C H A P T E R IV

CORRELATION ANALYSIS OF LINEAR MULTIVARIABLE SYSTEM DYNAMICS USING SHIFT REGISTER SEQUENCES AS TEST PERTURBATIONS

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4.1 INTRODUCTION

' System dynamic identification ' may be stated as the process of obtaining adequate information from input-output measurement on a system to determine its responses to any input. Although, systems in general are nonlinear, linearized versions may be thought of, when the operation is restricted to a limited range. Further, in case the statistics of the dynamics does not change with time, a 'linear time-invariant system' originates. In such a case, it is a familiar fact that the system response y(t) at a time 't' is given by a weighted sum of inputs x(t), which have occured in the past. In mathematical terms :

$$y(t) = \int_0^{t} h(s)\, x(t-s)\, ds \qquad \ldots \qquad \ldots \qquad (4.1)$$

where s is a time variable; and h(t), the response to a unit impulse δ (t), is called the weighting function of the system.

In a practical system, impulse response applied at ~~to any input applied at~~ t = 0, settles down by a finite time T_s, termed as system settling

these three must match!

Not cared as stated here!

time. Under this assumption, the response y(t) becomes :

$$y(t) \cong \int_0^{T_s} h(s).x(t - s)ds \quad \ldots \quad \ldots \quad (4.2)$$

This procedure can be repeated to determine y(t) for each value of time 't'.

Now, y(t) in the above equation gives true response to the input x(t) provided that no noise/disturbance is present in the process. However, any physical system does exhibit noise to a certain degree, which contaminates the useful output of the system. Representing the net effect of all such disturbances referred to the output by n(t), the measurable output z(t) of the system, then becomes :

$$z(t) = y(t) + n(t)$$

$$= \int_0^{T_s} h(s) \,.\, x(t - s)ds + n(t) \quad \ldots \quad (4.3)$$

This equation calls for a method of dynamic analysis that is capable of suppressing the noise effects. For this reason, correlation methods are favoured.

For the linear time-invariant system, there exists a relationship between the input-output crosscorrelation function $\phi_{xz}(\tau)$ and the impulse response h(t) as under : (Lee 1960)

$$\phi_{xz}(\tau) = \int_0^{T_s} h(s)\, \phi_{xx}(\tau - s)ds + \phi_{xn}(\tau) \quad \ldots \quad (4.4)$$

where τ is the relative time-shift between x(t) and z(t), and

$\phi_{xn}(\tau)$ and $\phi_{xx}(\tau)$ are the crosscorrelation between the input and noise signal, and the autocorrelation function of the input respectively.

Equation (4.4) indicates that an impulse-formic autocorrelation function of the input signal greatly facilitates the deconvolution necessary for determining the inpulse response ordinates $h(\tau)$. In addition to this, a signal having negligible correlation with the noise is highly preferable.

For the former reason, the signal 'white-noise' has come into existence. White noise has the characteristic that its autocorrelation is zero everywhere except at one value of its argument. If w(t) represents white noise, then

$$\left.\begin{array}{lll} \phi_{ww}(\tau) = 0, & \text{for} & \tau \neq 0 \\ \phi_{ww}(\tau) = 0, & \text{for} & \tau = 0 \end{array}\right\} \quad \ldots \quad (4.5)$$

Such a waveform is apparently a mathematical fiction, and hence generation of approximate white noise has been experimented. In this context, it has been stated in Chapter 2 that use can be made of certain cyclic binary signals which possess properties in correlation similar to those of white noise, but which are deterministic. These have been called 'Pseudorandom Signals', the name being given in view of the manner in which the autocorrelation function of such signals approximate to that of white noise.

The use of pseudorandom binary signals as perturbation signals in the crosscorrelation method of obtaining the linearised impulse response of a system has been considered in some detail in Chapter 2, and a comprehensive list of references has been given in Section 2.1. Most of these investigations have been concerned with single input/single output linear system. From a practical viewpoint, the analysis of the single-input system is of limited importance as most systems are subjected to several inputs, yielding more than one output. However, because of the several practical advantages associated with the use of pseudorandom signals, a search for implementing the above correlation principle to arrive at the multivariable system dynamics has been necessitated. The successive single-input method is the early result of such an effort. Obviously, for systems having many inputs (outputs), such a serial approach is not to be preferred due to long observation time involved. So much so, attention has been later confined to developing correlation techniques, that provide quick results. Initially, the situation has been entertained for a two-input case, and soon ideas have been explored and refined to effect significant generalizations (Cummins 1966, Utsal 1965, Briggs et al. 1966, Godfrey et al. 1966, 1969, Douce and Ng. 1966, Ream 1967, Jaruis 1968). However, a satisfactory solution to the problem has not yet emerged. Apparently, there exists need for a more basic understanding of the underlying theories to advance a well-suited technique for the purpose.

The objective of this chapter is, therefore -

(i) To make a systematic exposition of both the qualitative and quantitative issues involved in the currently available correlation methods for the multivariable system dymanic analysis with shift register sequences as test perturbations, and

(ii) To present some improved schemes for uncorrelated system input signals with autocorrelation functions approximating to delta functions for use in the identification of multivariable system by the input/output crosscorrelation method. (The order of the content of this chapter is given at the end of the following section).

4.2 ESTIMATION OF MULTIVARIABLE LINEAR SYSTEM IMPULSE RESPONSES BY CROSSCORRELATION METHOD

The usual impulse response estimation scheme for a multivariable system having j-inputs and k-outputs is shown in Fig. (4.1). This linear system can be characterized by j independent impulse response functions. The input signals $x_1(t)$, $x_2(t)$,..., $x_j(t)$ together produce each of the output signals $y_1(t)$, $y_2(t)$,..., $y_k(t)$. In experimental work both $x_u(t)$, $1 \leq u \leq j$, and $y_v(t)$, $1 \leq v \leq k$ represent deviations from the normal operating levels. The response of the system

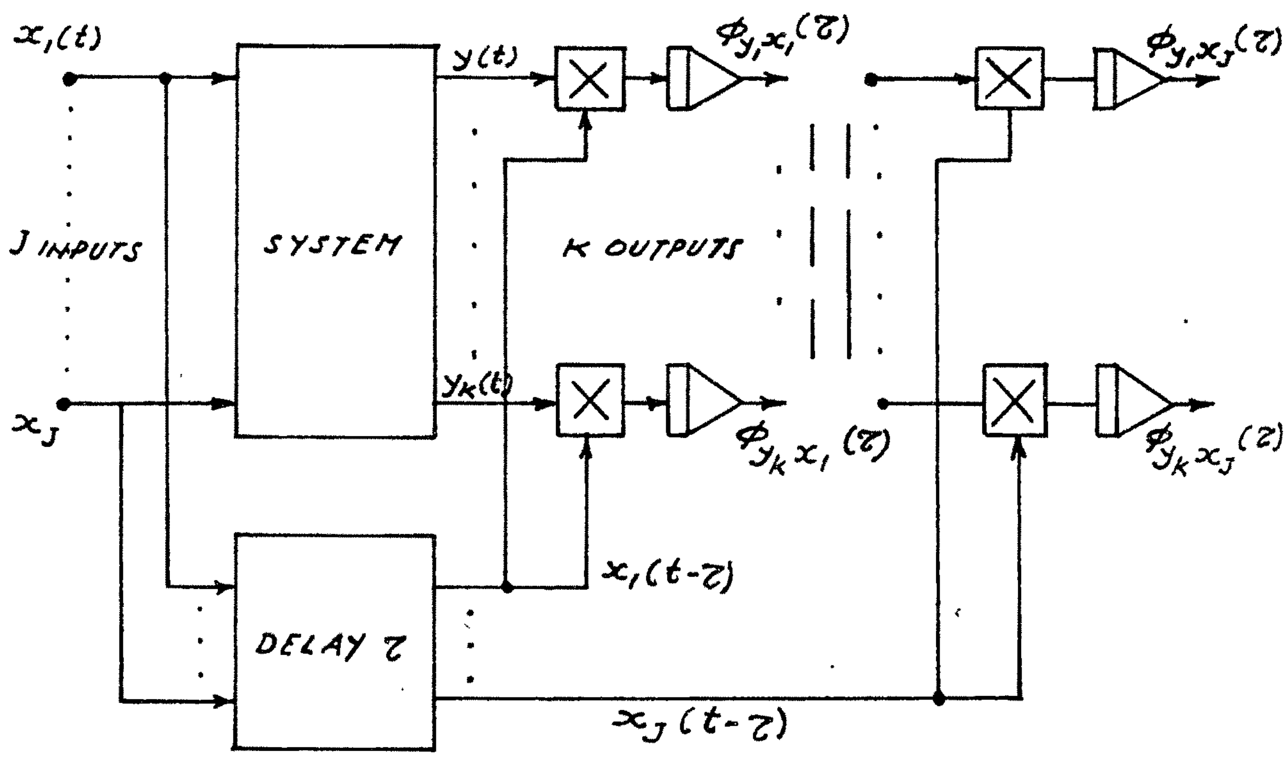


FIG. 4.1 WEIGHTING - FUNCTION ESTIMATION SCHEME OF A MULTI-VARIABLE SYSTEM

to inputs other than $x_u(t)$, together with the measurement disturbance, forms the noise signal $n(t)$, which is assumed to equally influence all the outputs.

In such a case, an output $y_v(t)$ of the system, (Fig.4.1) in the absence of noise is given by :

$$y_v(t) = \int_{-\infty}^{\infty} h_{1v}(s).x_1(t-s)ds + \int_{-\infty}^{\infty} h_{2v}(s).x_2(t-s)ds + \dots\dots + \int_{-\infty}^{\infty} h_{jv}(s).x_j(t-s)ds \quad \dots (4.6)$$

and the corresponding measurable output is

$$z_v(t) = y_v(t) + n(t)$$

$$= \sum_{r=1}^{j} \int_{-\infty}^{\infty} h_{rv}(s).x_r(t-s)ds + n(t), \quad 1 \leq v \leq k \quad .. (4.7)$$

This equation gives the time response of the vth output/in the presence of all j-inputs.

Now, the crosscorrelation function between input $x_u(t)$ and output $z_v(t)$ becomes :

$$\phi_{x_u z_v}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_u(t) \sum_{r=1}^{j} \int_{-\infty}^{\infty} h_{rv}(s).x_r(t+\tau-s)ds + n(t+\tau) \quad \dots (4.8)$$

By changing the order of integration and simplifying eqn. (4.8) leads to :

$$\phi_{x_u z_v}(\tau) = \int_{-\infty}^{\infty} h_{uv}(s).\phi_{x_u x_u}(\tau - s)ds$$

$$+ \sum_{r \neq u} \int_{-\infty}^{\infty} h_{rv}(s).\phi_{x_u x_r}(\tau - s)ds + \phi_{xn}(\tau) \quad .. (4.9)$$

As stated before, in practical systems, the present output is unaffected by inputs applied to the system at a time greater than the system settling time (T_s) in the past. With the assumption, eqn. (4.9) becomes :

$$\phi_{x_u z_v}(\tau) = \int_{0}^{T_s} h_{uv}(s).\ \phi_{x_u x_u}(\tau - s)ds$$

$$+ \sum_{r \neq u} \int_{0}^{T_s} h_{rv}(s).\phi_{x_u x_r}(\tau - s)ds + \phi_{x_r n}(s) \quad ..(4.10)$$

An estimate of the impulse response function $h_{uv}(\tau)$ can be easily obtained from the measured crosscorrelation function $\phi_{x_u z_v}(\tau)$ provided the input test signals :

(i) be repetitive,

(ii) possess an autocorrelation function which approximates to an impulse about $\tau = 0$, and which is small and constant or negligible for large τ, and

(iii) have zero crosscorrelation for τ less than the maximum system settling time T_{smax} (i.e. $\phi_{x_r x_v}(\tau) = 0$ for $\tau < T_{smax}$).

The advantages of periodicity of the test signal are stated in Chapter 2 when dealing with the single input / single output system dynamic analysis. With periodic inputs, crosscorrelation is usually perfomed over several periods of the input signal to keep the variance of the correlator output to a minimum, and thus improve the accuracy of the results.

Under the circumstances, a modified crosscorrelation function between x(t) and n(t) may be defined as below :

$$\phi^*_{xn}(\tau) = \frac{1}{mT} \int_0^T x(t).n(t+\tau)dt \quad \ldots \quad \ldots (4.11)$$

where T is the input signal repetetion period and m a chosen integer.

In case, the inputs hold property (iii) :

$$\phi_{x_u z_v}(\tau) = \int_0^{T_s} h_{uv}(s).\phi_{x_u x_u}(\tau - s)ds + \phi^*_{x_u n}(\tau) \quad .. (4.12)$$

And by property (ii) :

$$\phi_{x_u z_v}(\tau) = K.\ h_{uv}(\tau) + \phi^*_{x_u n}(\tau) \quad \ldots \quad (4.13)$$

where K is the area of the autocorrelation spike about $\tau = 0$.

As seen from this equation, in case the value of $\phi^{*}_{xn}(\tau)$ can be made small, the expected values of the correlator output become proportional to the ordinates of the system weighting function.

Now the main reasons for the presence of this noise-term n(t) are the following :

(i) Initial conditions and/or periodicity of the test signal.

(ii) Imperfections in the test signal characteristics.

(iii) Non-ideal characteristics of the input transducer.

(iv) Uncontrollable random fluctuations in the system.

(v) Slow change of the outputs from their desired operating levels (low frequency output drift).

(vi) Presence of nonlinearity over the range of amplitudes of the input.

(vii) Variations in the estimate of the impulse response h(c) with time.

(viii) Instrumentation errors.

Details of error analyses with random and pseudorandom binary test perturbations are well documented (Hughes and Noton 1962, Poortvliet 1963, Cummins 1964, Hazlerigg 1965, Hammond and Barber 1965, Briggs et al. 1965 and 1966, Godfrey 1965, 1966 and 1969). The main conclusions with reference to the above mentioned errors have been stated in Chapter 2 where the dynamic analysis of single input / single output system has been considered. The method of reducing/eliminating most of these errors has also been outlined in Chapter 2.

Hence, following the error reduction techniques, the value of $\phi^{*}_{x_u n}(\tau)$ can be made negligible in which case eqn.(4.13) becomes:

$$\phi_{x_u z_v}(\tau) \simeq k\, h_{uv}(\tau) \qquad \ldots \qquad \ldots \qquad (4.14)$$

Thus, the problem of linear multivariable system identification now settles down to one of finding uncorrelated signals with impulse like autocorrelation functions. For this purpose, a number of schemes have been communicated(Cummins, 1965; Utsal 1965, Briggs et. al (1966, Godfrey et. al 1966, 1969; Douce and Ng 1966, Ream 1967, Jaruis, 1968). But, a satisfactory solution to the problem has not emerged. This calls for a more basic understanding of the existing theories.

For this reason, at first a systematic exposition is made of the current status of the crosscorrelation art in multivariable system identification in secs (4.3 to 4.6). An effort is made here to present in an orderly manner both the qualitative and quantitative issues involved in the above referred existing schemes. Numerical examples are included to clear the theoretical concepts.

To overcome some of the shortcomings of the current schemes, in sections (4.7 and 4.8),the theory of p-level linear shift register sequences which possess impulse-like autocorrelation with zero bias is introduced. Such sequences are named as 'ideal sequences'. And in section (4.9) some improved schemes for uncorrelated signals, are broughtforth using transformed ideal as well as conventional shift register sequences.

The theory is developed in terms of sequences, signals are obtained from sequences and represent continuous versions of the respective sequences.

4.3. THE SIMULTANEOUS USE OF SEVERAL PSEUDORANDOM BINARY SEQUENCES IN THE IDENTIFICATION OF LINEAR MULTIVARIABLE DYNAMIC SYSTEMS

4.3.1 Pseudorandom binary sequences (PRBS)

In Chapter 1, it has been shown how to generate a pseudorandom binary sequence using a digital shift register with linear (modulo-2 adder) logic. For an n-stage shift register, activated by a master clock of period t_o, the maximum possible length of the sequence is $N = (2^n - 1)\ t_o$. The mathematical theory of the maximum length or m-sequences has been well advanced (Zierler 1959).

For present purposes, a pseudorandom binary sequence (PRBS) can be considered as an ordered cyclic sequence $\{c\}$ of binary elements $c_i = \pm 1$, with corresponding levels $\pm\alpha$. In the sequence, there is one more + 1 than - 1, over a period of the sequence, so that -

$$\sum_{i=1}^{N} c_i = 1 \qquad \ldots \qquad \ldots \qquad (4.15)$$

Thus, the mean value of the sequence over its period approaches zero as the value of N increases.

The autocorrelation function of such a sequence is defined as

$$\phi_{cc}(\tau) = \frac{1}{N} \sum_{i=1}^{N} c_i\, c_{i+\tau} \qquad \ldots \qquad \ldots \qquad (4.16)$$

$$= 1, \qquad \tau = 0 \text{ mod-N}$$

$$= \frac{-1}{N} \qquad \tau \neq 0 \text{ mod-N}$$

Eqn. (4.16) indicates that several pseudorandom binary sequences can be simultaneously used as test inputs for the dynamic analysis of a linear multivariable system, provided that the period of the sequence N assumes a large value (in accordance with properties of test signals for multivariable system identification by crosscorrelation as stated in Section (4.2), properties a, b, and c). In this context, the following possibilities ~~arise :~~ may be considered:

(i) Sequences of unequal period,

(ii) Sequences of equal period,

(iii) Sequences, whereby the basic interval of one sequence is twice of the other, and

(iv) Sequences with basic interval inversely proportional to the period of the sequence,

A discussion of each of these now follows :

4.3.2 Use of several pseudorandom binary sequences of unequal period

Let $\{c\}$ be a pseudorandom binary sequence (PRBS) of period P with levels $\pm\alpha$, and $\{d\}$ be a PRBS of period Q with levels $\pm\alpha$, satisfying the conditions in eqns. (4.15 and 4.16). The sequences $\{c\}$ and $\{d\}$, therefore, meet with the requirements of periodicity and impulse - like auto-correlation. Further, the sequences can be considered uncorrelated provided that the variance of the cross-correlation between them over their common period is zero.

Since P and Q are unequal, the situation may be examined in the following cases :

Case (1) : Periods P and Q are coprime.

Case (2) : Period P = ℓ Q, where ℓ is an integer.

Case (1) :

Here R = PQ becomes the correlation period. Hence, the crosscorrelation function $\phi_{cd}(\tau)$ is given by :

$$\phi_{cd}(\tau) = \frac{1}{R} \sum_{r=1}^{R} c_r \, d_{r+\tau} \qquad \ldots \qquad \ldots \qquad (4.17)$$

$$= \frac{1}{R} \text{ for all shifts.}$$

The mean $\bar{\phi}_{cd}$ of the crosscorrelation function over all shifts is then

$$\bar{\phi}_{cd} = \frac{1}{R} \sum_{\tau=1}^{R} \phi_{cd}(\tau)$$

$$= \frac{1}{R^2} \cdot \sum_{s=1}^{R} \sum_{r=1}^{R} c_r \, d_{r+\tau}$$

$$= \frac{1}{R^2} \cdot Q\left(\sum_{r=1}^{P} c_r\right) . P\left(\sum_{r=1}^{Q} d_r\right)$$

From eqn. (4.15), this becomes

$$\bar{\phi}_{cd} = \frac{1}{R} \qquad (\text{ since } R = PQ) \quad \ldots \quad (4.18)$$

Now, the variance σ^2 of the crosscorrelation function over all shifts is given by :

$$\sigma^2 = \frac{1}{R} \sum_{\tau=1}^{R} [\phi_{cd}(\tau)]^2 - [\bar{\phi}_{cd}]^2$$

$$= [\frac{1}{R} \cdot R \cdot \frac{1}{R^2}] - [\frac{1}{R}]^2$$

$$= 0 \qquad \ldots \qquad \ldots \qquad (4.19)$$

Hence, the sequences {c} and {d} whose periods are coprime may be used as test inputs in the multivariable system. However, this case is not of much practical importance due to very long integration time (= PQ) involved.

Example (4.1) :

Consider the m-sequences {c} and {d} given by :

{c} : 1 1 0, 1 1 0, 1 1 0, 1 1 0, 1 1 0, 1 1 0, 1 1 0, ...
with period P = 3.

