\left(\frac{\chi_2^2}{2b}, N\right) + 1 \right\} \\ - \left\{ I\left(\frac{\chi_1^2}{2b}, N\right) - I\left(\frac{\chi_2^2}{2b}, N\right) + 1 \right\} \end{array} \right]$$

_____(6.3.1.3)

Where $I(x;p) = (1/\Gamma p) \int_0^x e^{-x} x^{p-1} dx$ refers to the standard incomplete gamma function and b = (1/β) and

$$I_1 = \frac{e^{a(b-1)}}{\Gamma(N)} e^{-abk} \int_{\frac{x_1^2}{2b}}^{\frac{x_2^2}{2b}} e^{\left[\frac{ak(N-1)}{t}\right]} e^{-t} (t)^{N-1} dt$$

$$I_2 = \frac{e^{-a}}{\Gamma(N)} \int_0^{\frac{x_1^2}{2b}} e^{\left[\frac{a(N-1)}{t}\right]} e^{-t} (t)^{N-1} dt$$

$$I_3 = \frac{e^{-a}}{\Gamma(N)} \int_{\frac{\chi_2^2}{2b}}^{\infty} e^{\left[\frac{a(N-1)}{t}\right]} e^{-t} (t)^{N-1} dt$$

6.3.2 Risk of $\hat{\beta}_{ST_2}$

The risk of $\hat{\beta}_{ST_2}$ under L(Δ) is defined by

$$R(\hat{\beta}_{ST_2}) = E[\hat{\beta}_{ST_2} \mid L(\Delta)]$$

$$= E\left[\left(\frac{2Tr}{\chi^2}\left(\frac{N-1}{Tr} - 1\right) + 1\right) / \frac{\chi_1^2}{2} < Tr < \frac{\chi_2^2}{2}\right] \cdot P\left[\frac{\chi_1^2}{2} < Tr < \frac{\chi_2^2}{2}\right]$$

$$+ E\left[\left(\frac{N-1}{Tr}\right) / Tr < \frac{\chi_1^2}{2} \cup Tr > \frac{\chi_2^2}{2}\right] \cdot P\left[Tr < \frac{\chi_1^2}{2} \cup Tr > \frac{\chi_2^2}{2}\right]$$

______(6.3.2.1)

$$= e^{-a} \int_{\frac{\chi_1^2}{2}}^{\frac{\chi_2^2}{2}} e^{a\left(\frac{\frac{2Tr}{\chi^2}\left(\frac{N-1}{Tr}-1\right)+1}{\beta}\right)} f(Tr)dTr - a \int_{\frac{\chi_1^2}{2}}^{\frac{\chi_2^2}{2}} \left[\frac{\frac{2Tr}{\chi^2}\left(\frac{N-1}{Tr}-1\right)+1}{\beta} - 1\right] f(Tr)dTr$$

$$- \int_{\frac{\chi_1^2}{2}}^{\frac{\chi_2^2}{2}} f(Tr)dTr + e^{-a} \int_0^{\frac{\chi_1^2}{2}} e^{\left(\frac{N-1}{Tr}\right)} f(Tr)dTr + e^{-a} \int_{\frac{\chi_2^2}{2}}^{\infty} e^{\left(\frac{N-1}{Tr}\right)} f(Tr)dTr$$

$$- a \int_0^{\frac{\chi_1^2}{2}} \left(\frac{N-1}{Tr} - 1\right) f(Tr)dTr - a \int_{\frac{\chi_2^2}{2}}^{\infty} \left(\frac{N-1}{Tr} - 1\right) f(Tr)dTr - \int_0^{\frac{\chi_1^2}{2}} f(Tr)dTr - \int_{\frac{\chi_2^2}{2}}^{\infty} f(Tr)dTr$$

______(6.3.2.2)

$$f(Tr) = \frac{1}{b^N \Gamma(N)} e^{\frac{-Tr}{b}} (Tr)^{N-1} dTr \quad ; Tr > 0,\ b > 0$$

