C H A P T E R V

NONLINEAR SYSTEM IDENTIFICATION BY CROSSCORRELATION METHODS USING PSEUDORANDOM TEST PERTURBATIONS

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5.1 INTRODUCTION

The problem of system identification may be defined as that of determining, from input-output measurements on a system, sufficient information to enable the response to any input to be predicted. In case the system is linear time-invariant, its response y(t) to any input x(t) can be determined with the knowledge of the system impulse response h(t) by means of the convolution integral -

$$y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du \qquad \ldots \qquad \ldots \quad (5.1)$$

In general, all physical systems are nonlinear and have time-varying parameters in some degree. Where the effect of the nonlinearity is very small, or if the parameters vary only slowly with time, linear constant parameter methods of analysis can be applied to give an approximate answer. Where the experimental facts do not or would not correspond with any prediction of linear theory, nonlinear theory is essential to the description and understanding of physical phenomena.

Indeed, over the past few years, there has been considerable effort to improve the operation and control of existing industrial processes and the design of new plant.

This has led to an increasing interest in the measurement of system dynamics. The measurement of linear system dynamics by means of correlation methods has been considered for the single input/single output systems in Chapter II and for the multi-input/output systems in Chapter IV.

The present chapter is devoted to the problem of the measurement of dynamic response characteristics of certain nonlinear systems by means of crosscorrelation techniques.

As yet, methods of identifying the nonlinear systems using correlation principles are not well-developed. The theoretical work of Wiener (1958) laid the initial emphasis on determining a class of nonlinear system dynamics (Voltera, 1930) by means of crosscorrelation using Gaussian white noise as perturbation signal. Following Wiener's theory, for obtaining quick and accurate estimates of the system characteristics, several noteworthy theoretical as well as practical ideas have been explored, developed and refined, which have been well-documented. (Wiener, 1958; Lee and Schetzen, 1961, 1965; Widnall, 1962 Flake, 1963; Gyftopoulos and Hooper, 1964; Simpson, 1966; Brown and Goodwin, 1966; Godfrey, 1966, 1969; Briggs and Godfrey, 1966; Hooper and Gyftopoulos, 1967; Gardiner, 1966, 1967, 1968; Selway and Bell, 1968; Lubbock and Bandal, 1969; Brown, 1969, 1970; Bansal, 1969; Power and Simpson, 1970, 1970; Ream, 1970; Barker and Pradisthayon, 197 Kerlin, 1970; Kadri, 1971; Goodwin, 1971; Barker et al 1972; Buckner and Kerlin, 1972; Simpson, 1973; Crawford, 1970; Economakos, 1971; Simpson and Power, 1972 etc.).

Considerable research has also been devoted to extracting the impulse response of the linear approximation of the non-linear system when the system model is too complicated to analyze by standard mathematical models (linerization, Laplace transformation, root locus plots etc.) without making gross simplifying assumptions.) (Gardiner, 1966, 1967, 1968; Brown and Goodwin, 1966; Godfrey, 1966; Economakos, 1971). Since a linear approximant does not completely characterize a nonlinear system, efforts have been made to measure in some depth the dynamic responses of certain nonlinear systems based on Wiener's theory.

Despite all these efforts, a suitable correlation scheme which can give good measurements of the nonlinear system dynamics in a reasonably short time has not yet emerged. Evidently, there exist need for a better understanding of the existing theories as well as further work.

In view of the situation, an attempt is made in this chapter -

(i) to present briefly the development of the cross-correlation art in nonlinear system identification, pointing out the merits and shortcomings of the major schemes that are in current use, and

(ii) to describe some new correlation schemes which can give quick measurements with better accuracy, following Wiener's theory.

The content of this chapter is given as follows -

In section 5.2, the time-domain relationship between input and output of a class of nonlinear systems, is presented and the theory of crosscorrelation method applied to the nonlinear systems, is explained. In section 5.3, the development of crosscorrelation method for identifying the nonlinear system is given, pointing out the advantages and shortcomings of each. In section 5.4, some new correlation schemes are described which can give better as well as quick dynamic measurements as compared to those presently known.

5.2 NONLINEAR SYSTEM DYNAMIC ANALYSIS BY MEANS OF CROSS-CORRELATION METHOD

5.2.1 The relationship between input and the output of a non-linear system - The Volterra functional series expansion

The nonlinear system considered is shown in Fig.(5.1). The input signal x(t) produces a response signal y(t). In experimental work, both x(t) and y(t) will be perturbations added to the steady state input and output levels; thus input and output are to be taken to mean deviations from the operating levels.

According to Volterra (1930, 1959), the response y(t) of a time-invariant system of a very general type to an input x(t) may be represented by the following series of functionals :