{d} : 1 1 1 0 1 0 0, 1 1 1 0 1 0 0, 1 1 1 0 1 0 0, ...
with period Q = 7.

Hence, the common period R = PQ = 21.

The crosscorrelation function $\phi_{cd}(\tau)$ is (from eqn.4.17);

$$\phi_{cd}(\tau) = \frac{1}{R} = \frac{1}{21}$$

And, using eqn. (4.18),

$$\bar{\phi}_{cd} = \frac{1}{R} = \frac{1}{21}$$

Hence the variance σ^2 here is zero over all shifts (eqn. (4.19)).

<u>Case (II)</u> : Period $P = \ell Q$, ℓ is an integer.

Let $\{c\}$ be a PRBS of period P and $\{e\}$ be another PRBS of period Q, both with levels ± 1, and where $P = \ell Q$, ℓ is an integer.

The crosscorrelation function $\phi_{ce}(\tau)$ is given by :

$$\phi_{ce}(\tau) = \frac{1}{P} \sum_{r=1}^{P} c_r \, e_{r+\tau} \qquad \ldots \qquad \ldots \qquad (4.20)$$

The mean $\bar{\phi}_{ce}$ of $\phi_{ce}(\tau)$ over all shifts is :

$$\bar{\phi}_{ce} = \frac{1}{P} \sum_{\tau=1}^{P} \phi_{ce}(\tau) \qquad \ldots \qquad \ldots \qquad (4.21)$$

$$= \frac{1}{P^2} \cdot \sum_{\tau=1}^{P} \sum_{r=1}^{P} c_r \, e_{r+\tau}$$

$$= \frac{1}{P^2} \left(\sum_{r=1}^{P} c_r \right) \cdot \ell \left(\sum_{r=1}^{Q} e_r \right)$$

which, using eqn.(4.15) becomes , $\bar{\phi}_{ce} = \frac{1}{PQ}$. (Since $\ell = P/Q$)

And the variance σ^2 of the crosscorrelation function over all shifts is written as :

$$\sigma^2 = \frac{1}{P}\left[\sum_{\tau=1}^{P} (\phi_{ce}(\tau)\right]^2 - \left[\bar{\phi}_{ce}\right]^2$$

$$= \frac{1}{P^3}\left[\sum_{\tau=1}^{P}\sum_{r=1}^{P}\sum_{t=1}^{P} c_r c_t \, e_{r+\tau} \, e_{t+\tau}\right] - \frac{1}{P^2 Q^2} \qquad \ldots \qquad (4.22)$$

Summing with respect to τ

$$\sigma^2 = \frac{1}{P^2}\sum_{r=1}^{P}\sum_{t=1}^{P} c_r c_t \, \phi_{ee}(r-t) - \frac{1}{P^2 Q^2}$$

$$= \frac{1}{P^2}\sum_{r=1}^{P}\sum_{t=1}^{P} c_r c_t \left[\frac{Q+1}{Q}\sum_{m=0}^{\ell-1} \delta_{r-t+Qm} - \frac{1}{Q}\right] - \frac{1}{P^2 Q^2}$$

as there are ℓ spikes of $\phi_{ee}(\tau)$ over the period P.

Simplifying the above equation, we obtain -

$$\sigma^2 = \left[\frac{1}{P^2} \cdot \frac{Q+1}{Q}\sum_{m=0}^{\ell-1}\sum_{r=1}^{P} c_r c_{r+Qm} - \frac{1}{P^2 Q}\sum_{r=1}^{P}\sum_{t=1}^{P} c_r c_t - \frac{1}{P^2 Q^2}\right]$$

Now, by summation with respect to r -

$$\sigma^2 = \frac{Q+1}{PQ} \sum_{m=0}^{\ell-1} \phi_{cc}(Qm) - \frac{\ell}{P^2 Q} - \frac{1}{P^2 Q^2}$$

$$= \frac{Q+1}{PQ} \sum_{m=0}^{\ell-1} \left[\frac{P+1}{P} \delta_{Qm} - \frac{1}{P} \right] + \frac{\ell}{P^2 Q} - \frac{1}{P^2 Q^2}$$

$$\sigma^2 = \frac{Q+1}{PQ} \left[\frac{P+1}{P} - \frac{\ell}{P} \right] + \frac{\ell}{P^2 Q} - \frac{1}{P^2 Q^2} \quad \ldots (4.23)$$

since Qm is a multiple of P only for $m = 0 \bmod \ell$.

On simplification, the final result is :

$$\sigma^2 = \frac{Q + 1(PQ + Q - Q\ell - 1)}{P^2 Q^2} \quad \ldots \quad \ldots \quad (4.24)$$

Example 4.2 :

Let {c} and {e} be two m-sequences with repetition periods P = 15 and Q = 3, respectively, given by :

{c} : 1 1 1 1 0 0 0 1 0 0 1 1 0 1 0, ... (with period P = 15)

{e} : 1 1 0, 1 1 0, 1 1 0, 1 1 0, 1.1 0,...(with period Q = 3)

The crosscorrelation function $\phi_{ce}(\tau)$ is :

$$\phi_{ce}(\tau) = \frac{1}{15} \sum_{r=1}^{15} c_r e_{r+s}$$

The mean $\bar{\phi}_{ce}$ as obtained from eqn. (4.21) is :

$$\bar{\phi}_{ce} = \frac{1}{15(3)}$$

and, the variance σ^2 is (From eqn. (4.24)) :

$$\sigma^2 = \frac{(3 + 1)[(15)(3) + (3) - (3)(5) - 1]}{(15)^2(3)^2}$$

$$= 0.0632$$

i.e. $\sigma = 0.26$

As the variance over their common period is too large, the use of such sequences as system inputs is not a promising approach.

4.3.2(B) <u>Use of several pseudorandom binary sequences - equal periods</u>

Let $\{c\}$ and $\{c^*\}$ be two sequences with equal period P. Then, the crosscorrelation function $\phi_{cc^*}(\tau)$ is given by :

$$\phi_{cc^*}(\tau) = \frac{1}{P} \sum_{\tau=1}^{P} c_r \, c^*_{r+\tau} \quad \ldots \quad \ldots \quad (4.25)$$

Making use of eqn. (4.21) with P = Q, the mean $\bar{\phi}_{cc^*}$ is :

$$\bar{\phi}_{cc^*} = \frac{1}{P^2} \quad \ldots \quad \ldots \quad (4.26)$$

Similarly, the variance σ^2 of the crosscorrelation function over all shifts is given by :

$$\sigma^2 = \frac{1}{P} \sum_{\tau=1}^{P} \left[\phi_{cc^*}(\tau) \right]^2 - \left[\bar{\phi}_{cc^*} \right]^2$$

which, utilizing eqn. (4.24) becomes :

$$\sigma^2 = \frac{P-1}{P^2} + \frac{1}{P^3} - \frac{1}{P^4}$$

$$= (P^2 + 1)(P - 1) / P^4 \quad \ldots \quad \ldots \quad (4.27)$$

Example (4.3) :

Let the m-sequences under consideration be as follows :

{c} : 1 1 1 1 0 0 0 1 0 0 1 1 0 1 0 , ... Period P = 15

{c*} : 1 1 1 1 0 1 0 1 1 0 0 1 0 0 0 , ... Period P = 15

The crosscorrelation function $\phi_{cc^*}(\tau)$ is then -

$$\phi_{cc^*}(\tau) = \frac{1}{15} \sum_{r=1}^{15} c_r c^*_{r+\tau}$$

Now, using eqn. (4.26), the mean value is :

$$\bar{\phi}_{cc^*} = \frac{1}{15^2}$$

And by eqn. (4.27), the variance σ^2 over all shifts is given by :

$$\sigma^2 = \frac{(226)(14)}{(15)^4} = 0.0624$$

i.e. $\sigma = 0.26$.

4.3.2(C) Use of several pseudorandom binary sequences with their basic intervals in the ratio of 1 : 2

Let $\{c\}$ and $\{f\}$ be the two m-sequences having their basic intervals in the ratio of 1 : 2. In such a case, the elements of $\{f\}$ are given by :

$$f_{2r-1} = f_{2r} = c_r \quad \ldots \quad \ldots \quad (4.28)$$

The crosscorrelation function $\phi_{cf}(\tau)$ is given by :

$$\phi_{cf}(\tau) = \frac{1}{2P} \sum_{r=1}^{2P} c_r \, f_{r+\tau} \quad \ldots \quad \ldots \quad (4.29)$$

where P is the period of the sequence $\{c\}$.

Eqn. (4.29) may be simplified as :

$$\phi_{cf}(\tau) = \frac{1}{2P} \sum_{r=1}^{P} (c_{2r-1} \, f_{2r-1+\tau} + c_{2r} \, f_{2r+\tau})$$

$$= \frac{1}{2P} \sum_{r=1}^{P} (c_{2r-1} \, c_{r+s} + c_{2r} \, c_{r+\tau}) \quad \ldots \quad (4.30)$$

The mean value $\bar{\phi}_{cf}$ is given by :

$$\bar{\phi}_{cf} = \frac{1}{2P} \sum_{\tau=1}^{2P} \phi_{cf}(\tau) \quad \ldots \quad \ldots \quad (4.31)$$

and the variance of $\phi_{cf}(\tau)$ over all shifts is given by :

$$\sigma^2 = \frac{1}{2P} \sum_{\tau=1}^{2P} \left[\phi_{cf}(\tau)\right]^2 - (\bar{\phi}_{cf})^2 \quad \ldots \quad (4.32)$$

Example (4.4) :

Let $\{c\}$ be an m-sequence of period $P = 7$, given by

$\{c\}$: 1 1 1 0 1 0 0, 1 1 1 0 1 0 0, ... (Period P = 7)

Then $\{f\}$: 1 1 1 1 1 1 0 0 1 1 0 0 0 0, ... (Period = 2P)

From eqn. (4.29), the crosscorrelation function is :

$$\phi_{cf}(\tau) = +\frac{3}{7} \qquad \text{for } \tau = 1, 7, 8 \text{ and } 14.$$

$$= -\frac{1}{7} \qquad \text{for } \tau = 2, 3, 4, 5, 6, 9, 10, 11, 12, 13.$$

And, the mean $\bar{\phi}_{cf}$ obtained from eqn. (4.31) is :

$$\bar{\phi}_{cf} = \frac{1}{14} \; 4\left(\frac{3}{7}\right) - 10\left(\frac{1}{7}\right)$$

$$= \frac{4}{(14)^2}$$

Hence, the variance σ^2 over all shifts is (from eqn. 4.32)

$$\sigma^2 = \frac{1}{(14)^3}\left[(10)(4) + (4)(36)\right] = \frac{16}{(14)^4}$$

i.e. $\sigma = 0.25$

Here too, the variance is quite large. Further, with many inputs, the correlation period becomes prohibitively large for the use of such sequences. So much so, much attention has not been given for this case.

4.3.2(D) Use of several pseudorandom binary sequences in which the basic intervals are inversely proportional to the number of elements in the respective sequences

Let $\{c_1\}$ and $\{c_2\}$ be the 2 sequences with basic intervals t_1 and t_2, number of elements P_1 and P_2 such that

$$P_1 . t_1 = P_2 . t_2 \qquad \ldots \qquad \ldots \qquad (4.33)$$

Although of practical interest, this case has been found to be not promising as per the computer results obtained by Cummins (1964).

A more attractive approach is to use several phase shifted versions of PRBS as inputs to the multivariable system, which will be discussed in the following section.

4.4 USE OF PHASE-SHIFTED PRBS FROM THE SAME SOURCE IN THE IDENTIFICATION OF LINEAR MULTIVARIABLE SYSTEM

4.4.1 (Method 1) Evaluation of impulse responses between all inputs and one output from a single crosscorrelation

The time autocorrelation function of a pseudorandom binary signal x(t), (signal being a continuous version of the sequence) over any period $T = Nt_o$, $N = 2^n - 1$, where t_o is the basic interval, is given by -

$$\phi_{xx}(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{t_o} & \text{for } |\tau| < t_o \\ -1/N & \text{otherwise} \end{cases} \qquad \ldots \qquad (4.34)$$

If such a signal is used for a system impulse response measurements, assuming the basic interval t_o to be very small compared with the time constant of the system under test, its autocorrelation function $\phi_{xx}(\tau)$ can be approximated by a series of impulse functions as :

$$\phi_{xx}(\tau - s) \simeq K\left[\mu(\tau - s) + \mu(\tau - s - T) + \mu(\tau - s - 2T) + \dots\right] \quad (4.35)$$

so that, the input/output crosscorrelation function is :

$$\phi_{xz}(\tau) = K\left[h(\tau) + h(\tau - T) + h(\tau - 2T) + \dots\right] \quad (4.36)$$

which is a string of impulse response functions 'multiplied' by K, the area under the autocorrelation triangular spike.

Further, when the test signal period T is made greater than the system settling time (as is usual), the individual responses of this string will not overlap. In such a case, the individual responses can be treated as if they were the results of several experiments.

This concept, together with that of the repetitive impulse like crosscorrelation functions between phase-shifted pseudorandom binary signals can be taken advantage of in multivariable system identification to determine the impulse response functions between various inputs and a chosen output by a single correlation calculation as shown below -

Consider a system with two inputs and one output. Let the system be perturbed by two pseudorandom binary signals $x_1(t)$ and $x_2(t)$, which are merely phased shifted versions of each other, obtain from the same source. This situation is depicted in Fig. (4.2).

The correlation function between the input signal $x_1(t)$ and the output signal $z_1(t)$ may be written as : (from eqn. 4.13)

$$\phi_{x_1z_1}(\tau) = \int_0^{T_s} h_{11}(s).\,\phi_{x_1x_1}(\tau - s)ds + \int_0^{T_s} h_{21}(s).\,\phi_{x_1x_2}(\tau - s)ds + \phi_{x_1n}(\tau) \quad \ldots (4.37)$$

Since the signal $x_2(t)$ is only a phase shifted version of the signal $x_1(t)$, the crosscorrelation between the two functions $\phi_{x_1x_2}(\tau)$ has the same form as the autocorrelation function of the signal $x_1(t)$, i.e. $\phi_{x_1x_1}(\tau)$, but has a negative phase shift the same as that between the signals themselves, as shown in Fig. (4.3). If this phase shift is λt_o,

$$\begin{aligned}\phi_{x_1z_1}(\tau) = K\Big[&h_{11}(\tau) + h_{11}(\tau - T) + h_{11}(\tau - 2T) + \ldots \\ &+ h_{21}(\tau - \lambda t_o) + h_{21}(\tau - \lambda t_o - T) \\ &+ h_{21}(\tau - \lambda t_o - 2T) + \ldots\Big] \quad \ldots (4.38a)\end{aligned}$$

$$\begin{aligned}= K\Big[&h_{11}(\tau) + h_{21}(\tau - \lambda t_o) + h_{11}(\tau - T) \\ &+ h_{21}(\tau - \lambda t_o - T) + h_{11}(\tau - 2T) + \ldots \\ &+ h_{21}(\tau - \lambda t_o - 2T) + \ldots\Big] \quad (4.38)\end{aligned}$$

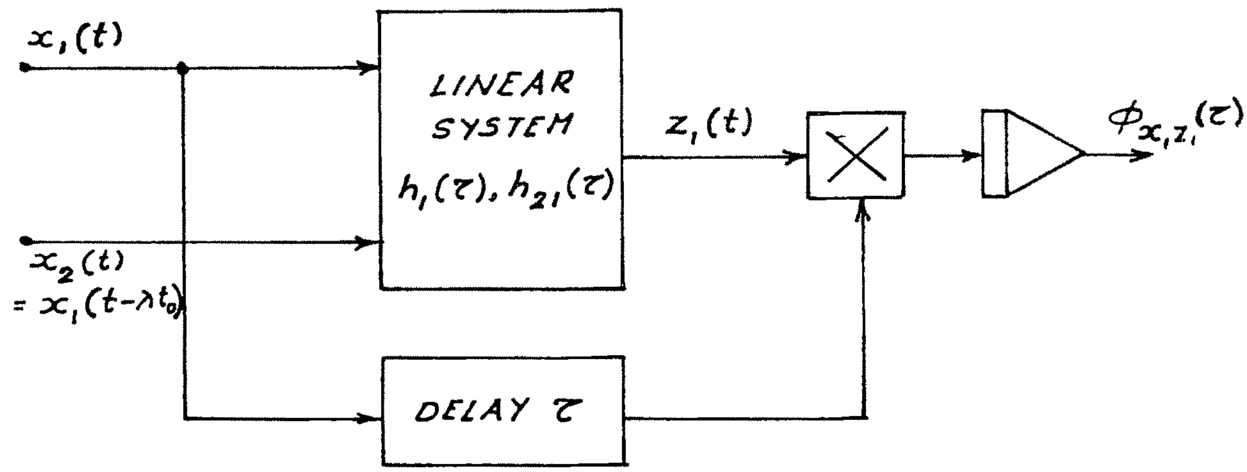


FIG. 4.2 TWO-INPUT LINEAR SYSTEM IDENTIFICATION IN ONE CROSS-CORRELATION USING PHASE-SHIFTED P.R.B.S. AS PERTURBATIONS

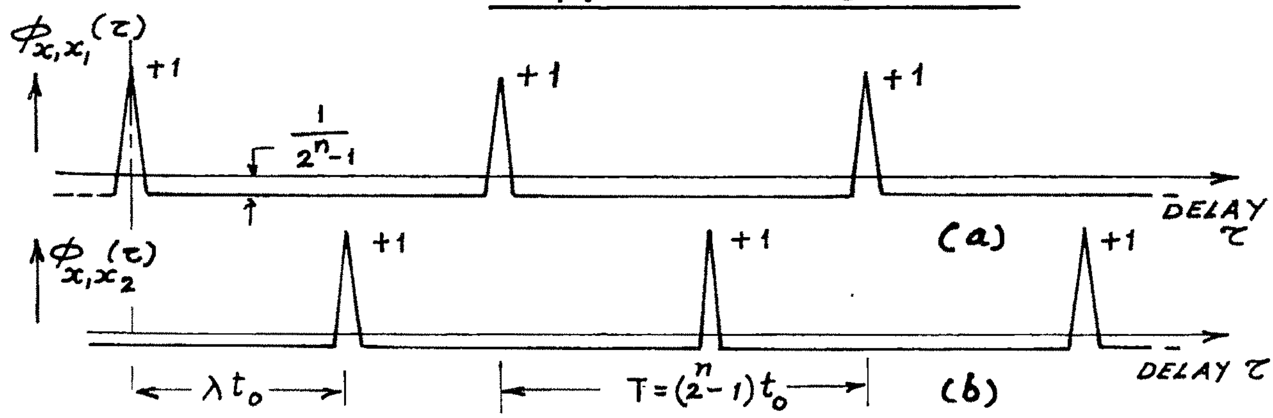


FIG. 4.3 (a) P.R.B.S. AUTOCORRELATION FUNCTION

(b) CROSS CORRELATION FUNCTION BETWEEN TWO-PHASE SHIFTED P.R.B.S.

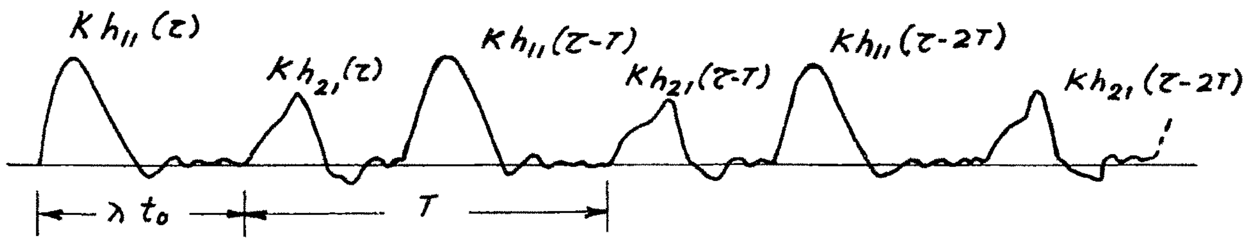


FIG. 4.3 (c) REPEATED PAIRS OF IMPULSE RESPONSE FUNCTIONS

Thus, the crosscorrelation function $\phi_{x_1 z_1}(\tau)$ is a chain of repeated pairs of individual impulse-response functions $h_{11}(\tau)$ and $h_{21}(\tau)$, multiplied by K, spread out along the τ - axis as shown in Fig. (4.3).