Straight forward integration of (6.3.2.2) gives

$$R(\hat{\beta}_{ST_2}) = \left[\begin{array}{c} \dfrac{e^{a(b-1)}}{\left[1+\dfrac{2ab^2}{\chi^2}\right]^N} e^{\left(\frac{2ab\,(N-1)}{\chi^2}\right)} \left\{ I\left(\dfrac{\chi_2^2}{2b}, N\right) - I\left(\dfrac{\chi_1^2}{2b}, N\right) \right\} \\ -\left\{ I\left(\dfrac{\chi_2^2}{2b}, N\right) - I\left(\dfrac{\chi_1^2}{2b}, N\right) \right\} a \left(\dfrac{2b(N-1)}{\chi^2} + b - 1 \right) \\ +\left\{ I\left(\dfrac{\chi_2^2}{2b}, N+1\right) - I\left(\dfrac{\chi_1^2}{2b}, N+1\right) \right\} \left(\dfrac{2ab^2 N}{\chi^2} \right) \\ -\left\{ I\left(\dfrac{\chi_2^2}{2b}, N\right) - I\left(\dfrac{\chi_1^2}{2b}, N\right) \right\} + I_1 + I_2 \\ -a\left\{ I\left(\dfrac{\chi_1^2}{2b}, N-1\right) - I\left(\dfrac{\chi_2^2}{2b}, N-1\right) + 1 \right\} \\ +a\left\{ I\left(\dfrac{\chi_1^2}{2b}, N\right) - I\left(\dfrac{\chi_2^2}{2b}, N\right) + 1 \right\} \\ -\left\{ I\left(\dfrac{\chi_1^2}{2b}, N\right) - I\left(\dfrac{\chi_2^2}{2b}, N\right) + 1 \right\} \end{array} \right]$$

_____(6.3.2.3)

Where $I(x;p) = (1/\Gamma p) \int_0^x e^{-x} x^{p-1}\, dx$ refers to the standard incomplete gamma function and b = (1/β) ; N = $n\, k_{r,n}$ and

$$I_1 = \frac{e^{-a}}{\Gamma(N)} \int_0^{\frac{x_1^2}{2b}} e^{\left[\frac{a(N-1)}{t}\right]} e^{-t}\, (t)^{N-1}\, dt$$

$$I_2 = \frac{e^{-a}}{\Gamma(N)} \int_{\frac{x_2^2}{2b}}^{\infty} e^{\left[\frac{a(N-1)}{t}\right]} e^{-t}\, (t)^{N-1}\, dt$$

6.4 Relative Risk of $\hat{\beta}_{ST_i}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator $\hat{\beta}$ in this case. For this purpose, we obtain the risk of $\hat{\beta}$ under $L_E(\hat{\beta},\beta)$ as:

$$R_E(\hat{\beta}) = E[\hat{\beta} \mid L(\hat{\beta},\beta)]$$

$$= e^{-a}\int_0^{\infty} e^{a\left(\frac{\hat{\beta}}{\beta}\right)} f(Tr)dTr - a\int_0^{\infty}\left(\frac{\hat{\beta}}{\beta}-1\right)f(Tr)dTr - \int_0^{\infty} f(Tr)dTr \qquad \text{\_\_\_\_(6.4.1)}$$

A straight forward integration of (6.4.1) gives

$$R_E(\hat{\beta}) = \frac{e^{-a}}{\Gamma(N)}\int_0^{\infty} e^{a\left(\frac{N-1}{t}\right)} e^{-t}\, t^{(N-1)}dt \ - 1 \qquad \text{\_\_\_\_\_(6.4.2)}$$

Now, we define the Relative Risk of $\hat{\beta}_{ST_i}$; $i = 1,2$ with respect to $\hat{\beta}$ under L(Δ) as follows

$$RR_1 = \frac{R_E(\hat{\beta})}{R(\hat{\beta}_{ST_1})} \qquad \text{\_\_\_\_\_(6.4.3)}$$

Using (6.4.2) and (6.3.1.3) the expression for RR_1 is given by (6.4.3). It is observed that RR_1 is a function of N, α, k, b and 'a'.