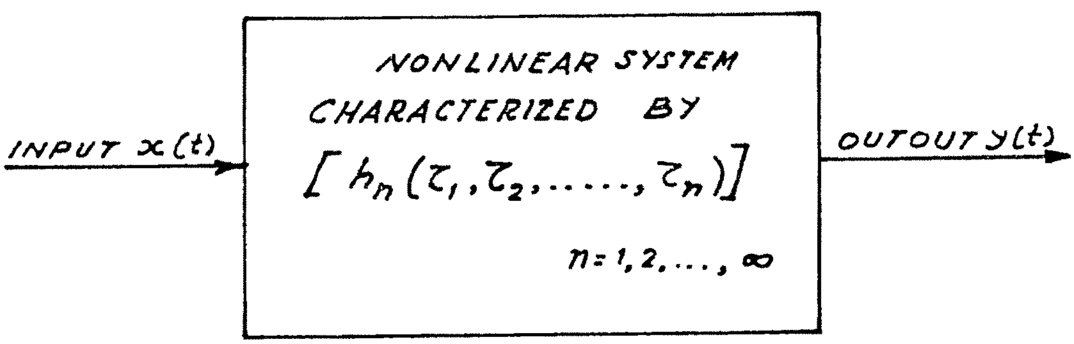

FIG. 5·1 BLOCK-DIAGRAM OF NONLINEARITY SYSTEM

$$
\begin{aligned}
y(t) = & \int_{-\infty}^{+\infty} h_1(\tau_1)\, x(t-\tau_1)\, d\tau_1 \\
& + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1, \tau_2)\; x(t-\tau_1)\; x(t-\tau_2)\, d\tau_1 d\tau_2 \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots \tau_n)\; x(t-\tau_1)\; x \,..\, x(t-\tau_n) \\
& \qquad\qquad .d\tau_1 \ldots d\tau_n \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
= \; & y_1(t) + y_2(t) + \ldots + y_n(t) + \ldots \qquad \ldots \qquad (5.2)
\end{aligned}
$$

The function $h_n(\tau_1, \tau_2, \ldots \tau_n)$ is termed the Volterra Kernel of order n of the system. Each of the kernels h_n is characteristic of the system; consequently, the set of kernels $\{h_n\}$ identifies the system. The first term in the expansion is the familiar convolution integral; the nth term is an n-dimensional convolution integral. Linearizing the system amounts to retaining only the first term in the expansion. This type of functional expansion is applicable to a wide class of nonlinear systems.

The Volterra Kernels are symmetric in the sense that the Kernel value remains unchanged if the values of any two independent variables τ_i are interchanged.

For a non-anticipative system, the values of the Volterra kernels are zero for any argument $\tau_i < 0$.

Further, the kernels are continuous in their arguments for all $\tau_i > 0$.

In a practical system, inputs occuring at times greater than T_s in the past have no effect on the present output, the time T_s being known as the settling time of the system. Further, in the present case, the settling time of all the kernels $\{h_n\}$ is not the same. In fact, the higher the order of the kernel, the sooner it settles down. This of course implies that the maximum settling time of the system ($T_{s\ max}$) is dictated by the first order kernel.

Denoting the settling time of the ith order kernel by T_{s_i}, the input-output Volterra relation in eqn.(5.2) for a practical system is modified to the form -

$$y(t) = \sum_{j=1}^{\infty} \int_{o}^{T_{s_j}} \cdots \int_{o}^{T_{s_j}} h_j(\tau_1 \ldots, \tau_j) \sum_{k=1}^{\infty} x(t-\tau_k)\, d\tau_k \qquad \ldots \quad (5.3)$$

This equation gives the time response of the nonlinear system to any input x(t), if the kernels, h_1, h_2, $h_3 \ldots$, are known. In theory, this equation could be solved to give the kernels from input/output records. In practice, this multi dimensional deconvolution is difficult unless suitable form of input is used.

Further, application of the input-output relationship under normal system operation requires a large amplitude perturbation signal or a long experimentation time for obtaining accurate measure of the dynamics. For this reason, correlation methods are currently finding much favour. The application of the correlation principle to the class of nonlinear systems which can be represented by the Volterra functional series expansion is presented in the following sub-section.

5.2.2 Crosscorrelation method applied to the nonlinear system represented by the Volterra functional series expansion

The crosscorrelation function $\phi_{xy}(\tau)$ between two signals x(t) and y(t) which are stationary (i.e. whose statistical properties such as mean level and mean squared amplitude remain constant with time) is defined by the equation -

$$\phi_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t-\tau)\, y(t)\, dt \qquad \ldots (5.4)$$

The physical meaning of this is that considerably long portions of the signal x(t) and y(t) are taken, and x(t) is shifted by an amount τ with respect to y(t). The signals $x(t-\tau)$ and y(t) are then multiplied together and the integral of this multiplication determined. The result is then divided by the time over which the integration was carried out. In case the two signals are one and the same, the autocorrelation function $\phi_{xx}(\tau)$ is obtained. Thus,

$$\phi_{xx}(\tau) = \lim_{T \to \infty} \int_{-T/2}^{T/2} x(t-\tau)\, x(t)\,dt \quad \ldots \quad (5.5)$$

When discussing periodic functions, the definition of the autocorrelation function is modified to read :

$$\phi_{xx}(\tau) = \frac{1}{T} \int_{o}^{T} x(t-\tau)\, x(t)\, dt \quad \ldots \quad \ldots \quad (5.6)$$

where T is the period of x(t). Similarly, the definition of the crosscorrelation function $\phi_{xy}(\tau)$ is modified to read :

$$\phi_{xy}(\tau) = \frac{1}{T} \int_{o}^{T} x(t-\tau)\, y(t)\, dt \quad \ldots \quad \ldots \quad (5.7)$$

Having defined the correlation functions, their use in nonlinear system dynamic analysis can now be examined. Let us compute the crosscorrelation function between the input x(t) and the output y(t) of a nonlinear system defined by the Volterra series expansion stated in Eqn. (5.3). It will be assumed that x(t) is periodic so that eqns.(5.6 and 5.7) apply. The expression is :

$$\phi_{xy}(\tau) = \frac{1}{T} \int_{o}^{T} x(t-\tau)\Big[\sum_{j=1}^{\infty} \int_{o}^{T_{s_j}} \ldots \int_{o}^{T_{s_j}} h_j(\tau_1,\ldots,\tau_j) \cdot \sum_{k=1}^{\infty} x(t-\tau_k)\, d\tau_k\Big]dt \quad ..(5.8)$$

Changing the order of the integration,

$$\phi_{xy}(\tau) = \sum_{j=1}^{\infty} \int_{o}^{T s_j} \cdots \int_{o}^{T s_j} h_j(\tau_1,\ldots,\tau_j) \cdot \left[\frac{1}{T} \int_{o}^{T} x(t-\tau)\; x(t-\tau_1) \ldots x(t-\tau_j)\, dt \right] \cdot d\tau_1 \ldots d\tau_j \quad \ldots \quad \ldots \qquad (5.9)$$

The first term in the expansion is the convolution of the linear kernel h_1 with the autocorrelation function of the input signal x(t). Higher order terms involve other system kernels and higher order correlation functions of the input. In practice, a physical system is considered to be characterized by a finite number of Volterra kernels.