If

$$\left.\begin{array}{llll} & h_{11}(\tau) \simeq 0 , & \text{for} & \tau > \lambda t_o \\ \text{and} & h_{21}(\tau) \simeq 0 , & \text{for} & \tau > T - \lambda t_o \end{array}\right\} \quad \ldots \quad (4.39)$$

the impulse responses will not overlap; and hence the individual response functions can be seperated out from the result of one crosscorrelation. Averaging over repeated pairs improves the accuracy.

In principle, the technique can be extended to the multiple-input / output system to estimate the impulse responses concerning one output and all the inputs from a single cross-correlation calculation. But, the phase shifts of the input perturbations must be chosen carefully so that overlapping of the respective response functions will not occur. Where such a possibility is ruled out, the following parallel approach may be considered.

4.4.2. (Method 2) Measurement of impulse responses using phase shifted pseudorandom binary signals - parallel approach

In case, the settling times of all channels of a multiple input system (j inputs) are the same, then, the input (prbs) to the uth input channel can be delayed by Nt_o/j relative to the input to the $(u-1)$ th input channel. Evidently, the crosscorrelation function between any two inputs is the same as the autocorrelation function of the input but for the phase shift which equals the difference in phase shifts of the two inputs. And, the crosscorrelation function $\phi_{x_u y_v}(\tau)$ for $0 \leq \tau \leq Nt_o/j$ directly yields the impulse response $h_{uv}(\tau)$.

If the impulse responses of the system settle at different times, inter-channel delay may be selected accordingly.

Several improvements have been suggested based on the above ideas and these will be discussed in the next section.

5. USE OF TRANSFORMED PSEUDORANDOM BINARY SIGNALS IN THE IDENTIFICATION OF LINEAR MULTIVARIABLE SYSTEM

5.1 Use of inverse-repeat sequences for a 2-input system

A sequence $\{g\}$ obtained by multiplying the elements of a pseudorandom binary sequence $\{c\}$ of period P alternatively by -1 and +1, is called an 'inverse-repeat sequence'. The elements of $\{g\}$ are therefore given by :

$$g_r = (-1)^r c_r \qquad \ldots \qquad \ldots \qquad (4.40)$$

Clearly, the period of $\{g\}$ equals 2P. Further, the sequence $\{g\}$ is such that its second half is the negative of the first half. This antisymmetric property leads to zero crosscorrelation between the sequences $\{c\}$ and $\{g\}$ over all shifts τ in the period 2P.

The autocorrelation function of $\{g\}$ is :

$$\phi_{gg}(\tau) = \frac{1}{2P} \sum_{r=1}^{2P} (-1)^{r} \cdot c_r (-1)^{r+\tau} \cdot c_{r+\tau}$$

$$= (-1)^{\tau} \phi_{cc}(\tau) \quad \ldots \quad \ldots \quad (4.41)$$

Thus, over the period 2P :

$$\begin{aligned} \phi_{gg}(\tau) &= +1 \quad .. \quad \tau = 0 \\ &= -1 \quad .. \quad \tau = P \\ &= -1/P \quad .. \quad \tau = 2m, \; m = 1, 2, .. \\ &= +1/P \quad .. \quad \tau = 2m+1, \; m = 0, 1, 2, .. \end{aligned} \quad (4.42)$$

Hence, if the period P is considerably large, and the system settling time is less than the period of the sequence $\{c\}$ the two sequences may be used as inputs to a 2-input system, whereby the measured crosscorrelation function for values of τ less than P will give the impulse responses directly.

For a system having greater number of inputs, the following method may be made use of.

4.5.2 Use of transformed pseudorandom binary sequences for a multiple input system dynamic analysis

In Section 4.2, it is shown that the crosscorrelation between two m-sequences of periods coprime to each other is zero over their common period. Based on this principle, for a system with many inputs, it is possible to form several uncorrelated sequences (transformed pseudorandom binary sequences) from a given pseudorandom binary sequence by the use of Hadamard matrices and Rademacher functions as discussed below.

Consider a pseudorandom binary sequence $\{c\}$ of period P. Let $\{d_i\}$ be a sequence, the elements of which are given by-

$$d_{i,r} = b_{i,r} c_r \quad \ldots \quad \ldots \quad (4.43)$$

where c_r are the elements of $\{c\}$, and $b_{i,r}$ are the elements of sequence $\{b_i\}$ of period Q (P and Q are coprime).

The crosscorrelation function between two such sequences $\{d_i\}$ and $\{d_v\}$ over the common period PQ is :

$$\phi_{iv}(\tau) = \frac{1}{PQ} \sum_{r=1}^{PQ} d_{i,r}\, d_{v,(r+\tau)}$$

$$= \frac{1}{PQ} \sum_{r=1}^{PQ} b_{i,r} c_r\, b_{v,r+\tau}\, c_{r+\tau} \quad \ldots \quad (4.44)$$

By considering the period PQ as P sums, each of length Q, eqn. (4.44) can be shown to be :

$$\phi_{iv}(\tau) = \frac{1}{P} \sum_{t=1}^{P-1} b_{i,t}\, b_{v,t+\tau}\, \frac{1}{Q} \sum_{r=1}^{Q} c_r c_{r+\tau}$$

$$= F_{iv}(\tau) \quad \phi_{cc}(\tau) \quad \ldots \quad \ldots \quad (4.45)$$

In order that the sequences $\{d_v\}$ be used as test signals to the multiple input system, it may be mow said that the crosscorrelation function between the sequences $\{b_i\}$ and $\{b_v\}$ namely, $F_{iv}(c)$ should satisfy the following requirements :

$$F_{ii}(0) = 1$$

$$F_{iv}(c) = 0, \quad i = v \quad \ldots \quad \ldots \quad (4.46)$$

This conditions can be satisfied provided that the sequences $\{b_i\}$s are chosen from the rows of Hadamard matrix of order Q as shown below :

An Hadamard matrix (Baumert, 1962), is a square matrix whose elements are 1s and -1s and whose rwo vectors are mutually orthogonal (equivalently, whose column vectors are mutually orthogonal). From this definition, it is clear that one may

(1) interchange rows
(2) interchange columns
(3) change the sign of every element in a row
(4) change the sign of every element in a column

without disturbing the Hadamard property. Using these operations it is possible to establish a normal form for Hadamard matrices by insisting that the first row and the first column contain only 1s. Hadamard matrixes exist for $Q = 1, 2$, and most multiples of 4. Particularly simple methods of construction are available when $Q = 2^k$ (k being an integer), when the rows may taken as Walsh functions (Henderson 1964) and, in such a case, (k + 1) of the rows are Rademacher functions (Handerson 1964), the crosscorrelation between which is zero for all shifts. (For a $2^k \times 2^k$ Hadamard matrix of standard form, rows 0, 1, 2, 4 ... 2^p ... 2^{k-1} are Rademacher functions). This means that, for sequences $\{d_j\}$ formed from sequences $\{b_j\}$, which are Rademacher functions of length 2^k, in the manner indicated in eqn. (4.43), the crosscorrelation function $\phi_{ij}(\tau)$ over the period 2^kN will be zero for all values of τ. This is a useful point.

The Hadamard matrices of order 2, 4 and 8 are given below :

row		
0	1	1
1	1	-1

row				
0	1	1	1	1
1	1	-1	1	-1
2	1	1	-1	-1
3	1	-1	-1	1

row								
0	1	1	1	1	1	1	1	1
1	1	-1	1	-1	1	-1	1	-1
2	1	1	-1	-1	1	1	-1	-1
3	1	-1	-1	1	1	-1	-1	1
4	1	1	1	1	-1	-1	-1	-1
5	1	-1	1	-1	-1	1	-1	1
6	1	1	-1	-1	-1	-1	1	1
7	1	-1	-1	1	-1	1	1	-1

The dotted lines indicate how the matrix of order 2^k may be built up from the order 2^{k-1}. The signals for the 2-input system discussed in section (4.5.1) are obtained using the Hadamard matrix of order 2. For the matrix of order 4 and 8 the sequences $\{b_i\}$ (rows of the matrix) have autocorrelation and crosscorrelation functions as set out in Table 4.1 and 4.2, using these values and substituting in eqn. (4.45), the autocorrelation and crosscorrelation functions of the sequences $\{d_i\}$ are as shown in Table (4.3 and 4.4). It may be noted here that the sequence $\{d_o\}$ is the original PRBS $\{c\}$ repeated four times.

Table 4.1 :

Autocorrelation and crosscorrelation functions of the rows of a 4 x 4 Hadamard matrix - the sequences $\{b_o\}$, $\{b_1\}$, $\{b_2\}$, and $\{b_3\}$

Correlation function	Shift τ			
	0	1	2	3
$F_{oo}(\tau)$	1	1	1	1
$F_{11}(\tau)$	1	-1	1	-1
$F_{22}(\tau)$	1	0	-1	0
$F_{33}(\tau)$	1	0	-1	0
$F_{01}(\tau)$	0	0	0	0
$F_{02}(\tau)$	0	0	0	0
$F_{03}(\tau)$	0	0	0	0
$F_{12}(\tau)$	0	0	0	0
$F_{13}(\tau)$	0	0	0	0
$F_{23}(\tau)$	0	-1	0	1

Table 4.2 :

Autocorrelation and crosscorrelation functions of the rows of 8 x 8 Hadamard matrix - the sequences $\{b_0\}$ to $\{b_7\}$

Correlation function	Shift τ							
	0	1	2	3	4	5	6	7
$F_{00}(\tau)$	1	1	1	1	1	1	1	1
$F_{11}(\tau)$	1	-1	1	-1	1	-1	1	-1
$F_{22}(\tau)$	1	0	-1	0	1	0	-1	0
$F_{33}(\tau)$	1	0	-1	0	1	0	-1	0
$F_{44}(\tau)$	1	0.5	0	-0.5	-1	-0.5	0	0.5
$F_{55}(\tau)$	1	-0.5	0	0.5	-1	0.5	0	-0.5
$F_{66}(\tau)$	1	-0.5	1	-0.5	-1	-0.5	0	0.5
$F_{77}(\tau)$	1	-0.5	0	0.5	-1	0.5	0	-0.5
$F_{01}(\tau)$ to $F_{07}(\tau)$	0	0	0	0	0	0	0	0
$F_{10}(\tau)$ to $F_{17}(\tau)$	0	0	0	0	0	0	0	0
$F_{23}(\tau)$	0	1	0	-1	0	1	0	-1
$F_{24}(\tau)$ to $F_{27}(\tau)$	0	0	0	0	0	0	0	0
$F_{34}(\tau)$ to $F_{37}(\tau)$	0	0	0	0	0	0	0	0
$F_{45}(\tau)$	0	0.5	0	0.5	0	-0.5	0	-0.5
$F_{46}(\tau)$	0	0.5	1	0.5	0	-0.5	-1	-0.5
$F_{47}(\tau)$	0	-0.5	0	0.5	0	0.5	0	-0.5
$F_{56}(\tau)$	0	0.5	0	-0.5	0	-0.5	0	0.5
$F_{57}(\tau)$	0	-0.5	1	-0.5	0	0.5	-1	0.5
$F_{67}(\tau)$	0	0.5	0	0.5	0	-0.5	0	-0.5

Table 4.3 :

Autocorrelation and crosscorrelation functions of the sequences $\{d_o\}$, $\{d_1\}$, $\{d_2\}$, and $\{d_3\}$ formed by multiplication of a PRBS of period N bits by the rows of a 4th order Hadamard matrix

Shift τ	0	1	2	3	4 . .	.N-1	N	N+1 . .	2N-1	2N	2N+1 . .	3N-1	3N	3N+1 . .	4N-1	4N
$\phi_{oo}(\tau)$	1	$-\frac{1}{N}$	$-\frac{1}{N}$	$-\frac{1}{N}$	$-\frac{1}{N}$	$-\frac{1}{N}$	1	$-\frac{1}{N}$	$-\frac{1}{N}$	1	$-\frac{1}{N}$	$-\frac{1}{N}$	1	$-\frac{1}{N}$	$-\frac{1}{N}$	1
$\phi_{11}(\tau)$	1	$\frac{1}{N}$	$-\frac{1}{N}$	$\frac{1}{N}$	$-\frac{1}{N}$	$-\frac{1}{N}$	-1	$-\frac{1}{N}$	$\frac{1}{N}$	1	$\frac{1}{N}$	$-\frac{1}{N}$	-1	$-\frac{1}{N}$	$\frac{1}{N}$	1
$\phi_{22}(\tau) = \phi_{33}(\tau)$	1	0	$\frac{1}{N}$	0	$-\frac{1}{N}$	$\frac{1}{N}$	0	$-\frac{1}{N}$	0	-1	0	$-\frac{1}{N}$	0	$\frac{1}{N}$	0	1
$\phi_{01}(\tau) = \phi_{02}(\tau)$ $= \phi_{03}(\tau) = \phi_{12}(\tau) = \phi_{13}(\tau)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\phi_{23}(\tau)$	0	$\frac{1}{N}$	0	$-\frac{1}{N}$	0	0	1	0	$\frac{1}{N}$	0	$-\frac{1}{N}$	0	-1	0	$-\frac{1}{N}$	0

Table 4.4 :

Autocorrelation and crosscorrelation functions of the sequences $\{d_0\}$, $\{d_1\}$ .. ,$\{d_7\}$ formed by multiplication of a PRBS of period N bits by the rows of an 8th order Hadamard matrix

Shift τ	0	1	2	3	4	..N-1	N	N+1..	2N-1	2N	2N+1..	3N-1	3N	3N+1..	4N-1	4N
$\phi_{oo}(\tau)$	1	$-\frac{1}{N}$	$-\frac{1}{N}$	$\frac{1}{N}$	$-\frac{1}{N}$	$-\frac{1}{N}$	$\frac{1}{N}$	$-\frac{1}{N}$	$-\frac{1}{N}$	$\frac{1}{N}$	$-\frac{1}{N}$	$-\frac{1}{N}$	$\frac{1}{N}$	$-\frac{1}{N}$	$-\frac{1}{N}$	$\frac{1}{N}$
$\phi_{11}(\tau)$	1	$\frac{1}{N}$	$-\frac{1}{N}$	-1	$-\frac{1}{N}$	$-\frac{1}{N}$	0	$-\frac{1}{N}$	$\frac{1}{N}$	0	$\frac{1}{N}$	$-\frac{1}{N}$	-1	$-\frac{1}{N}$	$\frac{1}{N}$	0
$\phi_{22}(\tau)$	1	0	$-\frac{1}{N}$	0	$\frac{1}{N}$	$-\frac{1}{N}$	0	$\frac{1}{N}$	0	0	0	0	0	$-\frac{1}{N}$	0	$\frac{1}{N}$
$\phi_{33}(\tau)$	1	0	$-\frac{1}{N}$	0	$\frac{1}{N}$	$-\frac{1}{N}$	0	$\frac{1}{N}$	0	-1	0	$\frac{3}{4N}$	0	$-\frac{1}{N}$	0	0

Shift τ	4N+1	..5N-1	5N	5N+1..	6N-1	6N	6N+1..	7N-1	7N	7N+1...	8N-1	8N
$\phi_{oo}(\tau)$	$-\frac{1}{N}$	$-\frac{1}{N}$	$\frac{1}{N}$	$-\frac{1}{N}$	$-\frac{1}{N}$	$\frac{1}{N}$	$-\frac{1}{N}$	$-\frac{1}{N}$	$\frac{1}{N}$	$-\frac{1}{N}$	$-\frac{1}{N}$	1
$\phi_{11}(\tau)$	$\frac{1}{N}$	$-\frac{1}{N}$	-1	$-\frac{1}{N}$	$\frac{1}{N}$	0	$\frac{1}{N}$	$-\frac{1}{N}$	-1	$-\frac{1}{N}$	$\frac{1}{N}$	1
$\phi_{22}(\tau)$	0	0	0	0	0	$-\frac{1}{N}$	0	$\frac{1}{N}$	0	0	0	1
$\phi_{33}(\tau)$	0	$-\frac{1}{N}$	0	$\frac{1}{N}$	0	-1	0	$\frac{3}{4N}$	0	$-\frac{1}{N}$	0	1

Table 4.4 Contd.

Shift τ	0	1	2	3	4 ..	N-1	N	N+1 ..	2N-1	2N	2N+1 ..	3N-1	3N	3N+1 ..	4N-1	4N
$\phi_{44}(\tau)$	1	$-\frac{3}{4N}$	$-\frac{1}{N}$	$-\frac{3}{4N}$	0 ..	$\frac{1}{N}$	$\frac{3}{4N}$	0	$\frac{1}{2N}$	$-\frac{3}{2N}$	$\frac{1}{2N}$	0	$-\frac{3}{2N}$	$\frac{1}{N}$	$\frac{1}{2N}$	-1
$\phi_{55}(\tau)$	1	$\frac{1}{2N}$	$-\frac{1}{N}$	$-\frac{3}{2N}$	0	$-\frac{1}{N}$	$-\frac{3}{2N}$	0	$-\frac{1}{2N}$	0	$-\frac{1}{2N}$	$-\frac{3}{4N}$	$\frac{3}{2N}$	$\frac{1}{N}$	$-\frac{1}{2N}$	-1
$\phi_{66}(\tau)$	1	$-\frac{1}{2N}$	0	$\frac{3}{2N}$	0	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0	$\frac{1}{2N}$	0	$\frac{3}{2N}$	$\frac{1}{N}$	$\frac{1}{2N}$	-1
$\phi_{77}(\tau)$	1	$\frac{3}{4N}$	$\frac{1}{N}$	$\frac{3}{2N}$	0	$\frac{1}{N}$	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0	$\frac{1}{2N}$	$\frac{1}{2N}$	$\frac{3}{2N}$	0	$\frac{1}{2N}$	-1

Shift τ	4N+1 ..	5N-1	5N	5N+1 ..	6N-1	6N	6N+1 ..	7N-1	7N	7N+1 ..	8N-1	8N
$\phi_{44}(\tau)$	$\frac{1}{2N}$	$\frac{1}{N}$	$-\frac{9}{4N}$	0	$\frac{1}{2N}$	0	$-\frac{1}{2N}$	0	$\frac{3}{2N}$	$-\frac{1}{N}$	$-\frac{1}{2N}$	1
$\phi_{55}(\tau)$	$-\frac{61}{2N}$	$\frac{1}{N}$	$\frac{3}{2N}$	$-\frac{3}{4N}$	$-\frac{1}{2N}$	0	$\frac{1}{2N}$	0	$-\frac{3}{2N}$	$-\frac{1}{N}$	$\frac{1}{2N}$	1
$\phi_{66}(\tau)$	$\frac{1}{2N}$	$\frac{1}{N}$	$-\frac{3}{2N}$	0	$\frac{1}{2N}$	0	$-\frac{1}{2N}$	0	$\frac{3}{2N}$	0	$-\frac{1}{2N}$	1
$o_{77}(\tau)$	$-\frac{1}{2N}$	0	$\frac{3}{2N}$	$-\frac{1}{2N}$	$-\frac{1}{2N}$	0	$\frac{1}{2N}$	0	$-\frac{3}{2N}$	$\frac{1}{N}$	$\frac{3}{4N}$	1

Table 4.4 : Contd.

Shift τ	0	1	2	3	4 ..	N-1	N	N+1..	2N-1	2N	2N+1..	3N-1	3N	3N+1..	4N-1	4N
$\phi_{01}(\tau)=..=\phi_{07}(\tau)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\phi_{23}(\tau)$	0	$-\frac{1}{N}$	0	1	0	0	1	0	$-\frac{1}{N}$	0	$\frac{1}{N}$	0	-1	0	$\frac{1}{N}$	0
$\phi_{24}(\tau)=..=\phi_{27}(\tau)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\phi_{34}(\tau)$	0	0	0	0	$\frac{1}{2N}$	0	0	$\frac{1}{2N}$	0	$\frac{3}{2N}$	$\frac{1}{2N}$	$\frac{1}{2N}$	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0

Shift τ	4N+1 ..	5N-1	5N	5N+1 ..	6N-1	6N	6N+1 ..	7N-1	7N	7N+1 ..	8N-1	8N
$\phi_{01}(\tau)=..\phi_{07}(\tau)$ $=\phi_{12}(\tau)=..\phi_{12}(\tau)$	0	0	0	0	0	0	0	0	0	0	0	0
$\phi_{23}(\tau)$	$=\frac{1}{N}$	0	1	0	$-\frac{1}{N}$	0	$\frac{1}{N}$	0	-1	0	$-\frac{1}{N}$	0
$\phi_{24}(\tau)=..=\phi_{27}(\tau)$	0	0	0	0	0	0	0	0	0	0	0	0
$\phi_{34}(\tau)$	0	0	0	$-\frac{1}{2N}$	0	$\frac{3}{2N}$	$-\frac{1}{2N}$	$-\frac{1}{2N}$	$\frac{3}{2N}$	0	$-\frac{1}{2N}$	0

Table 4.4: Contd.