Again, we define the Relative Risk of $\hat{\beta}_{ST_2}$ by

$$RR_2 = \frac{R_E(\hat{\beta})}{R(\hat{\beta}_{ST_2})} \qquad \text{\_\_\_\_\_\_(6.4.4)}$$

The expression for RR_2 is given by (6.4.4) which can be obtained by using (6.4.2) and (6.3.2.3). Again we observed that RR_2 is a function of N, α, k, b and 'a'.

We have calculated N from the values of $k_{r,n}$ given in Bain (1972). For n = 10, 20, 30 the values of N are respectively.

$^r/_n$	0.2	0.3	0.4	0.5	0.6
N	1.0540	2.1720	3.3690	4.6670	6.0980
N	3.1666	5.4420	7.8880	10.5540	13.5120
N	5.2770	8.7120	12.4110	16.4460	20.9370

6.5 Recommendations for $\widehat{\beta}_{ST_i}$

In this section we wish to compare the performance of $\hat{\beta}_{ST_1}$ and $\hat{\beta}_{ST_2}$ with respect to the best available (unbiased) estimator of $\hat{\beta}$.

6.5.1 Recommendations for $\widehat{\beta}_{ST_1}$

The risk of $\hat{\beta}_{ST_1}$ with respect to $\hat{\beta}$ the best available estimator of β is a function of 'k', a, b, 'α', N (i.e. n *and* $k_{r,n}$). For different values of $^r/_n$ these values are given by Bain (1972). Now we have taken k = 0.2 (0.2) 0.8, a = -0.5, -0.75, -1.0, -1.25, -1.50 and b = 0.2 (0.2) 1.8, N is given in table and α = 1% and 5%.

For a = -1.5, it performs better for the whole range of 'k' and 'b' considered here. The values of RR_1 are better for almost all the values of N obtained for different censoring fractions however as N increases i.e. '$^r/_n$' increases there. We observe that RR_1 values are still better. Implying that one can take larger censoring factions. However for other values of 'a' up to – 0.5 $\hat{\beta}_{ST_1}$ performs better than $\hat{\beta}$ but the magnitude of RR_1 values decrease slightly (but greater than unity).

When we change 'α' to 5% the similar kind of pattern of RR_1 values is observed as observed for α =1% . But the magnitude of relative risk values lowers down but still greater than unity for the whole range of shrinkage factor (i.e. $0.2 \leq k \leq 0.8$) and the whole range of 'b' i.e. b = 0.2 (0.2) 1.8. Again changing the censoring fraction to some higher values, it is observed that the magnitude of RR_1 increases with larger values of 'N' i.e. again it may be suggested that a larger censoring faction can be considered.

We have also tried some positive values of 'a' the degree of asymmetry but it is observed that RR_1 values are not good in such situations indicating that the asymmetric loss function is more useful for those situations where under-estimation is more serious.

6.5.2 Recommendations for $\widehat{\beta}_{ST_2}$

It is observed that RR_2 is a function of 'a', 'b', 'α' and N (i.e. n *and* $k_{r,n}$) we have considered a= -0.5, -0.75, -1.0, -1.25, -1.5 b= 0.2 (0.2) 1.8, N is again tabulated for different values of $^r/_n$ in table and α = 1% and α = 5%. There will be several tables for RR_2 values for the above values considered. We have assembled some of the tables at the end of the chapter. However our recommendations based on all the computations are as follows.

1. $\hat{\beta}_{ST_2}$ performs better than $\hat{\beta}$ for almost all the values considered as above. However for a = - 1.5 and α = 1% its performance is at its best for the whole range of 'b'.

2. When the value of $\alpha = 5\%$ still the performance of $\hat{\beta}_{ST_2}$ is good though there is a slight decrease to in the values of RR_2.