In case the system is characterized by the first n-kernels h_1 to h_n , then input-output crosscorrelation function in eqn.(5.9) becomes :

$$\phi_{xy}(\tau) = \sum_{j=1}^{n} \int_{o}^{T s_j} \cdots \int_{o}^{T s_j} h_j(\tau_1, \ldots, \tau_j) \cdot \left[\frac{1}{T} \int_{o}^{T} x(t-\tau)\; x(t-\tau_1) \ldots x(t-\tau_j)\, dt \right] \cdot d\tau_1 \ldots d\tau_j \quad \ldots \quad \ldots \qquad (5.10)$$

Now, the rth order autocorrelation function of x(t) of period T is defined by -

$$\theta(\tau_1, \tau_2, \ldots, \tau_r) = \frac{1}{T} \int_0^T x(t-\tau_1)\, x(t-\tau_2) \ldots x(t-\tau_r)\, dt \qquad \ldots \ldots (5.11)$$

Substituting eqn. (5.11) into (5.10), $\phi_{xy}(\tau)$ is,

$$\phi_{xy}(\tau) = \sum_{j=1}^{n} \int_0^{T_{s_j}} \ldots \int_0^{T_{s_j}} h_j(\tau_1, \ldots, \tau_j) \cdot \theta(\tau_1, \ldots, \tau_j)\, d\tau_1 \ldots d\tau_j \qquad \ldots (5.12)$$

Thus, the identification problem of a nonlinear system represented by means of a Volterra series settles down to one of extracting the set of kernels $\{h_n\}$ from the measured values of the input-output correlation function.

Apparently, such a method of identification is based on the higher order autocorrelation function properties of the input signal. For instance, in measuring the linear kernel h_1, it is necessary to choose x(t) so that all the correlation functions of x(t) of order greater than or equal to three together contribute negligibly to eqn. (5.12).

If one considers a system for which the Volterra series contains only the isolated term $y_n(t)$ (see eqn. 5.2), multi-dimensional crosscorrelation with n delay lines and Gaussian noise input identifies the kernel

$$h_n(\tau_1, \tau_2, \ldots, \tau_n) = \frac{1}{n!\,A^n}\ \phi_{\underbrace{yx\ldots x}_{n\ \text{times}}}(\tau_1, \tau_2, \ldots, \tau_n) \qquad \ldots (5.13)$$

with $\tau_i \neq \tau_j$ for $i \neq j$

where

$$\phi_{yx\ldots x} = \lim_{T \to \infty} \frac{1}{T} \int_{t_o}^{t_o+T} y(t)\cdot x(t-\tau_1)\ldots x(t-\tau_n)\,dt \qquad \ldots (5.14)$$

and the autocorrelation function of the input is $\phi_{xx}(\lambda) = A\delta(\lambda)$. When this technique is applied to a system containing other terms in the expansion, $\phi_{yx\ldots x}(\tau_1, \tau_2, \ldots \tau_n)$ contains, in addition to the above, contributions from higher order terms in the Volterra series, these being from even order terms if n is even and odd if n is odd.

In brief, the theoretical basis of the crosscorrelation method of Volterra series type of nonlinear system identification involves proper choice of input signals with desired higher order autocorrelation functions.

To this end, some noteworthy correlation schemes have been proposed (References are stated in the introduction of this chapter) and the following section is meant to give briefly the development of the crosscorrelation art in nonlinear system dynamic analysis.

5.3 THE DEVELOPMENT OF CROSSCORRELATION ART IN NONLINEAR SYSTEM IDENTIFICATION

5.3.1 Measurement of higher order kernels

Wiener (1958) first showed that, for a system describable by the Volterra series, the response y(t) to a zero mean white Gaussian input x(t) could be written as -

$$y(t) = \sum_{n=0}^{\infty} G_n [g_n, x(t)] \quad \ldots \quad \ldots \quad (5.15)$$

where $\{G_n\}$ is a complete set of orthogonal functionals. Orthogonality is expressed by the condition -

$$\int_{-\infty}^{\infty} G_n[g_n, x(t)] \cdot G_m [g_m, x(t)] = 0 \text{ for } m \neq n \quad ..(5.16)$$

The first terms in the Wiener series for an input whose autocorrelation function is $A\delta(t)$ are listed below :

$$G_o [g_o, x(t)] = g_o$$

$$G_1 [g_1, x(t)] = \int_{-\infty}^{\infty} g_1(\tau_1)\, x(t-\tau_1)\, d\tau_1$$

$$G_2 [g_2, x(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2 - A \int_{-\infty}^{\infty} g_2(\tau_2, \tau_2)\, d\tau_2$$

$$G_3 [g_3, x(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_3(\tau_1, \tau_2, \tau_3)\, x(t-\tau_1)\, x(t-\tau_2)\, x(t-\tau_3)\, d\tau_1 d\tau_2 d\tau_3 - 3A \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_3(\tau_1, \tau_2, \tau_2)\, x(t-\tau_1)\, d\tau_1 d\tau_2$$

..(5.17)

In general, the leading term of G_n is the homogeneous functional of the nth degree

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g_n(\tau_1,\ldots,\tau_n)\; x(t-\tau_1)\ldots x(t-\tau_n)\,d\tau_1\ldots d\tau_n$$

and the remaining terms are lower degree functionals whose kernels are derived in a systematic manner from $g_n(\tau_1, \tau_2 \ldots \tau_n)$.