Shift τ	0	1	2	3	4..	N−1	N	N+1..	2N−1	2N	2N+1..	3N−1	3N	3N+1..	4N−1	4N
$\phi_{35}(\tau)=\phi_{36}(\tau)=\phi_{37}(\tau)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\phi_{45}(\tau)$	0	$-\frac{1}{2N}$	0	$\frac{3}{2N}$	0	0	$\frac{3}{2N}$	0	$-\frac{1}{2N}$	0	$-\frac{1}{2N}$	0	$\frac{3}{2N}$	0	$-\frac{1}{2N}$	0
$\phi_{46}(\tau)$	0	$-\frac{1}{2N}$	0	$-\frac{3}{3N}$	$-\frac{1}{N}$	0	$\frac{3}{2N}$	$-\frac{1}{N}$	$-\frac{1}{2N}$	1	$-\frac{1}{2N}$	$-\frac{1}{N}$	$\frac{3}{2N}$	0	$-\frac{1}{2N}$	0
$\phi_{47}(\tau)$	0	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0

Shift τ	4N+1..	5N−1	5N	5N+1..	6N−1	6N	6N+1..	7N−1	7N	7N+1..	8N−1	8N
$\phi_{35}(\tau)=\phi_{36}(\tau)=\phi_{37}(\tau)$	0	0	0	0	0	0	0	0	0	0	0	0
$\phi_{45}(\tau)$	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0
$\phi_{46}(\tau)$	$\frac{1}{2N}$	0	$\frac{3}{2N}$	$\frac{1}{N}$	$\frac{1}{2N}$	−1	$\frac{1}{2N}$	$\frac{1}{N}$	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0
$\phi_{47}(\tau)$	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0	$\frac{1}{2N}$	0	$\frac{1}{2N}$	0	$\frac{1}{2N}$	0

Table 4.4 Contd.

Shift τ	0	1	2	3	4 ..	N-1	N	N+1 ..	2N-1	2N	2N+1 ..	3N-1	3N	3N+1 ..	4N-1	4N
$\phi_{56}(\tau)$	0	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0
$\phi_{57}(\tau)$	0	$\frac{1}{2N}$	0	$\frac{3}{2N}$	$\frac{1}{N}$	0	$\frac{3}{2N}$	$\frac{1}{N}$	$\frac{1}{2N}$	1	$\frac{1}{2N}$	$\frac{1}{N}$	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0
$\phi_{67}(\tau)$	0	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0

Shift τ	4N+1 ..	5N-1	5N	5N+1 ..	6N-1	6N	6N+1 ..	7N-1	7N	7N+1 ..	8N-1	8N
$\phi_{56}(\tau)$	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0
$\phi_{57}(\tau)$	$\frac{1}{2N}$	0	$\frac{3}{2N}$	$\frac{1}{N}$	$\frac{1}{2N}$	1	$\frac{1}{2N}$	$\frac{1}{N}$	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0
$\phi_{67}(\tau)$	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0	$\frac{1}{2N}$	0	$\frac{3}{2N}$	0	$\frac{1}{2N}$	0

4.6 USE OF 3-LEVEL PSEUDORANDOM SEQUENCES FORMED FROM PRBSS IN THE IDENTIFICATION OF MULTIVARIABLE SYSTEMS

This section deals with the formation of 3-level pseudorandom sequences from the conventional pseudorandom binary sequences, which may be used as test signals in the multivariable system dynamic analysis.

Let $\{c\}$ be a pseudorandom binary sequence with elements +1 and 0 and is of period N. Then, the antisymmetric difference sequence $\{a\}$ formed in accordance with the following equation :

$$a_r = \frac{1}{2}(-1)^r (c_r - c_{r-1}) \quad \ldots \quad \ldots \quad \ldots \quad (4.47)$$

is a 3-level sequence with elements ± 1 and 0. Since N is odd, $\{a\}$ has period 2N, while its second half is the negative of the first half.

The autocorrelation function of $\{a\}$ is given by :

$$\phi_{aa}(\tau) = \frac{1}{2N}\sum_{r=1}^{2N}\left[\frac{1}{2}(-1)^r(c_r - c_{r-1})\,\frac{1}{2}(-1)^{r+\tau}\cdot(c_{r+\tau} - c_{r-1+\tau})\right]$$

$$= \frac{1}{8N}(-1)^{\tau}\sum_{r=1}^{2N}(c_r c_{r+\tau} - c_r c_{r-1+\tau} - c_{r-1}c_{r+\tau} + c_{r-1}c_{r-1+\tau})$$

$$= \frac{1}{4}(-1)^{\tau}\left[2\phi_{cc}(\tau) - \phi_{cc}(\tau-1) - \phi_{cc}(\tau+1)\right] \quad \ldots \quad (4.48)$$

This gives

$$\phi_{aa}(\tau) = \frac{1}{4}\,\frac{N+1}{N}\,, \quad \tau = \pm 1 \mod 2N$$

$$= \frac{1}{2}\,\frac{N+1}{N}\,, \quad \tau = 0 \mod 2N$$

$$= -\frac{1}{4}\,\frac{N+1}{N}\,, \quad \tau = (N \pm 1) \mod 2N$$

$$= -\frac{1}{2}\,\frac{N+1}{N}\,, \quad \tau = N \mod 2N$$

$$= 0 \quad \text{for other cases .} \qquad \ldots \qquad \ldots. \quad (4.49)$$

Hence, the antisymmetric 3-level sequence $\{a\}$ exhibits pseudorandom properties. Also $\phi_{ac}(\tau)$ is zero for all τ (over the period 2N). The continuous autocorrelation function of the signal corresponding to $\{a\}$ is shown in Fig. (4.4). The positive and negative spikes about $\tau = kNt_o$, k being an integer, are of double the width and approximately half the amplitude of those of the autocorrelation function of the signal corresponding to $\{c\}$. This drawback could be compensated by careful choice of the basic interval t_o of $\{c\}$ with respect to the system time constants.

With the signal corresponding to $\{a\}$ as the system input, the impulse response may be obtained directly from the measured input-output correlation function, if the system settling time is less than Nt_o.

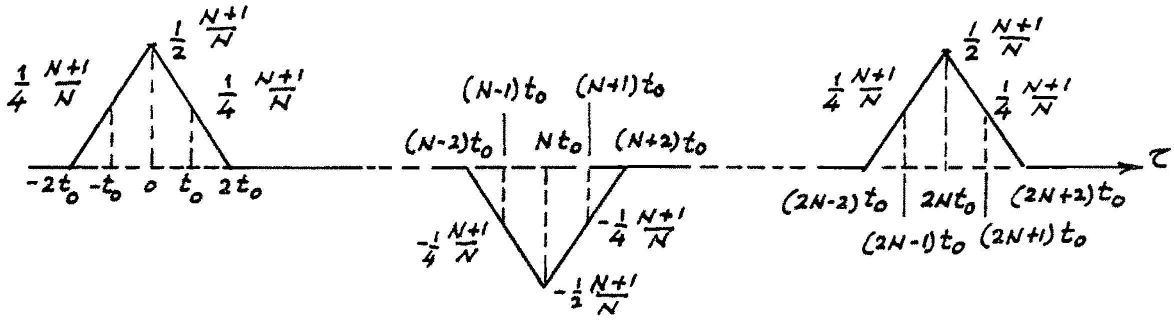


FIG. 4.4 AUTOCORRELATION FUNCTION OF SIGNAL CORRESPONDING TO THE ANTISYMMETRIC DIFFERENCE SEQUENCE {a}

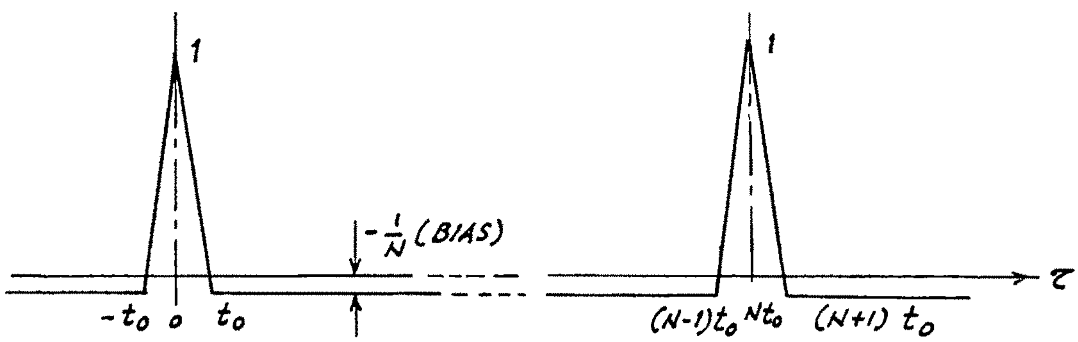


FIG. 4.5 (a) AUTOCORRELATION FUNCTION OF 2-LEVEL m-SEQUENCE SIGNAL

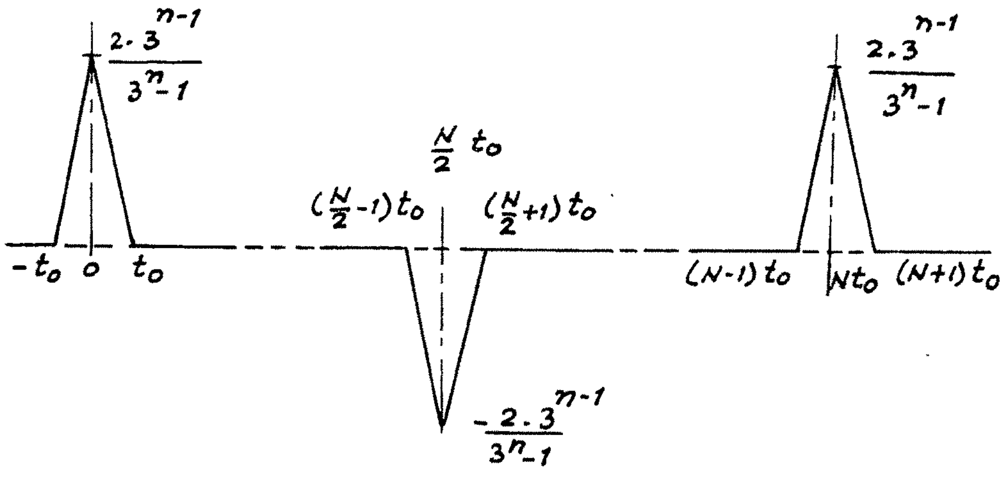


FIG. 4.5 (b) AUTO CORRELATION FUNCTION OF 3-LEVEL m-SEQUENCE SIGNAL

An example showing the formation of the ternary sequence {a} from {c} (formed in accordance with eqn.(4.47), where {c} of period 7 bits generated from a 3-stage binary shift register with feedback from second and third stage is shown in Table 4.5

Table 4.5 :

2-level m-sequence {c}	1	1	1	-1	-1	1	-1	1	1	1	-1	-1	1	-1
{c} shifted by one digit	-1	1	1	1	-1	-1	1	-1	1	1	1	-1	-1	1
Antisymmetric difference sequence {a}	-1	0	0	-1	0	1	1	1	0	0	1	0	-1	-1

In fact, the concept of the antisymmetric difference sequence {a} can be extended to formulate additional uncorrelated system - input signals in a similar manner to that described for a PRBS in section (4.5.2), by multiplying the difference sequence whose elements are $\frac{1}{2}(c_r - c_{r-1})$ by the rows of a Hadamard matrix and appropriately increasing the integration time. The autocorrelation function of such a sequence may be determined from the autocorrelation function of the difference sequence making use of eqn. (4.45).

4.7. IDEAL PSEUDORANDOM SIGNALS AS DERIVED FROM LINEAR SHIFT REGISTER SEQUENCES FOR USE IN SYSTEM IDENTIFICATION BY CROSS-CORRELATION

4.7.1. Statement of the Problem

So far, in this chapter, the current status of the cross-correlation art in the linear multivariable system identification is discussed. Schemes have been studied for uncorrelated system input signals with autocorrelation functions approximating to delta functions for use in the dynamic analysis of multi-input/output sys6ems by correlation method. So far, both two-level and three-level signals have been considered for the purpose.

With regard to the schemes studied thus far, some points of concern in the present section are the following :

1. In the schemes employing pseudorandom binary test perturbations, effort has not been made to compensate the bias present in the autocorrelation function of the binary signal (Fig. 4.5a). Although the error due to the bias effects vary inversely as N (Nt_o is the period of the signal, t_o being the basic interval), it cannot be made suitably small by choosing N large because the mean square errors due to wide-band noise are proportional to N. As a result a correction to the results must be made, based on measurement of input/output cross-correlation function over regions of its argument where the system weighting function is expected to be negligible. However,

in case the noise in the system is much significant, the estimation of bias should be made using several independent values of input/output correlation functions.

2. Further, in schemes making use of ternary pseudorandom test signals, the system settling time should be less than half the period of the signal. This is so because the ternary pseudorandom derived from the linear shift register m-sequence exhibits a negative spike in its autocorrelation for the lag $\tau = T/2$ (T being the period of the signal) due to the antisymmetric property of the m-sequence. (Fig. 4.5b). Thus the negative spike at the lag $\tau = T/2$ puts an upper limit to the use of the entire period of the ternary signal.

So much so, in the estimation of system weighting function, (i) to eliminate bias errors associated with conventional pseudorandom binary signals, and (ii) to make use of the entire period of the ternary signals, it is necessary that their autocorrelation function be of the form shown in Fig.(4.6b) which more closely approximates to that of the white noise (over one period of the signal) (Fig. 4.6a) as compared to their respective conventional counterparts. In practical terms, such an autocorrelation function may be termed as ideal autocorrelation. Accordingly, the signals which possess such an ideal autocorrelation may be called ' ideal pseudorandom signals.

This section is meant -

(i) to obtain the condition for a linear binary m-sequence to possess the said ideal autocorrelation, and

FIG. 4·6 (a) AUTOCORRELATION FUNCTION OF WHITE NOISE

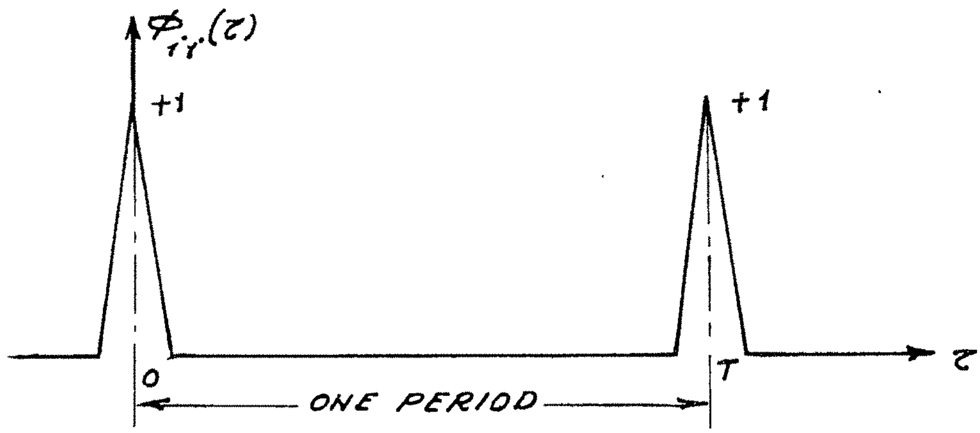


FIG. 4·6 (b) IDEAL AUTOCORRELATION FUNCTION, $\phi_{ii'}(\tau)$

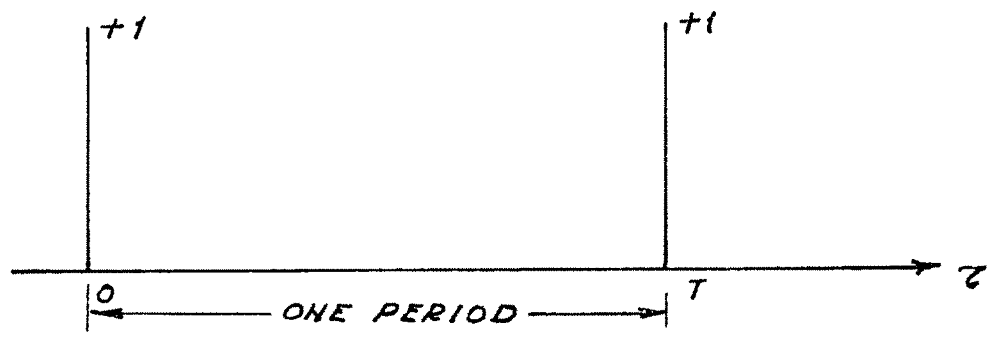


FIG. 4·6 (c) DISCRETE VERSION OF $\phi_{ii'}(\tau)$ OF FIG. (b)

(ii) to derive a simple criterion to determine when a linear p-nary shift register sequence ($p > 2$ and prime) exhibits ideal autocorrelation.

Some interesting properties of the p-nary shift register sequences having the ideal autocorrelation are also broughtforth. It needs no mention that such ideal sequences will be better-suited for system testing by cross-correlation. The use of these ideal sequences in the dynamic analysis of multiple input systems will be presented in the subsequent section.

The theory is developed here in terms of sequences. Signals are obtained from the sequences, and are held, during the interval $(r - 1)t_o < t < rt_o$, at a constant level determined by the rth element of the corresponding sequence (t_o being the basic interval or the clock pulse interval, of the signal). Thus, the autocorrelation function of any particular sequence $\{c\}$ is discrete, and the autocorrelation function of the corresponding signal is continuous with values at integral multiples of t_o given by the corresponding values of the sequence autocorrelation function and being linear between these points. (See Figs. 4.6b and 4.6c). Similar relationship exist for cross-correlation functions between different signals and between different sequences.

4.7.2. Ideal Pseudorandom Binary Sequences

Consider a pseudorandom binary sequence $\{c\}$ of N ($= 2^n - 1$, n an integer) digits, having $\frac{N+1}{2}$ digits in state 1 with α level and $\frac{N-1}{2}$ digits in state 0 with β level in one period. For any delay τ, the value of the autocorrelation function $\phi_{cc}(\tau)$ is :

$$\phi_{cc}(\tau) = \frac{1}{N} \sum_{i=0}^{N-1} c_i \; c_{i+\tau} \qquad \ldots \qquad \ldots \qquad (4.50)$$

Thus, $N\phi_{cc}(\tau)$ is the algebraic sum of N terms of the product sequence $\{c_i \; c_{i+\tau}\}$ each term of which being either α^2 or β^2 or $\alpha\beta$ depending on the aligning digits in $\{c_i\}$ and $\{c_{i+\tau}\}$.

Since $\{c_{i+\tau}\}$ is only a shifted version of the original sequence $\{c_i\}$, there are three different combinations of the aligning digits in the product sequence, viz. (1,1); (0,0); and (0,1). We assume here that the order is unimportant and hence the combination (0,1) is not different from (1,0).