3. As N increases (i.e. r/n increases) implying that a higher censoring fraction is admissible the RR_2 values increase indicating a better control over the risk of $\hat{\beta}_{ST_2}$. This is in contrast to the behaviour of MSE of $\hat{\beta}$ under the 'SELF'. As the recommendations with MSE criterion is to use small censoring fraction.

4. As the degree of asymmetry becomes positive the RR_2 values are not good in the sense that they are less in magnitude even lesser than unity. So, it is suggested that only negative degree(s) of asymmetries be considered.

5. RR_2 values are higher in magnitude compared to RR_1.

<u>CONCLUSIONS:</u>

We have proposed two shrinkage testimators $\hat{\beta}_{ST_1}$ and $\hat{\beta}_{ST_2}$ for the shape parameter β . It is suggested to use $\hat{\beta}_{ST_1}$ as it (k = k) performs better than the shrinkage factor dependent on test statistics i.e. $\hat{\beta}_{ST_2}$. In particular smaller level of significance i.e. α =1% coupled with proper censoring fraction is recommended for various degrees of asymmetry. The best performance is achieved at a = -1.5.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 6.5.1.1 **Relative Risk of $\widehat{\beta}_{ST_1}$** $\alpha = 1\%$, $n = 10$, k = 0.2

b	a = -0.5	a = -0.75	a = -1	a = -1.25	a = -1.5
0.2	1.016	1.048	1.076	1.101	1.126
0.4	1.099	1.176	1.253	1.333	1.416
0.6	1.174	1.297	1.427	1.569	1.724
0.8	1.233	1.395	1.573	1.771	1.997
1.0	1.272	1.459	1.666	1.902	2.175
1.2	1.287	1.48	1.695	1.94	2.223
1.4	1.278	1.461	1.663	1.891	2.151
1.6	1.249	1.411	1.586	1.783	2.004
1.8	1.205	1.339	1.484	1.646	1.828

Table : 6.5.1.2 **Relative Risk of $\widehat{\beta}_{ST_1}$** $\alpha = 1\%$, $n = 10$, k = 0.4

b	a = -0.5	a = -0.75	a = -1	a = -1.25	a = -1.5
0.2	1.009	1.037	1.06	1.081	1.101
0.4	1.073	1.135	1.195	1.254	1.315
0.6	1.134	1.231	1.329	1.433	1.543
0.8	1.187	1.316	1.453	1.602	1.766
1.0	1.229	1.385	1.554	1.743	1.956
1.2	1.257	1.431	1.623	1.838	2.085
1.4	1.271	1.452	1.652	1.878	2.136
1.6	1.271	1.449	1.644	1.864	2.112
1.8	1.258	1.425	1.606	1.807	2.032

Table : 6.5.1.3 **Relative Risk of $\widehat{\beta}_{ST_1}$** $\alpha = 5\%$, $n = 10$, k = 0.2

b	a = -0.5	a = -0.75	a = -1	a = -1.25	a = -1.5
0.2	1.026	1.062	1.094	1.124	1.153
0.4	1.095	1.173	1.25	1.329	1.414
0.6	1.158	1.276	1.402	1.538	1.689
0.8	1.205	1.355	1.52	1.705	1.916
1.0	1.23	1.397	1.583	1.793	2.033
1.2	1.232	1.4	1.584	1.791	2.025
1.4	1.214	1.368	1.535	1.719	1.923
1.6	1.181	1.313	1.455	1.608	1.776
1.8	1.138	1.246	1.361	1.485	1.621

Table : 6.5.1.4 **Relative Risk of $\widehat{\beta}_{ST_2}$** $\alpha = 5\%$, $n = 20$

b	a = -0.5	a = -0.75	a = -1	a = -1.25	a = -1.5
0.2	0. 633	0.594	0.558	0.526	0.497
0.4	0.859	0.834	0.808	0.783	0.76
0.6	1.438	1.474	1.489	1.495	1.496
0.8	2.497	2.744	2.935	3.094	3.234
1.0	3.729	4.3	4.782	5.212	5.611
1.2	3.597	3.91	4.12	4.279	4.411
1.4	2.445	2.456	2.444	2.433	2.429
1.6	1.574	1.525	1.487	1.464	1.453
1.8	1.078	1.033	1.005	0.991	0.986