The zero order kernel g_o is the average value of y(t). For identification of the remaining kernels, Wiener proposed that the stochastic input signal be expanded in terms of Laguerre functions and the kernels represented by the series of Hermite polynomials. But, the evaluation of the coefficients of the polynomial is rather involved.

Subsequently, Lee and Schetzen (1961, 1965) have shown that an n-dimensional crosscorrelation with n delay settings $\tau_1, \tau_2, \ldots, \tau_n$ and $\tau_i \neq \tau_j$ for $i \neq j$ gives the result

$$g_n(\tau_1, \tau_2, \ldots \tau_n) = \frac{1}{n!\,A^n}\,\phi_{\underbrace{yx\ldots x}_{n\ \text{times}}}(\tau_1, \tau_2, \ldots, \tau_n) \qquad \ldots \quad (5.18)$$

Recalling eqn.(5.13) and comparing it with eqn.(5.18), it is clear that the contributions which the Volterra kernels beyond the nth make to the n-dimensional crosscorrelation measurement represent the difference between the nth Wiener and Volterra kernels.

Recently Crawford (1970) has shown the procedure for deriving the relationships between Wiener and Volterra kernels in a systematic manner, and suggested that provided the crosscorrelation delay settings are considered in the space sufficiently far from the origin, Wiener and Volterra kernels are given by the relation -

$$g_n(\tau_1, \tau_2, \ldots \tau_n) = h_n(\tau_1, \tau_2, \ldots, \tau_n) \quad \ldots \qquad (5.19)$$

Using this theory of Schetzen and Lee, Widnall (1962) has determined the 2nd order kernel of a simulated system. Such experiments, however, require very long crosscorrelation times and may even be impractical for some applications.

To avoid long experimentation times and practical difficulties associated with the use of Gaussian white noise as test signal, Gyftopoulos (1964) suggested the use of periodic input signals, whose correlation functions over a finite time interval and upto some order K (say) are approximately equal to those of Gaussian white noise. Hooper and Gyftopoulos (1967) noted that the ternary pseudorandom signals based on m-sequences satisfy the requirements to some extent as can be seen from the properties of the ternary signal stated in brief below :

A ternary pseudorandom signal based on m-sequences assumes at any instant of time only one of the three possible (normalized) values +1, 0, or -1.

The signal is discontinuous and may change value only at event points having uniform spacing t_o.

The signal is periodic with period $T = Nt_o$ where $N = 3^n-1$, and n = integer. The signal is generated using linear recursion formula :

$$c_i = a_1c_{i-1} + a_2c_{i-2} + \dots + a_nc_{i-n} \qquad \text{(mod-3 addition)}$$

For the ternary periodic m-sequence signals, the following has been found :

1. The second order autocorrelation function $\theta(\tau_1, \tau_2) = \phi_{xx}(\tau_2 - \tau_1)$ of the ternary chain of period T has the form of a succession of triangular peaks of width t_o and height 2/3 (normalized value), alternating in sign and centered at $\tau = \pm m\frac{T}{2}$, m = integer. The function is identically zero between the peaks. When $T >> t_o$, the approximate analytic expression for $\theta(\tau_1, \tau_2)$ is -

$$\theta(\tau_1, \tau_2) = \phi_{xx}(\tau_2 - \tau_1) = \frac{2.3^{n-1}}{3^n-1}\, t_o \sum_{m=-\infty}^{+\infty} (-1)^m \delta(\tau + \frac{mT}{2}) \qquad \dots (5.20)$$

Hence, $\theta(\tau_1, \tau_2)$ over the interval $0 \leq \tau_1, \tau_2 < T/2$ is approximately equal to the corresponding correlation function of Gaussian white noise.

2.

2. The odd order autocorrelation functions are identically equal to zero, viz.

$$\theta(\tau_1, \tau_2, \ldots, \tau_{2k+1}) = 0 \qquad \text{all} \quad k \geq 1.$$

These correlations are identically equal to the corresponding correlations of Gaussian white noise.

3. Over one period, the 4th order autocorrelation function $\theta(\tau_1, \tau_2, \tau_3, \tau_4)$ assumes its maximum value at $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ and has the property -

$$\theta(0,0, \tau_3 = \tau_4) = \frac{2}{3}\,\theta(0, 0, 0, 0) \qquad (\tau_3 = \tau_4 \neq 0)$$

Further, throughout the ($\tau_i \neq \tau_j$ for $i \neq j$, $1 \leq i,j \leq 4$)-space, $\theta(\tau_1, \tau_2, \tau_3, \tau_4)$ is identically equal to zero except for a relatively small number of 'anomolous' regions of size $8t_o^3$, where it varies from zero to either of the values $\pm\theta(0, 0,0,0)/3$. Also these anamolous regions are fairly uniformly distributed in the ($\tau_1 \neq \tau_2 \neq \tau_3 \neq \tau_4$) - space.

Thus, the ternary periodic signals have second, third - and fourth order autocorrelation functions approximating to corresponding correlations of Gaussian white noise. Consequently, they are suitable for the measurement of the kernels of nonlinear system that can be characterized by only the first three Wiener functionals. Equivalently, these signals are suitable for the measurement of the kernels $h_1(\tau)$ and

$h_2(\tau_1, \tau_2)$ of nonlinear systems represented by the input/output Volterra relation :

$$y(t) = \int_0^{T_{s_1}} h_1(\tau)\, x(t-\tau)\,dt$$
$$+ \int_0^{T_{s_2}} \int_0^{T_{s_2}} h_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1 d\tau_2 \qquad \ldots (5.21)$$

where x(t) is the input signal and y(t) is the output signal.