Let the number of occurances of these alignments be denoted as follows -

$$\begin{aligned} L_{11} &= \text{Number of } (1,1) \\ L_{o,o} &= \text{Number of } (0,0) \qquad \ldots \qquad \ldots \qquad (4.51) \\ L_{o,1} &= \text{Number of } (0,1) \end{aligned}$$

These definitions mean that, considering only $0 \leq \tau \leq N-1$

$$\phi_{cc}(\tau) = \alpha^2 L_{1,1}(\tau) + \beta^2 L_{o,o}(\tau) + 2\alpha\beta L_{o,1}(\tau) \qquad \ldots (4.52)$$

Since a pseudorandom binary sequence (m-sequence) possesses the 'shift and add' property, it may be noted that $L_{11}(\tau)$, $L_{oo}(\tau)$ and $L_{o,1}(\tau)$ are independent of the lag $\tau \neq 0$, their values in one period of the sequence (Nt_o) being given by -

$$L_{1,1} = \frac{N+1}{4}$$
$$L_{o,o} = \frac{N+1}{4} - 1 \qquad \ldots \qquad \ldots \qquad (4.53)$$
$$L_{o,1} = \frac{N+1}{2}$$

Hence, the value of the autocorrelation function of the sequence $\{c\}$ becomes :

$$\phi_{cc}(\tau = 0) = \frac{1}{N}\left[\frac{N+1}{2}\alpha^2 + \frac{N-1}{2}\beta^2\right]$$
$$= K_o \text{ (constant) } \ldots \qquad \ldots \qquad (4.54)$$

$$\phi_{cc}(\tau \neq 0) = \frac{1}{N}\left[\frac{N+1}{4}\alpha^2 + \left(\frac{N+1}{4} - 1\right)\beta^2 - \frac{N+1}{2}\alpha\beta\right]$$
$$= K \text{ (constant) } \qquad \ldots \qquad (4.54b)$$

In the above equation, unless N assumes a very large value, K is not zero and this bias in the autocorrelation spoils the noise-like feature of the sequence (i.e. of the corresponding signal).

Now, let an ideal pseudorandom binary sequence $\{c'\}$ be defined as one, the autocorrelation function of which consists of a series of impulses of strength, say K_1 ($= \phi_{c'c'}(\tau = 0)$) spaced by the signal repetition period $T = Nt_o$, with zero off-peak value.

In accordance with this definition, for an ideal sequence we may write -

$$\phi_{c'c'}(\tau = 0) = K_1\left(1 - \frac{|\tau - mT|}{t_o}\right), \text{ if } |\tau - mT| < t_o, \quad m \text{ an integer}$$

$$= 0 \text{ , for other values} \quad \ldots \quad (4.55)$$

Such an ideal binary sequence can be realised from the conventional sequence $\{c\}$ provided the levels corresponding to its states, viz. α, β have the following relation -

$$\alpha = \left(1 \pm \frac{2\beta}{\sqrt{N+1}}\right) \quad \ldots \quad \ldots \quad (4.56)$$

and the autocorrelation function of the ideal sequence for zero shift then becomes :

$$\phi_{c'c'}(\tau = 0) = \frac{\beta^2}{N}\left(1 \pm \sqrt{N+1}\right)^2 \ldots \quad (4.57)$$

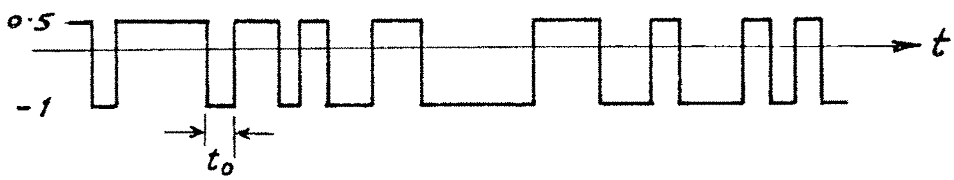
As N assumes very large values, the autocorrelation function for zero shift takes the value β^2 .

Thus, by proper adjustment of the levels α and β, it is possible to realise an ideal pseudorandom binary sequence. In

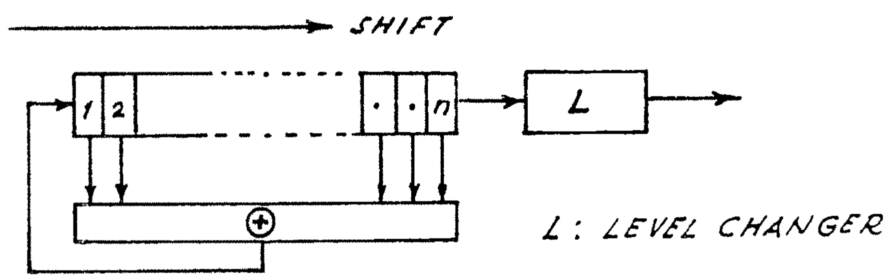
In order to limit the mean and peak to peak value of the signal, the negative sign in the eqn. (4.56) is preferable. The ideal pseudorandom binary sequence as generated by a 4-stage linear feedback shift register (fsr) along with its autocorrelation function is shown in Fig. (4.7). Setting $\beta = 1$, the value of α, and the mean-and mean-squared values of this ideal sequence are given in Table 4.6, for values of n from 2 to 16, n being the number of stages in the binary fsr generating the sequence.

Table 4.6

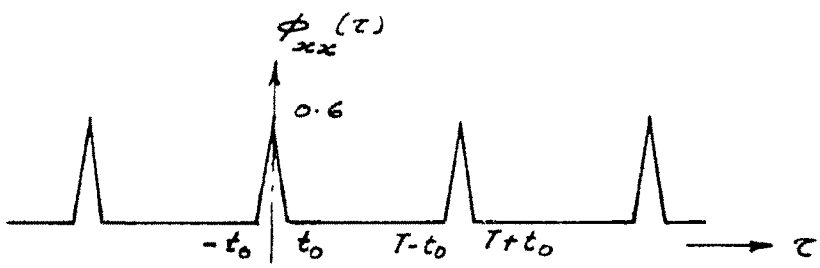
n	$N = 2^n - 1$	α	Mean of c(t)	$\phi_{cc}(0)$
2	3	0	- 0.33	0.33
3	7	0.29	- 0.26	0.48
4	15	0.50	- 0.20	0.60
5	31	0.65	- 0.15	0.70
6	63	0.75	- 0.11	0.77
7	127	0.82	- 0.08	0.84
8	255	0.86	- 0.06	0.88
9	511	0.91	- 0.04	0.92
10	1023	0.94	- 0.03	0.94
11	2047	0.95	- 0.02	0.96
12	4095	0.96	- 0.01	0.97
13	8191	0.97	- 0.009	0.98
14	16383	0.98	- 0.005	0.985
15	32767	0.99	0	0.989
16	65535	0.99	0	0.999



(a) TIME-VARIATION OF BINARY SIGNAL GENERATION



(b) DIGITAL SHIFT-REGISTER GENERATOR OF PERIODIC IDEAL PRBS.



(c) AUTO-CORRELATION FUNCTION FOR $n = 4$

FIG. 4·7 IDEAL PSEUDORANDOM BINARY SIGNAL GENERATION

4.7.3. Linear P-nary Shift Register Sequences with Ideal autocorrelation (p > 2 and prime)

A simple criterion is derived here to determine when a linear p-nary (p > 2 and prime) feedback shift register sequence possesses ideal autocorrelation function like that shown in Fig. (4.6b). Such an ideal sequence, when used as test perturbation in system-testing by cross-correlation facilitates the use of full period of the sequence unlike a p-nary m-sequence.

Consider a sequence $\{f_i\} = f_o, f_1, f_2, \ldots f_{M-1}; \ldots$ of period M digits where $f_i \in (o, \pm 1, \pm 2, \ldots, \pm \frac{p-1}{2})$, from a linear p-nary feedback shift register (fsr).

The autocorrelation function of this sequence is defined as :

$$M\phi_{ff}(\tau) = \sum_{i=0}^{M-1} f_i f_{i+\tau} \quad \ldots \quad \ldots \quad (4.58)$$

This means, for any time lag τ, the value of $M\,\phi_{ff}(\tau)$ is the algebraic sum of M terms formed by shifting $\{f_i\}$ relative to itself and obtaining the product of the corresponding aligning digits. Let this product sequence be denoted by $\{f_i'\}$ i.e.

$$\{f_i\} : \quad f_o,\ f_1,\ f_2,\ \ldots\ldots,\quad f_{M-2},\ f_{M-1}$$

$$\{f_{i+c}\} : \quad f_c,\ f_{c+1},\ f_{c+2},\ \ldots,\quad f_{c-2},\ f_{c-1}$$

$$\{f_i'\} = \{f_i \cdot f_{i+c}\} : \quad f_o',\ f_1',\ f_2',\ \ldots\ldots,\quad f_{M-2}',\ f_{M-1}'$$

Evidently, a term f_i' $(0 \leq i \leq M - 1)$ in the product sequence $\{f_i'\}$ will be positive or negative depending on whether the aligning digits are of same or opposite sign.

Let 's' be a positive integer which assumes any value in the range 1 to $\frac{p-1}{2}$.

In view of the formation of the product sequence we see that the following combinations occur :

(0 , 0) ;	(0 , s) ;	(0 , s*)	Here –
(s , s) ;	(s*, s*) ;	(s , s*)	$r \neq s$
(r , s) ;	(r*, s*) ;	(r , s*)	r, s $(1, 2, \ldots, \frac{p-1}{2})$
			$s^* = -s$
			$r^* = -r$

We assume that the order is uminportant and hence there is no distinction between the combinations (r , s) and (s , r)

Further, let λ_{ij} denote the number of occurances of the alignments (i , j) i.e.

$$\lambda_{ij} = \text{Number of alignments of } (i,j) \quad \ldots \quad \ldots \quad (4.59)$$

Now, from eqn. (4.54), the autocorrelation function of the sequence $\{f_i\}$ is :

$$\phi_{ff}(\tau) = \sum_{i=0}^{M-1} f_i f_{i+\tau}$$

$$= \sum_{i=0}^{M-1} f_i'$$

$$= D_1 - D_2 \quad \ldots \quad \ldots \quad \ldots \quad (4.60)$$

whereby

D_1 = algebraic sum of all positive integers in the product sequence $\{f_i'\}$, and

D_2 = algebraic sum of all negative integers in the product sequence $\{f_i'\}$

Hence, the necessary and sufficient condition for the autocorrelation function to be zero at the lag τ is that :

$$D_1 = D_2 \quad \ldots \quad \ldots \quad \ldots \quad (4.61)$$

which implies that -

$$[\Sigma \text{ integers formed from the alignments } \lambda_{ss}, \lambda_{s^*s^*}, \lambda_{rs} \text{ and } \lambda_{r^*s^*}]$$

$$= [\Sigma \text{ integers formed from the alignments } \lambda_{ss^*} \text{ and } \lambda_{rs^*}] \quad \ldots \quad \ldots \quad (4.62)$$

Now all the possible alignments stated above may be classified as under :

λ_{ss} , $\lambda_{s^*s^*}$ -- Alighments of digits with equal value and same sign.

λ_{ss^*} -- -- Alignments of digits with equal value and opposite sign.

λ_{rs} , $\lambda_{r^*s^*}$ -- Alignments of digits with unequal value and same sign.

λ_{rs^*} -- -- Alignments of digits of unequal value and opposite sign.

From this classification, autocorrelation function $\phi_{ff}(\tau)$ of the sequence $\{f_i\}$ will be zero at the lag τ provided that the product sequence satisfies the following requirements :

$$\text{(i)} \quad \lambda_{ss} + \lambda_{s^*s^*} = \lambda_{ss^*} \quad \text{(for each s considered independently)}$$

$$\text{(ii)} \quad \lambda_{rs} + \lambda_{r^*s^*} = \lambda_{rs^*} \quad \text{(for each combination r, s)} \qquad (4.63)$$

i.e. let s = 1, 2 and $s^* = -1, -2$ then

$$\lambda_{1,1} + \lambda_{-1,-1} = \lambda_{1,-1}$$

$$\lambda_{2,2} + \lambda_{-2,-2} = \lambda_{2,-2}$$

$$\lambda_{1,2} + \lambda_{-1,-2} = \lambda_{1,-2}$$

Thus, a linear p-nary shift register sequence which satisfies the requirements in Eqn. (4.63) possesses an ideal autocorrelation function of the form depicted in Fig. (4.6b).

In Chapter 3, the theory of multilevel shift register sequences is presented in some detail. It has been shown that the sequential behaviour of a linear shift register depends on the nature of its characteristic polynomial. When this polynomial is primitive modulo-p, an n-stage shift register generates a sequence of the largest possible period equal to $p^n - 1$ digits and is accordingly called as m- or maximal sequence. However, in case of an irreducible nonprimitive polynomial, several sequences of equal length will be generated. This of course implies that the longest sequence of such a polynomial is of length $\frac{p^n - 1}{2}$ digits. Finally, when the characteristic polynomial can be factored into irreducible polynomials, a number of sequences with different periods will be generated as per the factors.

In the correlation method of system identification, the test sequence period should be greater than the system settling time. For this reason, shift register sequences with larger period only are of current interest.

A p-nary m-sequence, for prime values of $p > 2$, has its second half as the modulo-p complement of its first half. This is known as the antisymmetric property of the sequence. As such, in one period, the autocorrelation function of an m-sequence will be zero for all values of its argument except

when the value of the argument equals '0' and $\frac{'p^n - 1'}{2}$.
Hence, m-sequence do not meet the requirements of the ideal autocorrelation as specified in eqn. (4.63). As an illustration, a quinary m-sequence is examined as shown below :

Example 4.5 :

(Test for zero autocorrelation at nonzero time-shifts)

Characteristic polynomial of feedback shift register ⁑ $4 + 2D + D^2$ (mod-5 addition)

(This means modulo-5 sum of outputs from stages 1 and 2 multiplied respectively by the coefficients 2 and 1 provides input to the 1st stage of the fsr)

Quinary m-sequence, say, $\{f_1\}$: .., 242230121140313320434410,..

(In determining the autocorrelation function of the m-sequence $\{f_1\}$ the states 0, 1, 2, 3, 4 are taken to correspond respectively to levels 0, 1, 2, -2, -1. This correspondance makes the signal derived from the sequence symmetric about the zero level).

Autocorrelation function of the above quinary m-sequence $\{f_1\}$ is given by :

$$\phi_{f_1 f_1}(\tau) = \frac{1}{24} \sum_{i=0}^{23} f_{1,i}\, f_{1,i+2} \quad , \text{(Here } p^n - 1 = 5^2 - 1 = 24 \text{)}$$

Results of the m-sequence are given in tabular form on the next page.

Results in the case of quinary m-sequence of Example (4.5)

Lag τ in digits	$\lambda_{1,1}$	$\underline{\lambda}_{1,-1}$	$\underline{\lambda}_{1,1}$	$\lambda_{2,2}$	$\underline{\lambda}_{2,-2}$	$\underline{\lambda}_{2,2}$	$\lambda_{1,2}$	$\underline{\lambda}_{1,-2}$	$\underline{\lambda}_{1,2}$	$\underline{\lambda}_{2,1}$	$\phi_{f_1 f_1}(\tau)$
$\tau = 0$	5	5	0	5	5	0	0	0	0	0	$\frac{20}{24} = \frac{2 \times 5^{n-1}}{5^n - 1}$
$\tau = 1$ to 11	1	1	2	1	1	2	2	2	2	2	0
$\tau = 12$	0	0	5	0	0	5	0	0	5	5	$-\frac{20}{24} = \frac{-2 \times 5^{n-1}}{5^n - 1}$
$\tau = 13$ to 23	1	1	2	1	1	2	2	2	2	2	0

As p-nary m-sequences do not exhibit ideal autocorrelation, it now remains to be seen whether a nonmaximal sequence has the desired autocorrelation. Since the present concern is with sequences that are of use in system identification, we examine only the longest nonmaximal sequences generated by a shift register with an irreducible nonprimitive characteristic polynomial. i.e. the sequences with period $\frac{p^n-1}{2}$ digits.

Table (4.7) gives some such irreducible nonprimitive polynomials and the sequences generated by corresponding shift registers for values of modulus-p from 3 to 7. A close look at the sequences reveals the following -

1. Sequences with period $\frac{p^n-1}{2}$ even exhibit antisymmetric property (e.g. sequences under serial numbers 1, 6, 7, 8, 9, 10, 11) and hence these cannot satisfy the requirements for ideal autocorrelation. It may be noted here that, since $\frac{5^n-1}{2}$ is always even for any n, no sequence associated with a quinary polynomial will have ideal autocorrelation.
2. Further, on testing sequences with period $\frac{p^n-1}{2}$ odd are seen to satisfy the requirements for ideal autocorrelation as specified in eqn. (4.63), (Sequences under serial numbers 2, 3, 4, 5 and 12).

The results obtained in case of (i) a ternary sequence and (ii) septenary sequence are given in Tables (4.8) and (4.9).

And in what follows in the next sub-section are given some interesting properties of such ideal sequences.

Table 4.7 : Some irreducible nonprimitive p-nary linear feedback shift register characteristic polynomials having exponent $\frac{p^n-1}{2}$

Form of characteristic equation of the fsr : $+\sum_{i=0}^{n} a_i D^i \{f_i\} = 0$

whereby -

'+' denotes modulo-p addition,

'a_i' (modulo field of integers 0, 1,.. p-1),

$a_o = p - 1$,

$\sum_{i=0}^{n} a_i D^i$ is the characteristic polynomial with 'D' as the Unit delay operator, and

$\{f_i\}$ is the generated nonmaximal sequence of period $\frac{p^n-1}{2}$ shift pulses.

Sr. No.	p	Polynomial		Nonmaximal sequences $\{f_i$	period
1	3	$2 + 2D^2$	(i)	1122, ... repeats	4
			(ii)	1020, ... repeats	4
2	3	$2 + D^2 + D^3$	(i)	1112201210010,...	13
			(ii)	2221102120020,...	13
3	3	$2 + 2D + D^3$	(i)	1110100121022,...	13
			(ii)	2220200212011,...	13
4	3	$2 + 2D + 2D^2+D^3$	(i)	1112110012020,...	13
			(ii)	2221220021010,...	13
5	3	$2 + D + D^2 + D^3$	(i)	1110202100112,...	13
			(ii)	2220101200221,...	13

(continued)

Sr. No.	p	Polynomial		Nonmaximal sequences f_i	period
6	3	$2 + 2D + 2D^2 + 2D^4$	(i)	1111011020001201120022222 222202201000210221000,..	40
			(ii)	11002012101012221121 22001021202021112212,..	40
7	3	$2 + 2D^2 + D^3 + 2D^4$	(i)	11112212202022201001 22221121101011102002,..	40
			(ii)	11000201202121002210 22000102101212001120,..	40
8	5	$4 + 3D + D^2$	(i)	143324412231,...	12
			(ii)	130340420210,...	12
9	5	$4 + 3D^2 + 4D^3$	(i)	1112204320334144134340412400103 4443301230221411421210143100402,..	62
			(ii)	2224403140113233213130324300201 3331102410442322342420231200304,..	62
10	7	$6 + D + 3D^2$	(i)	114055604420663022103350,...	24
			(ii)	431361512545346416265232,...	24
11	7	$6 + 3D^2 + 6D^3$	(i)	11144320346065516501321361403 1 61002044123634402146364462622 2 11640615053325302642652106252 0 04011246561104215651154544422 5 10523036643604514534205434001 0 2241535220142353 22313,..	171
			(ii)	66633450431012261206456416304 6 16005033654143305631413315155 5 66130162024452405135125601525 0 03066531216603562126623233355 2 60254041134103263243502343006 0 5536242550635424 55464,..	171

Table 4.8 : Test for ideal autocorrelation in the case of the nonmaximal sequence given in Table 4.7 with serial No.2

Let this sequence : 1112201210010,.. of period 13 digits be denoted by $\{f_2\}$.

In the evaluation of the autocorrelation function of a ternary sequence, the states 0,1 and 2 are respectively taken to correspond to the levels 0, 1 and -1. This correspondance makes the signal derived from the sequence symmetric about the zero level. So we write $\{f_2\}$ as -

$\{f_2\}$: 111-1-101-110010,... repeats

Lag τ in digits	$\lambda_{1,1}$	$\lambda_{-1,-1}$	$\lambda_{-1,1}$	$\phi_{f_2 f_2}(\tau) = \frac{1}{13} \sum_{i=0}^{13} f_{2,i} f_{2,i+\tau}$
$\tau = 0$	6	3	0	$\frac{9}{13} \left(= \frac{2 \times 3^{n-1}}{3^n - 1} \right)$
τ= 1,3,4, 9,10, 12	2	1	3	0
τ = 2,5,6, 7,8,11	3	0	3	0

* In the sequence $\{f_2\}$ the runs of the p-nary digits viz. 0, 1 and 2 are unequal. Evidently the sequence $\{f_2\}$ satisfies the requirements for ideal autocorrelation as specified in eqn. (4.63). In Table (4.9) a ternary sequence having equal runs for all the p-nary digits is examined.