Table : 6.5.2.1 **Relative Risk of $\widehat{\beta}_{ST_2}$** $\alpha = 1\%$, $n = 10$

b	a = -0.5	a = -0.75	a = -1	a = -1.25	a = -1.5
0.2	1.02	1.053	1.082	1.11	1.137
0.4	1.108	1.192	1.276	1.363	1.455
0.6	1.181	1.31	1.446	1.593	1.757
0.8	1.23	1.39	1.563	1.756	1.974
1.0	1.25	1.422	1.609	1.818	2.053
1.2	1.235	1.4	1.576	1.768	1.977
1.4	1.188	1.329	1.472	1.617	1.758
1.6	1.113	1.218	1.31	1.378	1.389
1.8	1.018	1.077	1.099	1.033	0.71

Table : 6.5.2.2 **Relative Risk of $\widehat{\beta}_{ST_2}$** $\alpha = 5\%$, $n = 10$

b	a = -0.5	a = -0.75	a = -1	a = -1.25	a = -1.5
0.2	1.034	1.074	1.109	1.143	1.177
0.4	1.106	1.188	1.27	1.356	1.446
0.6	1.159	1.275	1.397	1.529	1.674
0.8	1.18	1.312	1.454	1.609	1.778
1.0	1.158	1.286	1.42	1.56	1.706
1.2	1.095	1.198	1.295	1.38	1.443
1.4	1	1.061	1.094	1.074	0.948
1.6	0.883	0.887	0.817	0.58	0.515
1.8	0.755	0.679	0.422	0.44	0.416

6.6 Shrinkage Testimator of β by Shrinkage towards an Interval

Suppose 'n' items are put to life test and the experiment is continued until 'r' failures are observed. Let these failure times be $x_1, x_2 ...., x_r$ and suppose further that they follow Weibull distribution. Then following Thompson (1968,b) a shrinkage of β can be defined as

$$\hat{\beta}_{ST_1} = k\left(\frac{h-2}{t}\right) + (1-k)\left(\frac{\beta_1+\beta_2}{2}\right)$$

which was proposed by Pandey and Singh (1984). The properties of this estimator were studied by minimizing the Mean Square Error.

We propose another shrunken estimator of β as follows:

$$\hat{\beta}_{ST_3} = \begin{cases} \beta_1 & , \ if \ t > \dfrac{h-2}{\beta_1} \\ k\left(\dfrac{h-2}{t}\right) + (1-k)\left(\dfrac{\beta_1+\beta_2}{2}\right) & , if \ \dfrac{h-2}{\beta_2} \le t \le \dfrac{h-2}{\beta_1} \\ \beta_2 & , \ if \ t < \dfrac{h-2}{\beta_2} \end{cases}$$

Where (β_1, β_2) $(\beta_2 < \beta_1)$ is the guess interval in which β is supposed to lie. So, here we have proposed a shrinkage testimator for the shape parameter of Weibull distribution by shrinking the guess towards an interval. The shrinkage factor 'k' lies between '0' and '1'. We have derived the risk of $\hat{\beta}_{ST_3}$ under an asymmetric loss function. L(Δ) which is defined and discussed earlier.

As Weibull distribution finds applications in many real life problems. This estimation procedure could find its place whenever the guess value of β lies in some interval. One these could be estimation of pollutants in water (say) the arsenic contents in the water lies in some interval (β_1, β_2) and then utilizing this guess interval a better estimate of β can be proposed along the lines discussed in the chapter. The use of asymmetric loss will facilitate a proper control over the 'risk' by choosing the degrees of asymmetry appropriately.