Using the properties of ternary signal stated earlier, the following equations may be written :

$$h_1(\tau) = \frac{1}{2.3^{n-1}(t_o)^2} \int_0^T x(t-\tau)\, y(t)\,dt$$

and

$$\left[8 \times 3^{n-2}(t_o)^3\, h_2(\tau_1, \tau_2) + \text{contribution from anamolus regions}\right] = \int_0^T x(t-\tau_1)\, x(t-\tau_2)\, y(t)\,dt$$
$$\tau_1 \neq \tau_2 \qquad \ldots \quad \ldots (5.22)$$

To avoid the anamolous regions of $\theta(\tau_1, \tau_2, \tau_3, \tau_4)$, Hooper and Gyftopoulous have used ternary chain of period $T = 100T_{s_2}$, T_{s_2} = settling time of 2nd order kernel. However, the first order kernel h_1 is well-identified with $T = 2T_{s_1}$, T_{s_1} = settling time of the first order kernel.

However, because of the anamolous regions, $\theta(\tau_1, \tau_2, \tau_3, \tau_4)$ is not a good approximation to the corresponding correlation of Gaussian white noise. Similar 'anamolies' were also observed by Simpson (1966) in the fourth order autocorrelation functions of pseudorandom signals based on binary n-sequences (i.e. inverse repeat sequences formed from binary m-sequences).

Recently Ream (1970) has investigated these anamolies and has given a complete formal solution of the Wiener approximation problem for the case when the input is a ternary or an inverse repeat binary m-sequence, concluding that these sequences do not effectively identify kernels of order greater than two.

Barker et al (1970) have shown that this observed behaviour is neither anamolous nor confined to only the cases of modulo-2 and modulo-3 m-sequence signals, but is the characteristic of the higher order autocorrelation functions of all pseudorandom signals based on m-sequences. More recently, Barker et al (1972) have also derived criteria of performance of inverse repeat pseudorandom signals based on binary, ternary, and quinary m-sequences for the identification of second order Volterra kernels.

5.3.2 Measurement of the linear kernel in the presence of system nonlinearities

This sub-section is meant to discuss some of the ideas which have been explored and developed to identify the linear approximant to a nonlinear system.

(A) Preliminaries in the linear kernel identification

Whenever the linear kernel must be measured in the presence of system nonlinearities, two basic considerations influence the amplitude of the test signal (i.e. the severity of the perturbation). On the one hand, it is desirable that the signal be strong enough so that the system response due to the input signal be large compared with inherent noise fluctuations in the output. On the other hand, it is desirable to keep the perturbation level small so as to avoide large changes in the output that might take the system outside the linear range, thereby invalidating the assumed mathematical model. It is evident that if the input is such that the crosscorrelation function between input and output is relatively insensitive to the system nonlinearities, then such a signal permits greater latitude in the choice of level to be used.

The manner in which a crosscorrelation experiment is influenced by system nonlinearities is seen from eqn.(5.12) rewritten below for convenience :

$$\phi_{xy}(\tau) = \sum_{j=1}^{n} \int_{0}^{T_{s_j}} \dots \int_{0}^{T_{s_j}} h_j(\tau_1,..,\tau_j)\ \theta(\tau, \tau_1, ..\tau_j)\ d\tau_1..d\tau_j$$

where $\theta(\tau_1,...,\tau_j)$ is the jth order autocorrelation function of the test input $x(t)$.

The first term in the expansion is the convolution of the linear kernel with the second order autocorrelation function of $x(t)$. Higher order terms involve other system kernels and higher order correlation functions of the input. In measuring the linear kernel, it is necessary to choose $x(t)$ so that all of these higher order terms together contribute negligibly to eqn.(5.12). This is generally done by limiting the amplitude of the perturbation signal $x(t)$.

(B) Measurement of the linear kernel in the presence of an amplitude nonlinearity - Gardiner's Method

Without reference to the Volterra series expansion, Gardiner (1966,1967, 1968) described a method of determining the linear kernel of a system with an amplitude nonlinearity of the form -

$$y = a_1x + a_2x^2 + a_3x^3 + a_4x^4 \quad \dots \quad \dots \qquad (5.23)$$

where x is the input to the nonlinearity and y is the output of the nonlinearity, and a_1, a_2, a_3 and a_4 are constants.

Using different amplitude pseudorandom binary signals, Gardiner has shown that by performing four seperate cross-correlation experiments, terms involving the second, third and fourth terms on the right hand side of eqn. (5.23) can be eliminated and the impulse response of the linear channel through the system can be determined. The method was illustrated by determining the impulse response of the linear channel of a computer - simulated system, considering the case of the amplitude nonlinearity preceded and followed by a linear filter.

Recently Economakos (1971) considered the generalized view of Gardiner for linear kernel identification. The block-diagram of the scheme used by Economakos and Gardiner is shown in Fig.(5.2). For this system -

$$w(t) = \int_{-\infty}^{+\infty} y(u)\, \ell(t - u)\, du \quad \ldots \quad \ldots \quad (5.24)$$

since the nonlinear element considered in this scheme has a power law characteristic,

$$y(t) = \sum_{n=1}^{\infty} a_n \big(y'(t)\big)^n$$

where

$$y'(t) = \int_{-\infty}^{+\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1 \quad \ldots \quad \ldots \quad (5.25)$$

Hence w(t) can be written as -

$$w(t) = \int_{-\infty}^{+\infty} \Big[\sum_{n=1}^{\infty} a_n \{h(\tau_1)\, x(t-\tau_1)\, d\tau_1\}^n \Big] \cdot \ell(t - u)\, du \quad \ldots \quad (5.26)$$