Table 4.9 : Test for ideal autocorrelation in case of the nonmaximal sequence given in Table 4.7 with serial No.4

Let this sequence : 1112110012020,... of period 13 digits be denoted by $\{f_3\}$.

As before, in the evaluation of autocorrelation function, the states 0,1,2 are taken to correspond to levels 0, 1 and -1 respectively. So we write $\{f_3\}$ as -

$\{f_3\}$: 111-111001-10-10,.. repeats

Lag τ in digits	$\lambda_{1,1}$	$\lambda_{-1,-1}$	$\lambda_{-1,1}$	$\phi_{f_3 f_3}(\tau) = \frac{1}{13} \sum_{i=0}^{13} f_{3,i} f_{3,i+\tau}$
$\tau = 0$	6	3	0	$\frac{9}{13}$ $\left(= \frac{2 \times 3^{n-1}}{3^n - 1} \right)$
τ = 1,3,4, 9,10, 12	3	0	3	0
τ = 2,5,6, 7,8,11	2	1	3	0

* Evidently the sequence $\{f_3\}$ meets the requirements for ideal autocorrelation as specified in eqn. (4.63). It may be noted that the sequence $\{f_3\}$ has equal runs for all the ternary digits viz. 0, 1, 2 unlike $\{f_2\}$ (Table 4.8). We may, therefore, classify the ideal sequences into 2 groups - (1) sequences in which the total number of runs are equally distributed amongst all the p-nary digits (0 to p - 1) and (2) sequences in which the runs of digit are unequal.

4.7.4 Properties of ideal linear p-nary shift register sequences

From the results of the previous sub-section (4.7.3), some properties of p-nary linear shift register sequences possessing ideal autocorrelation function are broughtforth here as under :

It is convenient to take the p-levels corresponding to the p-states viz. 0,1,2,.., p-1 as 0, 1, 2,..., $\frac{p-1}{2}$, $-\frac{p-1}{2}$, $-\frac{p-3}{2}$,..., -2 , -1. Such a correspondance makes the signal derived from the sequence symmetrical about the zero-level.

Let -

$$s \in (1, 2, \ldots, \frac{p-1}{2})$$

then, $s^* \in (-1, -2, \ldots, -\frac{p-1}{2})$

1. For an n-stage shift register, the length of an ideal sequence is $\frac{p^n - 1}{2}$ shift pulses. A shift register whose characteristic polynomial is irreducible and nonprimitive with exponent $\frac{p^n - 1}{2}$ generates two such ideal sequences, provided the sequence period is odd. These two sequences are modulo-p complement of each other.

2. If any n consecutive digits (0s, 1s, ... p-1s) are considered as a p-nary number between 0 and $p^n - 1$, then in one period of the ideal sequence one half of all the p-nary

numbers in this range appear once and once only, except the all-zero state which does not appear at all.

3. An ideal sequence contains in its period $\frac{p^{n-1} - 1}{2}$ zeros.

4. The number of p-nary nonzero n-tuples appearing in an ideal sequence equals $\frac{p - 1}{2}$. The all-zero tuple does not appear just as in a p-nary m-sequence.

5. The non-zero digits, whose n-tuples have appeared in the sequence, are equal in number.

6. If 'A' denotes the number of any nonzero digits whose n-tuples has occured in the sequence, and if 'B' denotes the number of any nonzero digit whose n-tuples has not occured, then -

$$A - B = p, \text{ the value of the modulus (p prime)}$$

7. The total number of runs of all the p-nary digits (0, 1, 2,.., p - 1) in one period of an ideal sequence equals the difference between the total number of digits in the sequence and the number of zeros i.e.

$$\text{Total No. of runs} = \frac{p^n - 1}{2} - \frac{p^{n-1} - 1}{2}$$

$$= \frac{(p - 1)(p^{n-1})}{2}$$

8. Every p-nary nth degree irreducible nonprimitive characteristic polynomial with exponent $\frac{p^n - 1}{2}$, for $\frac{p^n - 1}{2}$ odd, generates two ideal sequences. One of these has equal number of runs for all its p-nary digits. But, in the other the number of runs of nonzero digit~~s~~ differs from the number of runs of zero digit by one. Based upon this consideration, the ideal sequences for given p and n can be classified into two types -

<u>Type - 1</u> : Ideal sequence with equal number of runs for all its p-nary digits (in one period). Hence - the number of runs of any p-nary digit is given by :

$$\frac{(p - 1)(p^{n-2})}{2} = \frac{(\text{Total No. of runs})}{p}$$

<u>Type - 2</u> : Ideal sequence wherein number of runs of any nonzero digit differs from the number of runs of the zero digit by one. Thus -

No. of runs of a nonzero digit whose n-tuple has appeared in the sequence = No. of runs of0s + 1

$$= \frac{p^{n-1} - p}{2} + 1$$

No. of runs of a nonzero digit whose n-tuple does not appear in the sequence = No. of runs of zeros - 1

$$= \frac{p^{n-1} - p}{2} - 1$$

9. The second order autocorrelation function of an ideal p-nary sequence is given by -

$$\phi_{ff}(\tau) = \frac{1}{M}\sum_{i=0}^{M-1} f_i f_{i+\tau}$$

$$= \frac{2.p^{n-1}}{p^n - 1}\ , \quad \tau = 0 \ (\text{mod}\ M = p^n - 1)$$

$$= 0 \qquad \text{for other values of } \tau.$$

The continuous autocorrelation function of the corresponding p-level signal is shown in Fig. (4.8).

A p-level ideal sequence is suitable for the determination of the system impulse response of a linear system, provided that the system settling time is less than Mt_o, where t_o is the basic interval or the digit period of the sequence.

10. With regard to the evaluation of the autocorrelation function of an ideal sequence $\{f_i\}$, if λ_{ij} denotes the number of occurances of the aligning digits (i, j), where $i, j \in (0, \pm 1, \pm 2, \ldots, \pm \frac{p-1}{2})$, then

$$\lambda_{o,o} = \frac{p-1}{2}$$

$\lambda_{s,s} + \lambda_{s^*,s^*} = \lambda_{s,s^*} = p$ for each value of s considered independently,

$\lambda_{r,s} + \lambda_{r^*,s^*} = {}_{r,s^*} = 2p$ For each combination (r, s and $r \neq s$) considered separately,

$$\sum_s (\lambda_{o,s} + \lambda_{o,s^*}) = p(p-1)$$

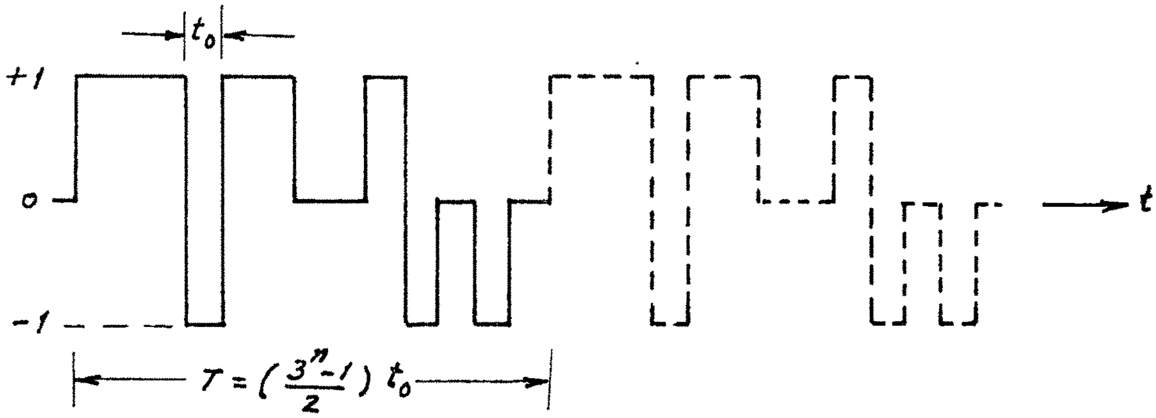


FIG. 4·8 (a) IDEAL-TERNARY SIGNAL -TYPE -1 ($n = 3$)

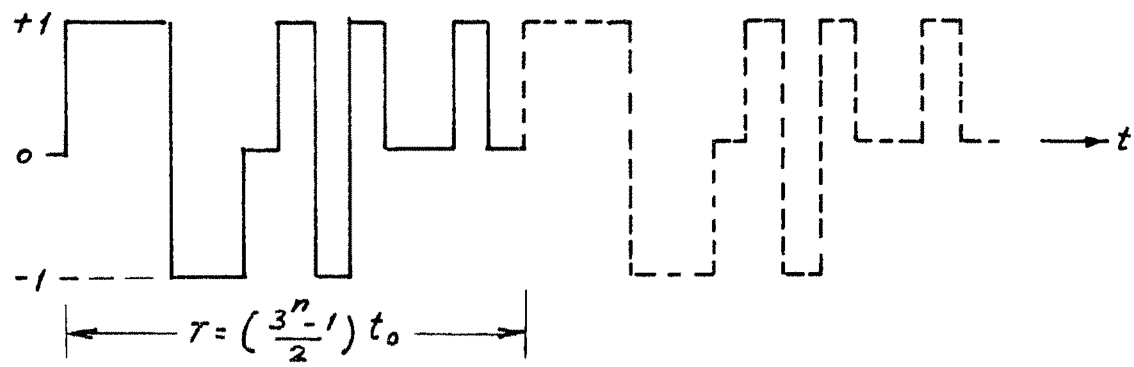


FIG. 4·8 (b) IDEAL-TERNARY SIGNAL - TYPE - 2 ($n = 3$)

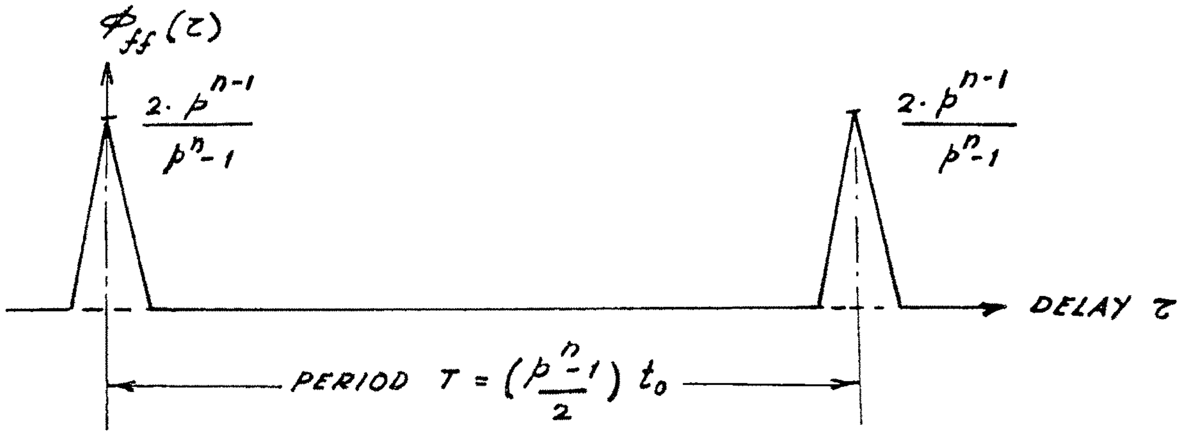


FIG. 4·8 (c) AUTO-CORRELATION FUNCTION OF
A p-NARY IDEAL SIQUENCE-SIGNAL

11. If a Type - 1 ideal p-level sequence (see property 8) $\{f_i\}$ is subtracted, modulo-p, from the same sequence delayed by k digits, the resulting sequence is the original sequence delayed by q digits where k and q are integers in the range $1 \leq k,q \leq \frac{p^n - 1}{2}$. Thus -

$$D^k(f_i) - (f_i) = D^q(f_i) .$$

This means, Type - 1 ideal sequence possesses shift and subract property.

Example :

Sequence (f_i) : 1112110012020,..

$D^3(f_i)$: 0201112110012,..

$D^3(f_i) - (f_i)$: 1211001202011,..

: $D^{11}(f_i)$

Since Type-2 ideal sequence, likewise, exhibits shift and add property.

12. If an ideal p-nary sequence $\{f_i\}$ is multiplied alternatively by -1 and +1, the resulting sequence is a p-nary m-sequence.

Example :

$\{f_i\}$: 1110202100112,.. (period = 13 digits)

$(-1)^r\{f_i\}$: 12102022001222120101100211,.. (period = 26 digits)

13. The delayed versions Df, D^2f, ..,D^nf of an ideal sequence $\{f\}$, are of course, available as outputs of the stages of the shift register. The delayed versions $D^{n+1}f$, $D^{n+2}f$, .. D^Mf ($M = \frac{p^n - 1}{2}$) of the sequence can be made available by adding or by subtracting, modulo-p, the outputs of two or more of the stages in the register. There is no explicit method of determining which of the outputs to add and which to subtract (modulo-p) to obtain a particular delay. For this purpose, in Chapter 3.7 a new method of evaluating shift register connections is described, which makes use of the familiar concept of the generating function associated with the sequence. As shown in Section 3.7, this generating function method yields quick results as the only calculation involved here is simple modulo-p addition (or subtraction).

14. Sampling an ideal sequence by a sampling period of j digits where j is relatively prime to the period length of the sequence $\frac{p^n - 1}{2}$, leads to an ideal sequence.

15. If in the ideal sequence $\{f\}$, we replace the nonzero terms by zero and the zero terms by one, the resulting binary sequence is again pseudorandom.

Let this transformed sequence be denoted by $\{f_t\}$

Example : (Ternary sequence)

$\{f\}$: 1110202100112,...

$\{f_t\}$: 0001010011000,...

Now, in the determination of the autocorrelation function of a binary sequence, the states 0 and 1 correspond to the levels -1 and 1 respectively. Under the circumstances, the autocorrelation function of the transformed sequence $\{f_t\}$ of period$(p^n-1)/2$ digits is : (in one period)

$$\phi_{f_t f_t}(\tau = 0) = 1$$

$$\phi_{f_t f_t}(\tau \neq 0) = 2/(p^n - 1) = 2/26$$

16. For every ideal sequence obtained from an n-stage shift register, there exists a related n-stage shift register which will produce a reverse ideal sequence. This reverse sequence is the original sequence read in the reverse order. The procedure for finding the reverse sequence shift register connections is as follows :

If an n-stage shift register has feedback connections from stages n, k, m, ... etc. the reverse sequence shift register will have feedback connections from stages n, n-k, n-m,... etc. with feedback coefficients (multipliers) being p - 1 for all stages except the nth stage.

17. It has been shown that p-level linear shift register maximal sequences exhibit a reference phase which makes the corresponding signal uncorrelated with constant and linear signal. (Barker, H.A. 1969). This concept can be extended to the present case of p-level linear nonmaximal ideal sequence.

If the analogue levels corresponding to the states of an ideal p-nary sequence are selected so that the mean value of the corresponding signal, f(t), obtained from the ideal sequence $\{f\}$ becomes zero, then there exists a characteristic or reference phase for the signal f(t) for which -

$$\int_0^T t\,f(t) = 0, \quad \text{and} \quad \int_0^T t^2\, f(t) = \text{a minimum.}$$

In the estimation of system weighting function by cross-correlation, use of a tabulated reference phase eliminates errors due to constant and linear drift in the system output, and minimizes errors due to quadratic drift.

Example : (Reference phase of an ideal Ternary sequence)

Consider the ideal ternary sequence $\{f\}$ given by -

$\{f\}$: 1110100121022,... repeats of period 13 digits.

In every ternary ideal sequence, there are $\frac{3^{n-1}-1}{2}$ zeros . Further, the digits 1s and 2s are in the ratio 2:1(or 1:2, see Table 4.7). Hence to make the mean of the signal f(t) obtained from sequence $\{f\}$ zero, the states 0, 1, and 2 in the sequence should correspond to levels 0, 0.5, -1 (or 0, 1, -0.5 in case of sequence in which 1s and 2s are in ratio 1:2). With these levels, the first moment taken over one period of the signal f(t) is given in tabular form below for all possible starting phases.

Sequence	$\frac{1}{T} \int_0^T t\, f(t)\,dt$
1110100121022,..	-2
2111010012102,..	+1
2211101001210,..	+0.5
0221110100121,..	+0.1
1022111010012,..	0
2102211101001,..	+1
1210221110100,..	+0.5
0121022111010,..	+0.5
0012102211101,..	+0.5
1001210221110,..	0
0100121022111,..	0
1010012102211,..	-0.5
1101001210221,..	-1

As can be seen, there are 3 starting phases of the signal for which the first time moment is zero. Further it is seen that the sequence underlined in the above table has a minimum value for its second order time moment over one period of the signal. Thus the sequence underlined is uncorrelated with constant and linear signals.

4.8 SINGLE AND MIXED-MODULI SHIFT REGISTER SEQUENCES AS TEST PERTURBATIONS IN MULTIVARIABLE SYSTEM IDENTIFICATION

For a J-input/K-Output lineqr system, the cross-correlation function, over an integration time T, between uth input (x_u) and vth output(y_v) is given by -

$$\phi_{x_u y_v}(\tau) = \frac{1}{T}\int_0^T x_u(t)\, y_v(t+\tau)\,dt \quad .. \quad .. \quad (4.64)$$

$$= \int_0^T h_{uv}(s)\, \phi_{x_u x_u}(\tau - s)\,ds$$

$$+ \sum_{r\neq u} \int_0^T h_{rv}(s)\, \phi_{x_u x_r}(\tau - s)\,ds \quad .. \quad (4.65)$$

where r, u = 1,2,...,J ; v = 1, 2,...,K ; and $h_{rv}(s)$ is the impulse response between rth input and vth output.

Provided the input signals are such that :

(i) The autocorrelation function $\phi_{x_r x_r}(\tau)$ is negligible except for this spike about $\tau = 0$, for $\tau < T_s$, the system settling time, and

(ii) the crosscorrelation function between any two inputs is negligible for $\tau < T_s$

Eqn. (4.65) simplifies to the form : $\phi_{x_u y_v}(\tau) = K_o h_{uv}(\tau)$,c $< T_s$ where K_o is the area of the autocorrelation spike about $\tau = 0$.

Hence, the problem of multivariable system identification is that of finding uncorrelated signals possessing impulse - like autocorrelation functions. Keeping this requirements in view, a number of schemes have been documented, the underlying theories of which have already been discussed earlier in the chapter.

In this section, some improved schemes for uncorrelated system input signals are presented using transformed ideal p-nary linear shift register sequences (the theory of which is developed in Secs. 4.7 and 4.8) as test perturbations for the multi-input system testing. The use of mixed-moduli shift register sequences as test inputs is also indicated.

4.9.1 Testing a 2-input system with transformed ideal shift register sequences

For a 2-input system, consider the situation when an ideal p-nary shift register sequence $\{f\}$ of period $M = \frac{p^n-1}{2}t_o$ ($p > 2$ and prime, n an integer denoting the number of stages in the register, t_o being a shift pulse interval) be used as one system input signal, and the sequence $\{f_t\}$ obtained by multiplying the elements of $\{f\}$ alternatively by -1 and +1 be used as the second input signal. Thus the transformed sequence $\{f_t\}$ is :

$$\{f_t\} = (-1)^i f \quad , \qquad \ldots \qquad \ldots \qquad (4.66)$$

It is shown in the previous section (Sec. 4.8 Property 12) that -

If an ideal p-nary sequence $\{f\}$ of period $\frac{p^n - 1}{2}$ is multiplied alternatively by -1 and +1, the resulting sequence is the p-nary m-sequence of period $p^n - 1$ digits.

Thus f_t is the p-nary m-sequence of period 2M shift pulse intervals. Since the m-sequence is antisymmetric (i.e. the second half is the negative of the first half), the cross-correlation function $\phi_{ff_t}(\tau)$ between the two sequences, taken over the period 2M, is zero for all values of τ. Hence, the sequences $\{f\}$ and $\{f_t\}$ are uncorrelated and have impulse like autocorrelation functions.

Therefore, provided that the system settling times are less than Mt_o ,the measured crosscorrelation function for values of τ less than Mt_o will yield the system impulse responses directly.