6.7 Risk of Testimators

The risk of $\hat{\beta}_{ST_3}$ under $L_E(\hat{\beta}, \beta)$ is defined by

$$R(\hat{\beta}_{ST_3}) = E[\hat{\beta}_{ST_3} | L_E(\hat{\beta}, \beta)]$$

$$= E\left[\beta_1 \Big/ t > \frac{h-2}{\beta_1}\right] \cdot p\left[t > \frac{h-2}{\beta_1}\right]$$
$$+ E\left[k\left(\frac{h-2}{t}\right) + (1-k)\left(\frac{\beta_1+\beta_2}{2}\right) \Big/ \frac{h-2}{\beta_2} \le t \le \frac{h-2}{\beta_1}\right] \cdot p\left[\frac{h-2}{\beta_2} \le t \le \frac{h-2}{\beta_1}\right]$$
$$+ E\left[\beta_2 \Big/ t < \frac{h-2}{\beta_2}\right] \cdot p\left[t < \frac{h-2}{\beta_2}\right]$$

______(6.7.1)

$$= e^{-a} \int_{\frac{h-2}{\chi_1^2}}^{\infty} e^{a\left(\frac{\beta_1}{\beta}\right)} f(t)dt - a \int_{\frac{h-2}{\chi_1^2}}^{\infty} \left[\frac{\beta_1}{\beta} - 1\right] f(t)dt - \int_{\frac{h-2}{\chi_1^2}}^{\infty} f(t)dt$$

$$e^{-a} \int_{\frac{h-2}{\chi_1^2}}^{\frac{h-2}{\chi_2^2}} e^{a\left(\frac{k\left(\frac{h-2}{t}\right)+(1-k)\left(\frac{\beta_1+\beta_2}{2}\right)}{\beta}\right)} f(t)dt - a \int_{\frac{h-2}{\chi_1^2}}^{\frac{h-2}{\chi_2^2}} \left[\frac{k\left(\frac{h-2}{t}\right) + (1-k)\left(\frac{\beta_1+\beta_2}{2}\right)}{\beta} - 1\right] f(t)dt$$

$$- \int_{\frac{h-2}{\chi_1^2}}^{\frac{h-2}{\chi_2^2}} f(t)dt + e^{-a} \int_0^{\frac{h-2}{\chi_2^2}} e^{a\left(\frac{\beta_2}{\beta}\right)} f(t)dt - a \int_0^{\frac{h-2}{\chi_2^2}} \left[\frac{\beta_2}{\beta} - 1\right] f(t)dt - \int_0^{\frac{h-2}{\chi_2^2}} f(t)dt$$

______(6.7.2)

Where $f(t) = \dfrac{\beta^{\left(\frac{h}{2}\right)} t^{\left(\frac{h}{2}-1\right)}}{2^{\left(\frac{h}{2}\right)} \Gamma\left(\frac{h}{2}\right)} e^{\left(-\frac{\beta t}{2}\right)} dt \ , \ t > 0$

Straight forward integration of (6.7.2) gives

$$R(\hat{\beta}_{ST_3}) = \begin{bmatrix} \left\{1 - I\left(\frac{h-2}{\chi_1^2}, \frac{h}{2}\right)\right\}\left[e^{a\left(\frac{\beta_1}{\beta}-1\right)} - a\left(\frac{\beta_1}{\beta} - 1\right) - 1\right] \\ + \left\{I\left(\frac{h-2}{\chi_2^2}, \frac{h}{2}\right)\right\}\left[e^{a\left(\frac{\beta_2}{\beta}-1\right)} - a\left(\frac{\beta_2}{\beta} - 1\right) - 1\right] \\ + \left\{I\left(\frac{h-2}{\chi_1^2}, \frac{h}{2}\right) - I\left(\frac{h-2}{\chi_2^2}, \frac{h}{2}\right)\right\} \\ \begin{bmatrix} e^{a\left(k\phi + (1-k)\left(\frac{\beta_1+\beta_2}{2}\right) - 1\right)} - \\ a\left(k\phi + (1-k)\left(\frac{\beta_1+\beta_2}{2}\right) - 1\right) - 1 \end{bmatrix} \end{bmatrix}$$