FIG. 5·2 SYSTEM DISCUSSED BY ECONOMAKOS AND GARDINER

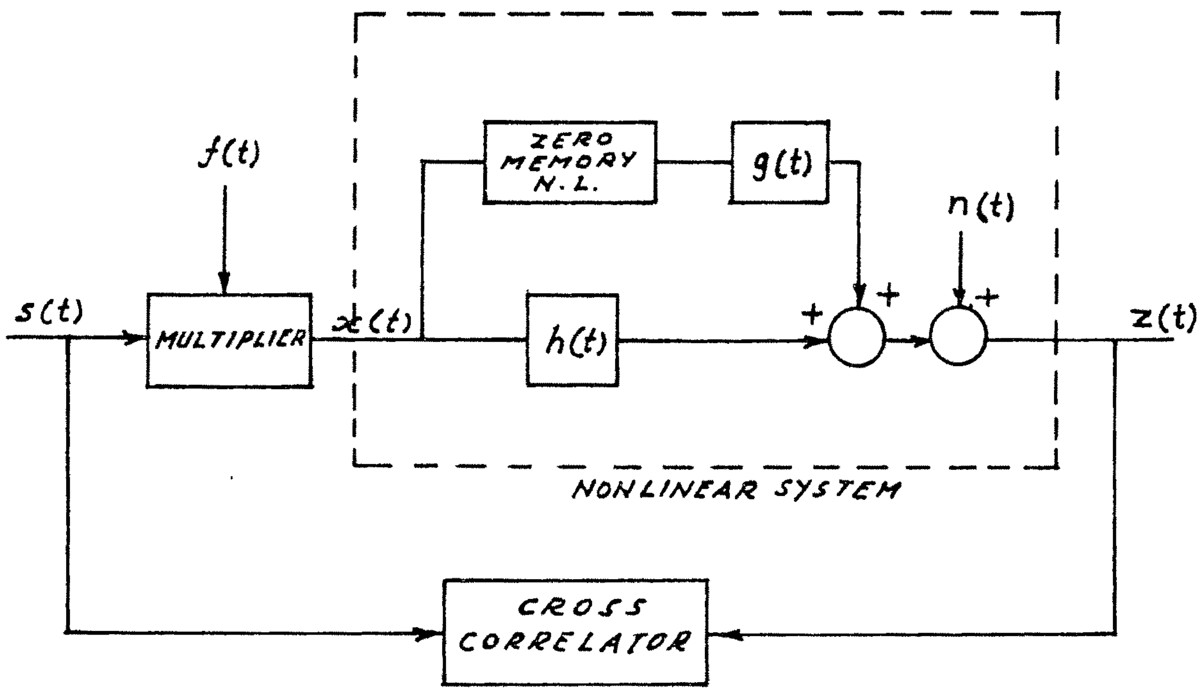

FIG. 5·3 SYSTEM CONSIDERED BY BROWN, AND BY POWER AND SIMPSON

Truncating the series at some value and performing crosscorrelations of w(t) with different amplitude pseudorandom binary signals, it is possible to extract $w_1(t)$ the first term in the expansion and hence the linear kernel $h_1(t)$.

(C) Gardiner's scheme viewed in the light of Volterra series approach by Simpson and Power

Recently, Simpson and Power (1972) have brought the Gardiner's scheme into the general framework based on Volterra series. As they state, since τ_1 in eqn. (5.26) is a dummy variable that appears on integration, the typical term in the eqn.(5.26) can be represented by the product of n similar integrals in terms of the variables $\tau_1, \tau_2, \ldots, \tau_n$. As limits of integration on these variables are independent, it is possible to collect them into a single multiple integral and write y(t) in eqn. (5.25) as -

$$y(t) = \sum_{n=1}^{\infty} a_n \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h(\tau_1) \ldots h(\tau_n) \prod_{k=1}^{\infty} x(t-\tau_k)\, d\tau_k$$

$$= y_1(t) + y_2(t) + \ldots + y_n(t) + \ldots \quad \ldots \qquad (5.27)$$

Recalling the general Volterra series expansion stated in eqn.(5.2), it is easy to see that the typical term $y_n(t)$ in eqn.(5.27) corresponds to the nth term of the Volterra series, but for the fact that the kernels are now seperable.

(D) Improvements on Gardiner's scheme suggested by Godfrey

Godfrey (1966) has discussed the identification problem of nonlinear systems and suggested that two out of the four crosscorrelation experiments, needed in Gardiner's method to extract the linear kernel, can be eliminated by using a 3-level m-sequence signal. This is evident by virtue of the anti-symmetric property of the ternary signal. Godfrey also suggested that the instrumentation difficulties associated with the use of 3-level signals can be overcome by using the binary n-sequences which have the same antisymmetric property and are of only two levels.

(E) Schemes demonstrating the superiority of antisymmetric signals over non-antisymmetric signals

Subsequently, a number of crosscorrelation experiments have been conducted for obtaining the linear kernel using the non-antisymmetric pseudorandom binary signals (PRBS) and the antisymmetric pseudorandom signals to examine the superiority of one class of signals over the other.

It has been shown (Buckner, 1970) that an antisymmetric signal eliminates the even-numbered terms from a Volterra expansion when either a Fourier - or crosscorrelation analysis is performed on the system output. In fact, it can be readily seen that for an antisymmetric signal, the value of the integral of the form -

$$\frac{1}{T}\int_{o}^{T} x(t-\tau_1)\ x(t-\tau_2)\ \ldots\ x(t-\tau_n)\ dt$$

is identically zero, regardless of the values of $\tau_1, \tau_2, \ldots, \tau_n$ when n is odd. This fact was demonstrated by Hooper and Gtftopoulos (1967) for pseudorandom ternary m-sequence signals. Rydin and Hooper (1969) have performed simulated crosscorrelation measurements which show that tests with pseudorandom ternary sequences (antisymmetric) gave estimates of the linear kernel which were better than estimates obtained with pseudorandom binary sequences (non-antisymmetric) in the nonlinear systems they considered.