Since the p-level ideal sequence $\{f\}$ and the corresponding transformed sequence $\{f_t\}$ (which is merely an m-sequence) have zero bias in the respective autocorrelation functions during the measurement interval Mt_o, the present scheme gives more accurate results as compared to the schemes employing :

(i) A pseudorandom binary sequence and its phase shifted version as the two system inputs, or

(ii) a pseudorandom binary sequence and the corresponding inverse repeat sequence as the two system inputs.

4.9.2 Testing a multi-input system with transformed p-level shift register ideal sequences

Here, two methods are presented for obtaining a greater number of uncorrelated signals with impulse-like autocorrelation for use in a multivariable system identification by cross-correlation. Of these -

The first method is based on the idea that multiplication of the elements of a pseudorandom sequence by the rows of a Hadamard matrix results in uncorrelated pseudo-random sequences, which is originally suggested by Godfrey et.al.(1966). It is shown here that the extension of the idea to p-level shift register ideal sequences leads to well-defined uncorrelated sequences with better correlation properties.

The second method is introduced here, whereby it is shown that by proper choising of the starting phases of sequences obtained by multiplying the elements of a p-level ideal sequences with the rows of a $k \times 2^k - 1$ matrix whose elements are 1 and -1 results in $2^k - 1$ uncorrelated sequences with impulse like autocorrelation function with zero bias. The theory is developed in general with reference to ideal sequences with p-levels. In practice, the methods can easily be implemented in the case of ideal ternary (p = 3) sequences. However, for more than three levels, the instrumentation becomes rather complex.

The extension of the second method to conventional pseudorandom binary sequences is also considered and the results obtained are found to be quite encouraging.

The advantages and shortcomings of the two methods are also stated.

Method I :

Consider an ideal p-level sequence $\{f\}$ of period M, and any binary sequence $\{\ell_j\}$ of period L, whereby the periods M and L are co-prime. Let $\{g_j\}$ be the product sequence defined by -

$$\{g_j\} = \{\ell_j\} \cdot \{f\} \quad \ldots \quad \ldots \quad \ldots \quad (4.67)$$

Evidently, the period of the sequence $\{g_j\}$ is LM. The crosscorrelation function $\phi_{ij}(\tau)$ between two such sequences $\{g_i\}$ and $\{g_j\}$ is written as -

$$\phi_{ij}(\tau) = \frac{1}{LM} \sum_{r=1}^{LM} g_{i,r}\, g_{j,r+\tau}$$

$$= \frac{1}{LM} \sum_{r=1}^{LM} \ell_{i,r}\, f_r\, \ell_{j,r+\tau}\, f_{r+\tau} \quad \ldots \quad (4.68)$$

This may be considered as L sums, each of length M. Thus -

$$\phi_{ij}(\tau) = \frac{1}{LM} \sum_{t=0}^{L-1} \sum_{r=1}^{M} \ell_{i,(r+Mt)}\, \ell_{j,(r+\tau+Mt)}\, f_{r+Mt}\, f_{r+\tau+Mt}$$

Since $\{f\}$ is of period M, and L,M are co-prime, the above eqn. simplifies to -

$$\phi_{ij}(\tau) = \frac{1}{L} \sum_{t=0}^{L-1} \ell_{i,t}\, \ell_{j,t+\tau}\; \frac{1}{M} \sum_{r=1}^{M} f_r\, f_{r+\tau} = F_{i,j}(\tau)\phi_{ff}(\tau) \quad \ldots (4.69)$$

whereby $F_{i,j}(\tau)$ is the crosscorrelation function between $\{l_i\}$ and $\{l_j\}$, and $\phi_{ff}(\tau)$ is the autocorrelation of $\{f\}$

Looking to eqn. (4.69), it can be said that signals corresponding to sequences $\{g_j\}$ become uncorrelated and possess impulse like autocorrelation with zero bias provided that -

$$F_{jj}(0) = 1$$

$$F_{ij}(0) = 0, \quad i \neq j \quad \dots \quad \dots \quad (4.70)$$

These are orthognality conditions, and they are satisfied if $\{l_j\}$ is chosen from the rows of a Hadamard matrix of order L.

A Hadamard matrix (Boumert L et. al 1962) is a square matrix whose elements are 1s and -1s and whose row vectors are mutually orthogonal. The Hadamard property is not disturbed by (1) interchanging rows (2) interchanging columns, (3) changing the sign of every element in a row and (4) changing the sign of every element in a column. Using these operations, it is possible to establish a normal form for Hadamard matrices by insisting that the first row and first column contain only 1s.

Hadamard matrices exist for L = 1,2 and most multiples of 4.

For L = 8, two Hadamard matrices (H_1 and H_2) in normal form are given below (These are equivalent) :

$$H_1 = \left[\begin{array}{cc|cc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ \hline 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{array}\right] \qquad H_2 = \left[\begin{array}{c|ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \end{array}\right]$$

For the 8th order Hadamard matrix (H_2), the sequences $\{\ell_j\}$ from j = 0 to 7 (The rows of the matrix) have autocorrelation and crosscorrelation function as given in Table (4.10). Using these values and substituting equation (4.69), the auto- and crosscorrelation function of the sequences $\{g_0\}$ to $\{g_7\}$ are stated in Table(4.11).

For the matrix considered (H_2) , the rows 2, 3 and 4; and the rows 5 and 6 are, respectively, phase-shifted versions of each other. Accordingly, the corresponding sequences $\{g_j\}$ also exhibit this phase-shift property.

Table 4.10 :

Autocorrelation and crosscorrelation functions of the rows of 8 x 8 Hadamard matrix - the sequences $\{f_0\}$ to $\{f_7\}$.

Correlation function	Shift τ							
	0	1	2	3	4	5	6	7
$F_{00}(\tau)$	1	1	1	1	1	1	1	1
$F_{11}(\tau)$	1	-0.5	0	+0.5	-1	+0.5	0	-0.5
$F_{22}(\tau)$ to $F_{44}(\tau)$	1	0	0	-0.5	0	-0.5	0	0
$F_{55}(\tau) = F_{66}(\tau)$	1	0	-0.5	0	0	0	-0.5	0
$F_{77}(\tau)$	1	-0.5	0	0	0	0	-0.5	-0.5
$F_{01}(\tau)$ to $F_{07}(\tau)$	0	0	0	0	0	0	0	0
$F_{12}(\tau)$	0	+0.5	0	0	0	-0.5	0	0
$F_{13}(\tau)$	0	0	+0.5	0	0	0	-0.5	0
$F_{14}(\tau)$	0	0	0	+0.5	0	0	0	-0.5
$F_{15}(\tau)$	0	-0.5	0	+0.5	-1	+0.5	0	+0.5
$F_{16}(\tau)$	0	0	-0.5	0	+0.5	-1	+0.5	0
$F_{17}(\tau)$	0	-0.5	+0.5	-0.5	0	+0.5	-0.5	+0.5
$F_{23}(\tau)$	0	1	0	0	-0.5	0	-0.5	0
$F_{24}(\tau)$	0	0	1	0	0	-0.5	0	-0.5
$F_{25}(\tau)$	0	0	0	+0.5	+0.5	-0.5	-0.5	0
$F_{26}(\tau)$	0	0	0	0	+0.5	+0.5	-0.5	-0.5
$F_{27}(\tau)$	0	0	0	-0.5	+0.5	0	+0.5	-0.5
$F_{34}(\tau)$	0	1	0	0	-0.5	0	-0.5	0
$F_{35}(\tau)$	0	0	+0.5	+0.5	-0.5	-0.5	0	0
$F_{36}(\tau)$	0	0	0	+0.5	+0.5	-0.5	-0.5	0
$F_{37}(\tau)$	0	0	-0.5	+0.5	0	+0.5	-0.5	0
$F_{45}(\tau)$	0	+0.5	+0.5	-0.5	-0.5	0	0	0
$F_{46}(\tau)$	0	0	+0.5	+0.5	-0.5	-0.5	0	0
$F_{47}(\tau)$	0	-0.5	+0.5	0	+0.5	-0.5	0	0
$F_{56}(\tau)$	0	1	0	-0.5	0	0	0	-0.5
$F_{57}(\tau)$	0	0	+0.5	0	-0.5	+0.5	0	-0.5
$F_{67}(\tau)$	0	+0.5	0	-0.5	+0.5	0	-0.5	0

Table 4.11 :

Autocorrelation and crosscorrelation functions of the sequences $\{g_o\}$ to $\{g_7\}$ formed by multiplication of an ideal p-level sequence of length $N = (p^n - 1)/2$ digits by the rows of a 8 x 8 Hadamard matrix

Shift τ	0	1	2 ..	M−1	M	M+1 ..	2M−1	2M	2M+1 ..	3M−1	3M	3M+1 ..	4M−1	4M	4M+1
$\phi_{oo}(\tau)$	A	0	0	0	A	0	0	A	0	0	A	0	0	A	0
$\phi_{11}(\tau)$	A	0	0	0	0	0	0	0	0	0	0	0	0	−A	0
$\phi_{22}(\tau)$ to $\phi_{44}(\tau)$	A	0	0	0	−A/2	0	0	0	0	0	0	0	0	0	0
$\phi_{55}(\tau)=\phi_{66}(\tau)$	A	0	0	0	0	0	0	−A/2	0	0	0	0	0	0	0
$\phi_{77}(\tau)$	A	0	0	0	0	0	0	0	0	0	−A/2	0	0	0	0
$\phi_{01}(\tau)$ to $\phi_{07}(\tau)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\phi_{12}(\tau)$	0	0	0	0	−A/2	0	0	0	0	0	0	0	0	0	0
$\phi_{13}(\tau)$	0	0	0	0	0	0	0	−A/2	0	0	0	0	0	0	0
$\phi_{14}(\tau)$	0	0	0	0	0	0	0	0	0	0	−A/2	0	0	0	0
$\phi_{15}(\tau)$	0	−A/2	0	0	A/2	0	0	0	0	0	A/2	0	0	−A	0
$\phi_{16}(\tau)$	0	0	0	0	−A	0	0	−A/2	0	0	0	0	0	A/2	0
$\phi_{17}(\tau)$	0	0	0	0	A/2	0	0	A/2	0	0	A/2	0	0	0	0
$\phi_{23}(\tau)=\phi_{34}(\tau)$	0	0	0	0	0	0	0	0	0	0	0	0	0	−A/2	0
$\phi_{24}(\tau)$	0	0	0	0	−A/2	0	0	A	0	0	−A/2	0	0	0	0
$\phi_{25}(\tau)=\phi_{36}(\tau)$	0	0	0	0	−A/2	0	0	0	0	0	0	0	0	+A/2	0
$\phi_{26}(\tau)$	0	0	0	0	A/2	0	0	0	0	0	−A/2	0	0	A/2	0
$\phi_{27}(\tau)$	0	0	0	0	0	0	0	0	0	0	−A/2	0	0	A/2	0
$\phi_{35}(\tau)=\phi_{46}(\tau)$	0	0	0	0	−A/2	0	0	A/2	0	0	0	0	0	−A/2	0
$\phi_{37}(\tau)$	0	0	0	0	A/2	0	0	−A/2	0	0	0	0	0	0	0
$\phi_{45}(\tau)$	0	0	0	0	0	0	0	A/2	0	0	0	0	0	−A/2	0
$\phi_{47}(\tau)$	0	0	0	0	−A/2	0	0	A/2	0	0	0	0	0	A/2	0
$\phi_{56}(\tau)$	0	0	0	0	0	0	0	0	0	0	−A/2	0	0	0	0
$\phi_{57}(\tau)$	0	0	0	0	A/2	0	0	A/2	0	0	−A/2	0	0	−A/2	0
$\phi_{67}(\tau)$	0	0	0	0	0	0	0	0	0	0	0	0	0	A/2	0

Table 4.11 (Continued)

Shift τ	5M-1 ..	5M	5M+1..	6M-1	6M	6M+1...	7M-1	7M	7M+1 ...	8M-1	8M
$\phi_{00}(\tau)$	0	A	0	0	A	0	0	A	0	0	A
$\phi_{11}(\tau)$	0	-A/2	0	0	0	0	0	A/2	0	0	A
$\phi_{22}(\tau)$	0	0	0	0	0	0	0	-A/2	0	0	A
$\phi_{33}(\tau)$	0	0	0	0	0	0	0	-A/2	0	0	A
$\phi_{44}(\tau)$	0	0	0	0	0	0	0	-A/2	0	0	A
$\phi_{55}(\tau)$	0	0	0	0	-A/2	0	0	0	0	0	A
$\phi_{66}(\tau)$	0	0	0	0	-A/2	0	0	0	0	0	A
$\phi_{77}(\tau)$	0	-A/2	0	0	-A/2	0	0	0	0	0	A
$\phi_{01}(\tau)$ to $\phi_{07}(\tau)$	0	0	0	0	0	0	0	0	0	0	0
$\phi_{12}(\tau)$	0	A/2	0	0	0	0	0	0	0	0	0
$\phi_{13}(\tau)$	0	0	0	0	- A/2	0	0	0	0	0	0
$\phi_{14}(\tau)$	0	0	0	0	0	0	0	A/2	0	0	0
$\phi_{15}(\tau)$	0	-A/2	0	0	0	0	0	A/2	0	0	0
$\phi_{16}(\tau)$	0	0	0	0	A/2	0	0	0	0	0	0
$\phi_{17}(\tau)$	0	-A/2	0	0	-A/2	0	0	-A/2	0	0	0
$\phi_{23}(\tau) = \phi_{34}(\tau)$	0	A	0	0	-A/2	0	0	0	0	0	0
$\phi_{24}(\tau)$	0	0	0	0	0	0	0	0	0	0	0
$\phi_{25}(\tau) = \phi_{36}(\tau)$	0	0	0	0	-A/2	0	0	A/2	0	0	0
$\phi_{26}(\tau) =$	0	0	0	0	-A/2	0	0	0	0	0	0
$\phi_{27}(\tau)$	0	0	0	0	A/2	0	0	-A/2	0	0	0
$\phi_{35}(\tau) = \phi_{46}(\tau)$	0	0	0	0	0	0	0	A/2	0	0	0
$\phi_{37}(\tau)$	0	0	0	0	-A/2	0	0	A/2	0	0	0
$\phi_{45}(\tau)$	0	0	0	0	0	0	0	-A/2	0	0	0
$\phi_{47}(\tau)$	0	-A/2	0	0	0	0	0	0	0	0	0
$\phi_{56}(\tau)$	0	A	0	0	0	0	0	-A/2	0	0	0
$\phi_{57}(\tau)$	0	0	0	0	0	0	0	0	0	0	0
$\phi_{67}(\tau)$	0	A/2	0	0	-A/2	0	0	-A/2	0	0	0

Looking to Table 4.11, we observe that : for time shift $\tau < Mt_0$, the sequences $\{g_o\}$ to $\{g_7\}$ have impulse like autocorrelation with zero bias and are absolutely uncorrelated. Therefore, provided that the maximum settling time of the system $T_{s\ max}$ is less than Mt_o (one period of the ideal sequence $\{f\}$), the sequences $\{g_o\}$ to $\{g_7\}$ can be employed as test perturbations in the multi-input system identification by input/output correlation method.

This method has some remarkable advantages and few shortcomings as stated below :

Advantages

1. Since the p-nary shift register sequence $\{f\}$ is ideal (i.e. its autocorrelation is a spike about $\tau = 0$ and is zero otherwise in one period), the sequences $\{g_o\}$ to $\{g_7\}$ have ideal autocorrelation with zero bias. Hence the problem of bias estimation does not arise.
2. Also, since $\{f\}$ is an ideal sequence, the sequences $\{g_o\}$ to $\{g_7\}$ have zero crosscorrelation for $0 < \tau < M$ and thus are truly uncorrelated over the measurement interval, viz. 0 to $(M - 1)t_o$.

Shortcomings

1. An n-stage p-level feedback shift register generates an ideal sequence only when $\frac{p^n - 1}{2}$ is an odd integer. Thus, for instance for p = 3, n must be odd. Further, for p = 5, $\frac{p^n - 1}{2}$ is always even and hence no ideal quinary sequence is possible. In practice, for more than three levels, the instrumentation becomes complex.

2. The sequences $\{g_j\}$ are obtained by multiplying the elements of a p-level ideal sequence with the rows of a Hadamard matrix. As stated earlier, Hadamard matrixes of order L exist for L = 1, 2, and some multiples of 4 . For instance, with a Hadamard matrix of order 4, it is only possible to derive 4 uncorrelated sequences.

In view of the above shortcomings (2), in what now follows, a new method for uncorrelated pseudo noise sequences is presented which has all the advantages of Method 1 and overcomes the shortcomings.

Method 2 :

Here it is shown that by proper choising of the starting phasing of the sequences obtained by multiplying the elements of a p-level ideal shift register sequence by the rows of a $K \times 2^k - 1$ matrix of binary elements 1 and -1 (rows representing all the k - bit possible nonzero binary numbers with 0 taken as -1) results in $2^k - 1$ uncorrelated sequences having impulse like autocorrelation with zero bias (i.e. ideal autocorrelation).

Let a k-column matrix be defined as one in which the rows described all the possible k-digit nonzero binary numbers (binary element 0 taken as -1). Evidently, this matrix will have $2^k - 1$ rows. Such a matrix is shown below for values of k = 2 and 3.

The order of the rows is arranged as shown - the binary numbers (rows of the matrix) are written in the decreasing sense for purposes of convenience in the present method.

$k \times 2^k - 1$ matrixes of binary elements 1 and -1

$$M_2 : \quad \begin{array}{r} \text{row} \\ 0 \\ 1 \\ 2^k-2 = 2 \end{array} \begin{array}{c} \\ \left[\begin{array}{rr} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{array}\right] \end{array} , \quad k = 2$$

$$M_3 : \quad \begin{array}{r} \text{row} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 2^k-2 = 6 \end{array} \begin{array}{c} \\ \left[\begin{array}{rrr} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{array}\right] \end{array} , \quad k = 3$$

For this arrangement, row ' 0 ' of such a matrix consists of only 1s. Further, some of the rows differ only in phase. For instance, in the matrix M_2 rows 1 and 2 are the same but for the phase-shift. Similarly, in M_3, rows 1, 2, 4 and 3, 5, 6 differ only in phase.

Now consider a sequence $\{s_j\}$, the elements of which are given by -

$$s_{j,r} = e_{j,\bar{r}} f_r \qquad \ldots \qquad \ldots \qquad (4.71)$$

where f_r are the elements of a p-level n-stage shift register ideal sequence $\{f\}$ of period $M = \frac{p^n - 1}{2}$, and $e_{j,r}$ are the elements of the jth row in the $k \times 2^k - 1$ matrix defined earlier. Let the periods k and M be coprime.

The product sequence $\{s_j\}$ is therefore of period kM. The crosscorrelation function $R_{ij}(\tau)$ between two such sequences $\{s_i\}$ and $\{s_j\}$ over their common period kM is :

$$R_{ij}(\tau) = \frac{1}{kM} \sum_{r=1}^{kM} s_{i,r} \, s_{j,r+\tau}$$

$$= \frac{1}{KM} \sum_{r=1}^{kM} e_{i,r} \, f_r \quad e_{j,r+c} \, f_{r+\tau} \quad \ldots \qquad (4.72)$$

Considering this as k sums of length M, it can be shown :

$$R_{ij}(\tau) = E_{i,j}(\tau) \, \phi_{ff}(\tau) \quad \ldots \quad \ldots \qquad (4.73)$$

where $E_{i,j}(\tau)$ is the crosscorrelation function between rows of the matrix $\{e_i\}$ and $\{e_j\}$ and $\phi_{ff}(\tau)$ is the autocorrelation function of the ideal sequence $\{f\}$.

For the k x $2^k - 1$ matrix, the rows have auto and cross-correlation functions as set out in Table (4.12) for k = 2 and in Table (4.13) for k = 3 respectively.