_____(6.7.3)

Where $I(x;p) = (1/\Gamma p)\int_0^x e^{-x} x^{p-1} dx$ refers to the standard incomplete gamma function and $\phi = \frac{\hat{\beta}}{\beta}$; $h = \frac{2}{Var\left(\hat{b}_s/_b\right)}$; $\hat{\beta} = \frac{h-2}{t}$; $\hat{b}_s = \frac{\sum \left(x_{(i)} - x_{(r)}\right)}{n\ k_{r,n}}$

6.8 Relative Risk of $\hat{\beta}_{ST_3}$

A natural way to compare the performance of $\hat{\beta}_{ST_3}$ is to compare its performance with respect to the unbiased estimator $\hat{\beta}$. For this we obtain the risk of $\hat{\beta}$ under L(Δ) and the risk of $\hat{\beta}_{ST_3}$ has already been obtained in the previous section. Now the risk of $\hat{\beta}$ is defined as

$$R_E(\hat{\beta}) = E[\hat{\beta} | \, L\,(\hat{\beta}, \beta)]$$

$$= e^{-a}\int_0^{\infty} e^{a\left(\frac{\hat{\beta}}{\beta}\right)} f(Tr)dTr - a\int_0^{\infty}\left(\frac{\hat{\beta}}{\beta}-1\right)f(Tr)dTr - \int_0^{\infty} f(Tr)dTr \quad \text{\_\_\_\_(6.8.1)}$$

A straight forward integration of (6.8.1) gives

$$R_E(\hat{\beta}) = \frac{e^{-a}}{\Gamma(N)}\int_0^{\infty} e^{a\left(\frac{N-1}{t}\right)} e^{-t}\, t^{(N-1)}dt \;-1 \quad \text{\_\_\_\_(6.8.2)}$$

Now, we define the Relative Risk of $\hat{\beta}_{ST_3}$ with respect to $\hat{\beta}$ under L(Δ) as follows

$$RR_3 = \frac{R_E(\hat{\beta})}{R(\hat{\beta}_{ST_3})} \quad \text{\_\_\_\_(6.8.3)}$$

The expression for RR_3 is given by (6.8.3) which can be obtained by using (6.8.2) and (6.7.3). It is observed that RR_3 expression is a function of 'k', β_1, β_2, $\emptyset$, a, n and α. In order to study its behaviour numerically we have taken k = 0.2 (0.2) 1.0, β_1, β_2, $\emptyset$ are taken as $0.2 \le \beta_1 \le 1.4$, $0.4 \le \beta_2 \le 1.6$ and $0.1 \le \emptyset \le 1.3$, a = ± 3 to ± 1 and α = 1% , 5%. There will be several tables of RR_3 for the above values considered. Some of these tables have been assembled at the end of the chapter. However our recommendations based on all these tables are summarized as follows.

6.9 Recommendations for $\hat{\beta}_{ST_3}$

1. $\hat{\beta}_{ST_3}$ dominates $\hat{\beta}$ for all the positive and negative values of 'a' i.e., it behaves well in both over / under estimation situations. However its performance is best for a = -3 and a = +1. These values are observed for the

almost the whole range of 'k' i.e. $0.2 \leq k \leq 0.8$ and $0.1 \leq \emptyset \leq 1.3$. The magnitude of RR_3 values are higher for n = 5.

2. As the level of significance is changed to 5% still $\hat{\beta}_{ST_3}$ performs better but now the RR_3 values are slightly lower than those obtained for α = 1%. Suggesting a lower level of significance should be preferred.
3. $\hat{\beta}_{ST_3}$ performs better than $\hat{\beta}$ for $0.3 \leq \frac{\beta_1+\beta_2}{2} \leq 1.5$ for all the values of 'a' considered here. However the range in this case is slightly increased as compared to the performance of $\hat{\beta}_{ST_3}$ under Minimum Mean Square Error (MMSE) criterion. (Ref: Pandey and Singh (1984)) where they have reported the range from (0.6 to 1.5). So the use of an asymmetric loss function improves the effective range where the proposed testimator performs better than the usual one.
4. The performance of $\hat{\beta}_{ST_3}$ is not good for higher values of 'n' and higher level of significance.