Since antisymmetry is easily obtained and since it has the potential for reducing errors in the test results, this type of signal is preferred over nonantisymmetric signals for general use. In this connection, the instrumentation difficulties can be overcome by using pseudorandom binary n-sequences signals which are also antisymmetric and relatively easily generated because only two signal levels are required. Thus the n-sequences formed by inverting every alternative digit of a pseudorandom binary m-sequence tend to circumvent some of the problems encountered in using pseudorandom signals in system identification.

(F) Reduction of number of crosscorrelations in measuring linear kernel - schemes suggested by Brown, and by Simpson and Power

The other researches on linear kernel measurement, without referring to Volterra series include those of Brown (1969,1970) and of Simpson and Power (1970, 1970) :

Since the implementation of Gardiner's method of correcting the effect of nonlinearities requires atleast two independent crosscorrelation experiments, it is found to be unsuitable in applications where drift is a problem. Brown (1969) presented a correlation method to estimate the linear kernel which requires only one crosscorrelation considering the system of Fig.(5.3) whereby perturbation signal x(t) is a 4-level sequence derived from a 2-level binary sequence s(t) and a signal f(t) where f(t) is periodic of period Δ and is defined over any period by -

$$
\begin{aligned}
f(t_1 + i\Delta) &= \xi \cdot a, \quad 0 < t_1 < \eta \frac{\Delta}{2} \\
&= -a, \quad \eta \frac{\Delta}{2} < t_1 < \Delta - \eta \frac{\Delta}{2} \\
&= \xi a, \quad \Delta - \eta \frac{\Delta}{2} < t_1 < \Delta
\end{aligned}
$$

considering a cubic nonlinearity Brown showed that by proper choice of ξ and η the crosscorrelation becomes insensitive to the nonlinear path and hence yields the impulse response of the linear portion $h(\tau)$.

By modifying the 4-level signal used in Brown's scheme, Power and Simpson (1970) have shown that it is possible to identify both the linear portions of the system of Fig.(5.3) and obtain the impulse responses $h(\tau)$ and $g(\tau)$ under the assumption that the linear portions have low - pass filter characteristic allowing the output to be almost periodic. Simpson and Power, subsequently, have also brought forward a generalized method for identification of a system of the type shown in Fig. (5.3) by expressing the output of the nonlinearity as the sum of the outputs due to odd and even components.

(G) Injection of high-frequency signal (dither) alongwith a pseudorandom binary test perturbation for measuring the linear kernel - Simpson's Method

Simpson has shown that the application of certain types of high frequency signals (dither) permits either linearization or total elimination of nonlinear channels in open and closed - loop systems. Under such conditions the linear portions of the system can be identified by the crosscorrelation techniques. The method is based upon the concept of equivalent nonlinearity as explained in brief below :

The basic open-loop system is shown in Fig.(5.4) whereby test signal W is assumed to have a spectrum containing significant components only upto the frequency $w_s << w_d$, where w_d is the angular frequency of the fundamental component of a high frequency signal, termed as dither.