Further, the theory of p-nary linear shift register ideal sequences ($p > 2$ and prime) is already presented in Secs. (4.7) and (4.8). The autocorrelation function of the ideal sequence $\{f\}$ is simply a spike at $\tau = 0$ and is zero for all other values of the argument in one period. In mathematical terms, $\phi_{ff}(\tau)$ is:

$$\phi_{ff}(\tau) = \frac{1}{M} \sum_{r=0}^{M-1} f_r \, f_{r+\tau} \, , \qquad M = (p^n - 1)/2$$

$$= \frac{2p^{n-1}}{p^n - 1} \qquad \text{for } \tau = 0 \qquad (\text{ mod } M)$$

$$= 0 \quad \text{elsewhere}$$

Table 4.12 : Auto and Crosscorrelation functions of the rows of the $k \times 2^k - 1$ matrix for $k = 2$ (The sequences $\{e_0\}$, $\{e_1\}$ and $\{e_2\}$)

Correlation function	Shift τ 0	1
$E_{00}(\tau)$	1	1
$E_{11}(\tau)$	1	-1
$E_{22}(\tau)$	1	-1
$E_{01}(\tau)$	0	0
$E_{02}(\tau)$	0	0
$E_{12}(\tau)$	-1	1

Table 4.13 : Auto-and Crosscorrelation functions of the rows of the $k \times 2^k - 1$ matrix for $k = 3$ (the sequences $\{e_0\}$, $\{e_1\}$, .. ,$\{e_6\}$)

Correlation function	Shift τ 0	1	2
$E_{00}(\tau)$	1	1	1
$E_{11}(\tau)$ to $E_{66}(\tau)$	1	-1/3	-1/3
$E_{01}(\tau) = E_{02}(\tau) = E_{04}(\tau)$	1/3	1/3	1/3
$E_{03}(\tau) = E_{05}(\tau) = E_{06}(\tau)$	-1/3	-1/3	-1/3
$E_{12}(\tau) = E_{24}(\tau) = E_{36}(\tau)$	-1/3	1	-1/3
$E_{13}(\tau) = E_{26}(\tau) = E_{45}(\tau)$	1/3	1/3	-1
$E_{14}(\tau) = E_{35}(\tau) = E_{56}(\tau)$	-1/3	-1/3	1
$E_{15}(\tau) = E_{23}(\tau) = E_{46}(\tau)$	1/3	-1	1/3
$E_{16}(\tau) = E_{25}(\tau) = E_{34}(\tau)$	-1	1/3	1/3

Using Tables (4.12) and (4.13), the autocorrelation and crosscorrelation functions of the sequences $\{s_j\}$, $j = 0$ to $2^k - 2$ are set out as shown in Tables (4.14) and (4.15) respectively.

Table 4.14 : Autocorrelation and crosscorrelation functions of the sequences $\{s_o\}$ to $\{s_2\}$ formed by multiplication of a p-level ideal sequence of length $M = (p^n - 1)/2$ digits by the rows of $k \times 2^k - 1$ matrix for k = 2. (Here $A = 2(p^{n-1}) / (p^n-1)$)

Shift τ	0	1	2 ..	M-1	M ..	2M-1	2M
$R_{oo}(\tau)$	A	0	0	0	A	0	A
$R_{11}(\tau) = R_{22}(\tau)$	A	0	0	0	-A	0	A
$R_{01}(\tau) = R_{02}(\tau)$	0	0	0	0	0	0	0
$R_{12}(\tau)$	-A	0	0	0	A	0	-A

Table 4.15 : Auto-and crosscorrelation functions of the sequences s_o to s_6 formed by multiplication of a p-level ideal sequence of length $M = (p^n - 1)/2$ digits by the rows of $k \times 2^k - 1$ matrix for k = 3. (Here $A = 2(p^{n-1}) / (p^n-1)$, and $B = A/3$)

Shift τ	0	1	2 ..	M-1	M	M+1 ..	2M-1	2M	2M+1	3M
$R_{oo}(\tau)$	A	0	0	0	A	0	0	A	0	A
$R_{11}(\tau)$ to $R_{66}(\tau)$	A	0	0	0	-B	0	0	-B	0	A
$R_{01}(\tau) = R_{o2}(\tau) = R_{o4}(\tau)$	B	0	0	0	B	0	0	B	0	B
$R_{03}(\tau) = R_{o5}(\tau) = R_{o6}(\tau)$	-B	0	0	0	-B	0	0	-B	0	-B
$R_{12}(\tau) = R_{24}(\tau) = R_{36}(\tau)$	-B	0	0	0	A	0	0	-B	0	-B
$R_{13}(\tau) = R_{26}(\tau) = R_{45}(\tau)$	B	0	0	0	B	0	0	-A	0	B
$R_{14}(\tau) = R_{35}(\tau) = R_{56}(\tau)$	-B	0	0	0	-B	0	0	A	0	-B
$R_{15}(\tau) = R_{23}(\tau) = R_{46}(\tau)$	B	0	0	0	-A	0	0	B	0	B
$R_{16}(\tau) = R_{25}(\tau) = R_{34}(\tau)$	-A	0	0	0	B	0	0	B	0	-A

Now, in order that the signals corresponding to the sequences $\{s_j\}$ serve as satisfactory test-perturbations in a multi-input system, their autocorrelation function must have a spike about $\tau = 0$ and otherwise be negligible for $\tau < T_{s\ max}$ (maximum system settling time) while their crosscorrelation functions must be negligible for $\tau < T_{s\ max}$.

From Tables 4.14 and 4.15 we observe that the sequences $\{s_j\}$ satisfy these requirements for time shifts $0 < \tau < M-1$ (i.e. one period of $\{f\}$) except when '$\tau = 0$'. Thus, provided $R_{ij}(\tau = 0)$ is made zero keeping other values unchanged, and $T_{s\ max} < Mt_o$, the sequences $\{s_j\}$ can be used as test signals.

Looking to Tables (4.14) and (4.15) (i.e. in view of eqns. (4.73 and 4.74) we see that $R_{ij}(\tau)$ can be made zero for $\tau = 0$ if the sequences $\{s_q\}$ for $q = 1, 2, .., 2^k-2$ are shifted relative to the sequence $\{s_o\}$ by q digits. This means -

The sequence $\{s_1\}$ is shifted relative to $\{s_o\}$ by 1 digit

The sequence $\{s_2\}$ is shifted relative to $\{s_o\}$ by 2 digits

....

....

The sequence $\{s_{2^k-2}\}$, is shifted relative to $\{s_o\}$ by $2^k - 2$ digits

By this phase shifting, the autocorrelation function remain undisturbed but the values of $R_{ij}(\tau)$ will be accordingly shifted by the number of digits by which the sequences $\{s_q\}$ is shifted relative to the sequence $\{s_j\}$, the maximum shift relative to the sequence $\{s_0\}$ being equal to $2^k - 2$ shift pulse intervals. Hence, the maximum settling time $T_{s\ max}$ of the system should now be less than $M - (2^k - 2)$ shift pulse interval rather than M shift pulse intervals. This is of no serious consequence, since in practice some safe margin is always kept between the system settling time and test sequence period. This time-margin easily facilitates the accommodation of the time-interval needed for the above referred phase shift of the respective sequences.

To illustrate the effect of this phase shifting process on the values of $R_{ij}(\tau)$, the auto-and crosscorrelation functions corresponding to the so phase - shifted sequences are set out in Tables (4.16) and (4.17) for values of k = 2 and 3. respectively (i.e. the values given in Tables (4.16) and (4.17) correspond to those given in Tables (4.14) and (4.15) but for the phase shift).

In view of the values of $R_{ij}(\tau)$ in Tables (4.16) and (4.17), it is clear that, provided the maximum settling time of the multi-input system is less than $M - (2^k - 2)$ shift pulse intervals, the signals corresponding to sequences $\{s_j\}$, $j = 0$ to $2^k - 2$, provide satisfactory test perturbations and the system impulse responses may be obtained directly from the measured input-output crosscorrelation functions in a similar to the familiar single input/single output case.

Table 4.16 :

Auto-and Crosscorrelation functions of the sequences $\{s_o\}$ to $\{s_2\}$ formed by the multiplication of a p-level ideal sequence of length $M = (p^n - 1) / 2$ digits by the rows of $k \times 2^k - 1$ matrix for $k = 2$, after subjecting the sequences to phase-shift. (Here $A = 2(p^{n-1}) / (p^n - 1)$.

Shift τ	0	1	2	3	4	. . . .	M-1	M	M+1	..	2M-1	2M
$R_{oo}(\tau)$	A	0	0	0	0		0	A	0		0	A
$R_{11}(\tau) = R_{22}(\tau)$	A	0	0	0	0		0	-A	0		0	A
$R_{01}(\tau) = R_{02}(\tau)$	0	0	0	0	0		0	0	0		0	0
$R_{12}(\tau)$	0	0	0	0	0		A	0	A		-A	0

The values of $R_{ij}(\tau)$ given in the above table are the same as those given in table (4.14) but for the phase shift as explained in the text. Here, the sequence $\{s_1\}$ is merely a p-nary (p, prime) m-sequence and hence possesses the antisymmetric property (i.e. its second half is the negative of the first half) and this property ensures zero crosscorrelation between sequences $\{s_1\}$ and $\{s_o\}$ over the period 2M digits. Further, $\{s_1\}$ and $\{s_2\}$ are only phase-shifted versions. To make $R_{12}(\tau)$ zero for $\tau = 0$, $\{s_2\}$ is shifted by one digit relative to $\{s_o\}$.

The effect of phase-shift on $R_{ij}(\tau)$ remains the same over all periods of sequence {f}; Hence the effect of phase-shift on $R_{ij}(\tau)$ is shown for 1 period only due to space-problem).

Table 4.17 : Auto-and crosscorrelation functions of the sequences $\{s_o\}$ to $\{s_6\}$ formed by the multiplication of a p-level ideal sequence of length $M = (p^n - 1) / 2$ digits by the rows of $k \times 2^k - 1$ matrix for $k = 3$, after subjecting the sequences to phase-shift. (Here $A = 2(p^{n-1}) / (p^n-1)$ and $B = A/3$)

Shift τ	0	1	2	3	4	..	M-6	M-5	M-4	M-3	M-2	M-1	M
$R_{oo}(\tau)$	A	0	0	0	0		0	0	0	0	0	0	A
$R_{11}(\tau)$; $R_{66}(\tau)$	A	0	0	0	0		0	0	0	0	0	0	-B
$R_{o1}(\tau)$	0	0	0	0	0		0	0	0	0	0	B	0
$R_{o2}(\tau)$	0	0	0	0	0		0	0	0	0	B	0	0
$R_{o3}(\tau)$	0	0	0	0	0		0	0	0	-B	0	0	0
$R_{o4}(\tau)$	0	0	0	0	0		0	0	B	0	0	0	0
$R_{o5}(\tau)$	0	0	0	0	0		0	-B	0	0	0	0	0
$R_{o6}(\tau)$	0	0	0	0	0		-B	0	0	0	0	0	0
$R_{12}(\tau)$	0	0	0	0	0		0	0	0	0	0	A	0
$R_{13}(\tau)$	0	0	0	0	0		0	0	0	0	B	0	0
$R_{14}(\tau)$	0	0	0	0	0		0	0	0	-B	0	0	0
$R_{15}(\tau)$	0	0	0	0	0		0	0	-A	0	0	0	0
$R_{16}(\tau)$	0	0	0	0	0		0	B	0	0	0	0	0
$R_{23}(\tau)$	0	0	0	0	0		0	0	0	0	0	-A	0
$R_{24}(\tau)$	0	0	0	0	0		0	0	0	0	A	0	0
$R_{25}(\tau)$	0	0	0	0	0		0	0	0	B	0	0	0
$R_{26}(\tau)$	0	0	0	0	0		0	0	B	0	0	0	0
$R_{34}(\tau)$	0	0	0	0	0		0	0	0	0	0	B	0
$R_{35}(\tau)$	0	0	0	0	0		0	0	0	0	-B	0	0
$R_{36}(\tau)$	0	0	0	0	0		0	0	0	A	0	0	0
$R_{45}(\tau)$	0	0	0	0	0		0	0	0	0	0	B	0
$R_{46}(\tau)$	0	0	0	0	0		0	0	0	0	-A	0	0
$R_{56}(\tau)$	0	0	0	0	0		0	0	0	0	0	-B	0

4.9.3 Transformed prbs for a multi-input system - Extension of the phase-shifting technique (Method 2) of Sec. 4.9.2)

Here, the extension of the phase-shifting technique viz. Method 2 of the last section is considered with regard to pseudo-random binary sequences (prbs). And it is shown that the results obtained are quite favourable.

Consider a sequence $\{m_j\}$, the elements of which are given by -

$$m_{j,r} = e_{j,r}\, c_r \qquad \ldots \qquad \ldots \qquad (4.75)$$

where c_r are the elements of the conventional 2-level n-stage shift register maximum length sequence of period $N = 2^n-1$ (prbs), and $e_{j,r}$ are the elements of the jth row in the $k \times 2^k - 1$ matrix of binary elements 1 and -1 defined in the last section. Let k and n be co-prime.

The product sequence $\{m_j\}$ is therefore of period kN. The crosscorrelation function $V_{ij}(\tau)$ between two such sequences $\{m_i\}$ and $\{m_j\}$ over their common period kN is given by -

$$V_{ij}(\tau) = \frac{1}{kN}\sum_{r=1}^{kN} m_{i,r}\, m_{j,r+\tau}$$

$$= \frac{1}{kN}\sum_{r=1}^{kN} e_{i,r}\, c_r\, e_{j,r+\tau}\, c_{r+\tau} \qquad \ldots \qquad \ldots \qquad (4.76)$$

Considering this as k sums of length N, it can be shown that-

$$V_{ij}(\tau) = E_{i,j}(\tau)\, \phi_{cc}(\tau) \qquad \ldots \qquad \ldots \qquad (4.77)$$

where $E_{i,j}(\tau)$ is the crosscorrelation function between the sequences $\{e_i\}$ and $\{e_j\}$ (rows of the $k \times 2^k - 1$ matrix) and

$\phi_{cc}(\tau)$ is the autocorrelation function of the prbs $\{c\}$.

The rows of the k x $2^k - 1$ matrix of binary elements 1 and -1 have auto and crosscorrelation functions as set out in Table 4.13 for k = 3 (Sec. 4.9.2).

The autocorrelation function of the prbs $\{c\}$is : (in one period)

$$\phi_{cc}(\tau) = \frac{1}{N}\sum_{r=0}^{N-1} c_r\, c_{r+\tau} \qquad N = 2^n - 1 \text{ bits}$$

$$= 1 \text{ for } \tau = 0$$

$$= -\frac{1}{N} \text{ for } \tau \neq 0 \qquad \ldots \quad \ldots \qquad (4.78)$$

Using Table (4.13 and eqn. (4.78), the auto-and cross-correlation functions of the sequences $\{m_j\}$, j = 0 to $2^k - 2$ are shown in Table 4.18 for k = 3.

As seen from Table (4.18), in case the sequences $\{m_r\}$ r = 2 to $2^k - 2$ are shifted relative to $\{m_o\}$ by r shift-pulses, the sequences $\{m_o\}$ to $\{m_{2^k-2}\}$ are uncorrelated and have impulse like autocorrelation function over the interval $0 < \tau < N - (2^k - 2)$. Thus provided $T_{s\,max}$ is less than $N - (2^k - 2)$ shift pulse intervals, signals corresponding to the sequences $\{m_j\}$ can serve as test perturbations in the multi-input system identification by crosscorrelation method.

<u>Advantages of the Method</u> :

(i) prbs can easily be generated by -stage shift register when its characteristic polynomial is primitive modulo-2.

(ii) With given N, value of k can be suitably chosen to make off-zero correlation values insignificant.

Table 4.18 : Autocorrelation and crosscorrelation functions of the sequences $\{m_j\}$, $j = 0$ to $2^k - 2$ for $k = 3$.

Shift τ	0	1	2	3	4 ..	N−1	N	N+1 ..	2N−1	2N	2N+1 ..	3N−1	3N
$V_{oo}(\tau)$	1	$\frac{-1}{N}$	$\frac{-1}{N}$	$\frac{-1}{N}$	$\frac{-1}{N}$	$\frac{-1}{N}$	1	$\frac{-1}{N}$	$\frac{-1}{N}$	1	$\frac{-1}{N}$	$\frac{-1}{N}$	1
$V_{11}(\tau)$ to $V_{66}(\tau)$	1	$\frac{1}{3N}$	$\frac{1}{3N}$	$\frac{-1}{N}$	$\frac{1}{3N}$	$\frac{-1}{N}$	$\frac{-1}{3}$	$\frac{1}{3N}$	$\frac{1}{3N}$	$\frac{-1}{3}$	$\frac{-1}{N}$	$\frac{1}{3N}$	1
$V_{o1}(\tau) = V_{o2}(\tau) = V_{o4}(\tau)$	$\frac{1}{3}$	$\frac{-1}{3N}$	$\frac{-1}{3N}$	$\frac{-1}{3N}$	$\frac{-1}{3N}$	$\frac{-1}{3N}$	$\frac{1}{3}$	$\frac{-1}{3N}$	$\frac{-1}{3N}$	$\frac{1}{3}$	$\frac{-1}{3N}$	$\frac{-1}{3N}$	$\frac{1}{3}$
$V_{o3}(\tau)=V_{o5}(\tau)=V_{o6}(\tau)$	$\frac{-1}{3}$	$\frac{1}{3N}$	$\frac{1}{3N}$	$\frac{1}{3N}$	$\frac{1}{3N}$	$\frac{1}{3N}$	$-\frac{1}{3}$	$\frac{1}{3N}$	$\frac{1}{3N}$	$-\frac{1}{3}$	$\frac{1}{3N}$	$\frac{1}{3N}$	$-\frac{1}{3}$
$V_{12}(\tau)=V_{24}(\tau)=V_{36}(\tau)$	$-\frac{1}{3}$	$-\frac{1}{N}$	$\frac{1}{3N}$	$\frac{1}{3N}$	$-\frac{1}{N}$	$\frac{1}{3N}$	1	$\frac{1}{3N}$	$-\frac{1}{N}$	$-\frac{1}{3}$	$\frac{1}{3N}$	$\frac{1}{3N}$	$-\frac{1}{3}$
$V_{13}(\tau)=V_{26}(\tau)=V_{45}(\tau)$	$\frac{1}{3}$	$\frac{-1}{3N}$	$\frac{1}{N}$	$\frac{-1}{3N}$	$\frac{-1}{3N}$	$\frac{-1}{3N}$	$\frac{1}{3}$	$\frac{1}{N}$	$\frac{-1}{3N}$	−1	$\frac{-1}{3N}$	$\frac{1}{N}$	$\frac{1}{3}$
$V_{14}(\tau)=V_{35}(\tau)=V_{56}(\tau)$	$-\frac{1}{3}$	$\frac{1}{3N}$	$-\frac{1}{N}$	$\frac{1}{3N}$	$\frac{1}{3N}$	$\frac{1}{3N}$	$-\frac{1}{3}$	$-\frac{1}{N}$	$\frac{1}{3N}$	1	$\frac{1}{3N}$	$-\frac{1}{N}$	$-\frac{1}{3}$
$V_{15}(\tau)=V_{23}(\tau)=V_{46}(\tau)$	$\frac{1}{3}$	$\frac{1}{N}$	$\frac{-1}{3N}$	$\frac{-1}{3N}$	$\frac{1}{N}$	$\frac{-1}{3N}$	−1	$\frac{-1}{3N}$	$\frac{1}{N}$	$\frac{1}{3}$	$\frac{-1}{3N}$	$\frac{-1}{3N}$	$\frac{1}{3}$
$V_{16}(\tau)=V_{25}(\tau)=V_{34}(\tau)$	−1	$\frac{-1}{3N}$	$\frac{-1}{3N}$	$\frac{1}{N}$	$\frac{-1}{3N}$	$\frac{1}{N}$	$-\frac{1}{3}$	$\frac{-1}{3N}$	$\frac{-1}{3N}$	$\frac{1}{3}$	$\frac{1}{N}$	$\frac{-1}{3N}$	−1

4.10 SUMMARY

In this chapter, the theory of the correlation method of multi-input system dynamic analysis has been advanced. At first, the current status of crosscorrelation art in a linear multivariable system identification is systematically discussed. To overcome the short comings in the characteristics of the input signal employed in current schemes, the theory of p-level shift register ideal sequences, which have impulse-like autocorrelation with zero off-peak value, is presented. Subsequently, making use of the ideal sequences, some schemes are proposed for uncorrelated system input signals, with autocorrelation function approximating to delta function, for use in the identification of multivariable system by the input / output crosscorrelation method. The choice of which scheme to use will depend on the type of the system and its associated equipment.

Although the signals derived here are considered from the view-point of their use in system identification, these may also be utilized in other fields particularly in the field of digital communications (Golomb, 1964).