CONCLUSION:

A shrinkage testimator for the shape parameter of Weibull distribution is proposed and its risk properties are studied under an asymmetric loss function. It has been observed that the proposed testimator dominates the usual unbiased estimator for fairly large range of departures of parameter(s). The proposed testimator is useful for larger negative values of degrees of asymmetry in particular a = -3 and different positive values of 'a' in particular a = +1. A lower sample size and smaller level of significance report the best performance of $\hat{\beta}_{ST_3}$.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 6.9.1 **Relative Risk of $\widehat{\beta}_{ST_3}$** $\alpha = 1\%$, $n = 5$, k = 0.2

Φ	**a = -1**	**a = -2**	**a = -3**	**a = 1**	**a = 2**	**a = 3**
0.1	1.333	1.403	1.493	1.243	1.216	1.196
0.3	1.438	1.507	1.593	1.341	1.308	1.283
0.5	1.66	1.733	1.819	1.549	1.507	1.473
0.7	2.436	2.532	2.639	2.275	2.207	2.147
0.9	3.923	3.858	3.913	3.388	2.772	2.249
1.1	2.245	2.253	2.261	2.228	2.219	2.21
1.3	1.566	1.583	1.599	1.527	1.506	1.482

Table : 6.9.2 **Relative Risk of $\widehat{\beta}_{ST_3}$** $\alpha = 1\%$, $n = 7$, k = 0.2

Φ	**a = -1**	**a = -2**	**a = -3**	**a = 1**	**a = 2**	**a = 3**
0.1	1.315	1.381	1.467	1.23	1.204	1.186
0.3	1.413	1.478	1.559	1.321	1.29	1.267
0.5	1.62	1.688	1.77	1.515	1.476	1.443
0.7	2.331	2.42	2.521	2.18	2.117	2.061
0.9	3.363	3.549	3.747	3.022	2.867	2.72
1.1	2.258	2.266	2.274	2.241	2.233	2.224
1.3	1.58	1.597	1.612	1.543	1.522	1.499

Table : 6.9.3 **Relative Risk of** $\widehat{\beta}_{ST_3}$ $\alpha = 5\%$; n = 5, k = 0.2

ϕ	**a = -1**	**a = -2**	**a = -3**	**a = 1**	**a = 2**	**a = 3**
0.1	1.319	1.386	1.472	1.233	1.206	1.188
0.3	1.418	1.484	1.566	1.325	1.294	1.27
0.5	1.628	1.698	1.78	1.522	1.482	1.449
0.7	2.352	2.443	2.545	2.199	2.136	2.079
0.9	2.986	2.799	3.164	2.868	2.499	2.548
1.1	2.255	2.263	2.271	2.238	2.23	2.221
1.3	1.577	1.594	1.609	1.539	1.518	1.496

Table : 6.9.4 **Relative Risk of** $\widehat{\beta}_{ST_3}$ $\alpha = 5\%$, n = 7, k = 0.2

ϕ	**a = -1**	**a = -2**	**a = -3**	**a = 1**	**a = 2**	**a = 3**
0.1	1.315	1.381	1.467	1.23	1.204	1.185
0.3	1.413	1.478	1.559	1.321	1.29	1.266
0.5	1.62	1.688	1.769	1.515	1.476	1.443
0.7	2.33	2.419	2.52	2.179	2.116	2.06
0.9	2.701	2.851	3.01	2.428	2.303	2.185
1.1	2.258	2.266	2.274	3.242	2.233	2.224
1.3	1.58	1.597	1.612	2.543	1.522	1.5