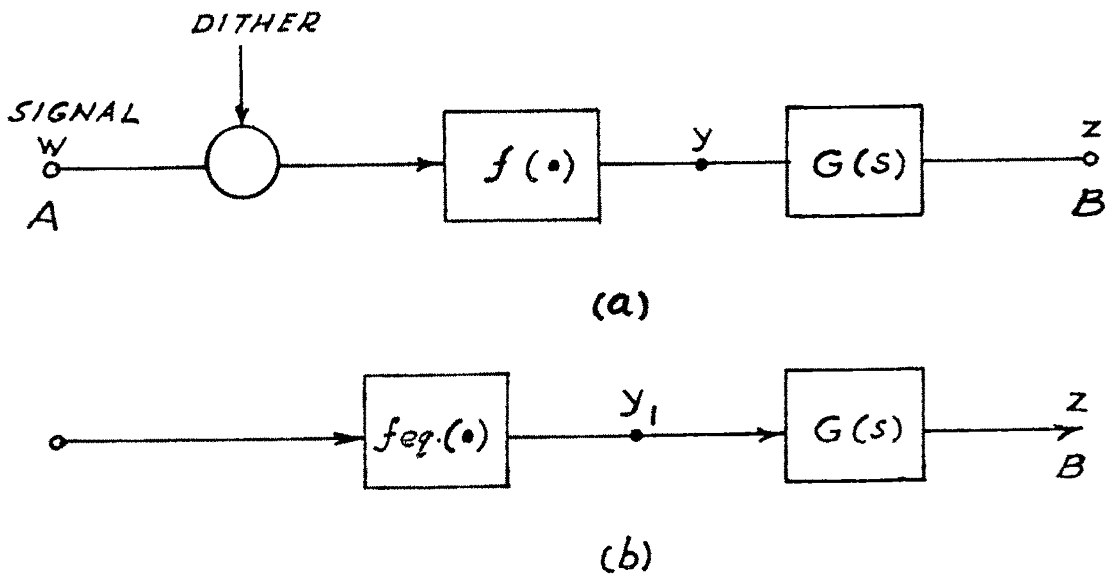

FIG.5.4 (a) BASIC SYSTEM AND
(b) IT'S EQUIVALENT

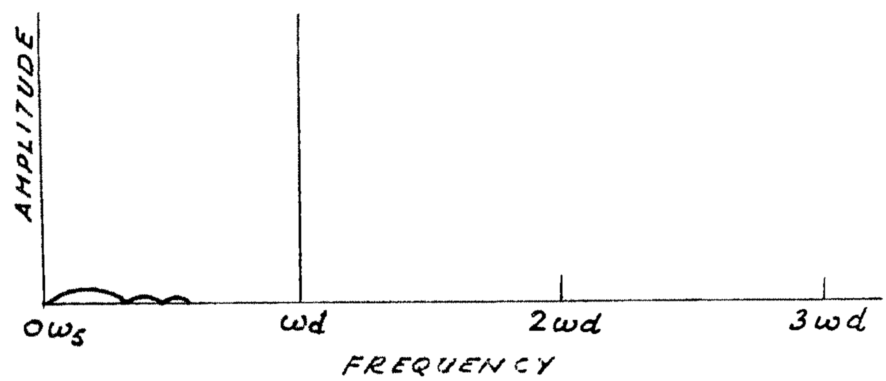

FIG.5.4 (c) SPECTRUM OF INPUT SIGNAL APPLIED TO NONLINEARITY

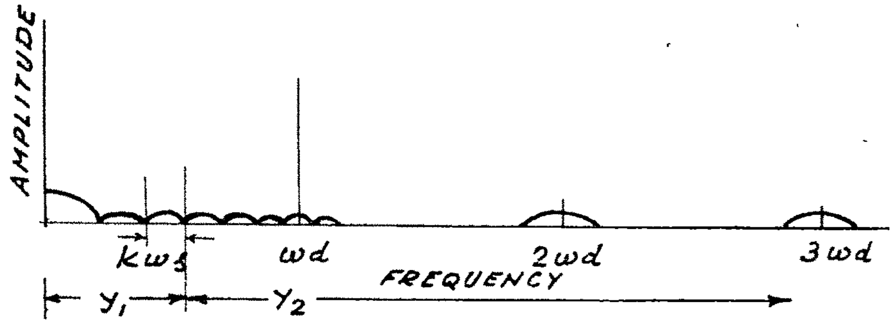

FIG.5.4 (d) SPECTRUM OF OUTPUT SIGNAL FROM NONLINEARITY

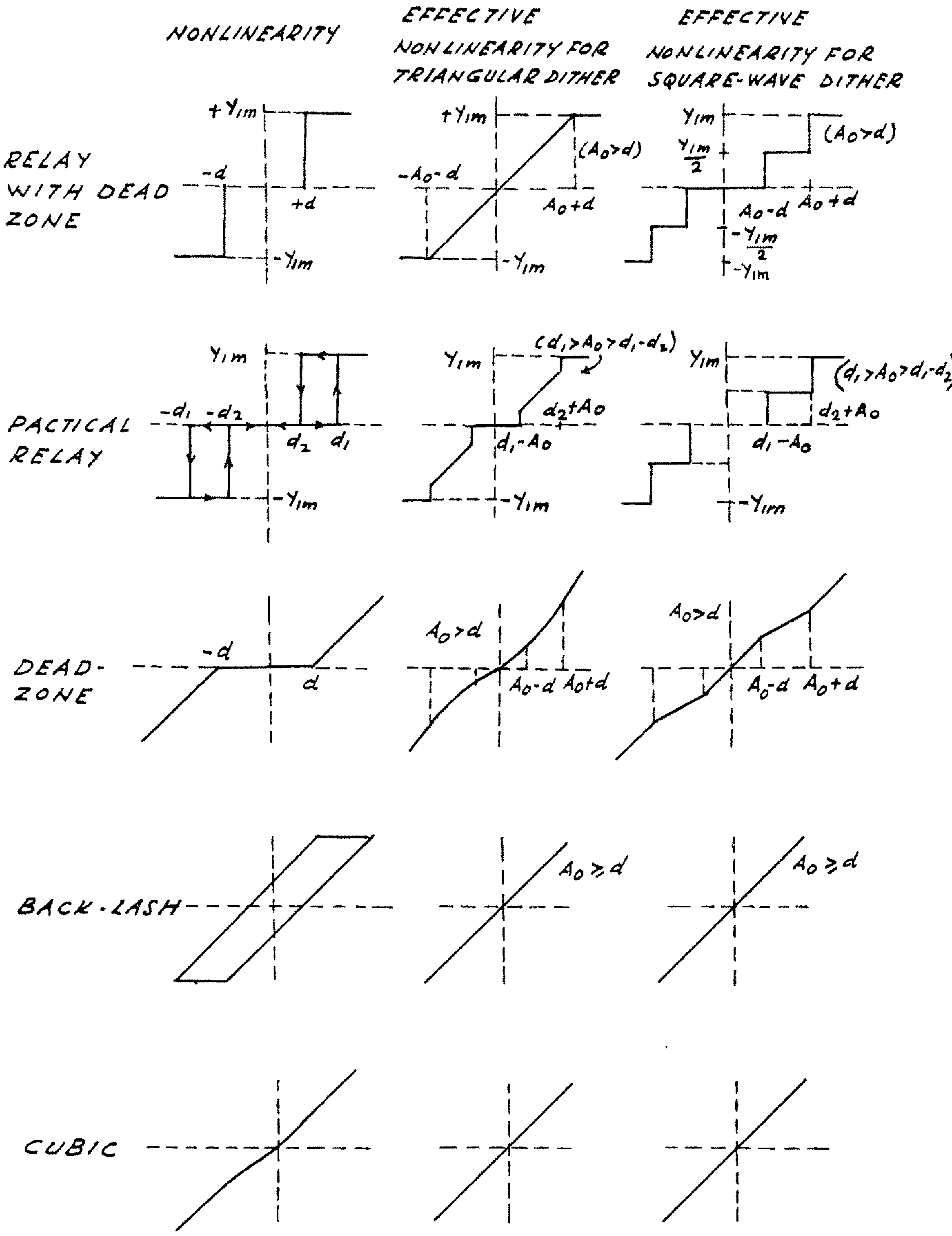

FIG. 5·5 SOME EQUIVALENT NONLINEARITIES

The output of the nonlinear element contains, in general, frequency components at all possible sums and differences of w_s and w_d. The spectrum for a pseudorandom binary signal and triangular dither are shown in Fig.(5.4). Provided $w_d - kw_s > kw_s$, y can be split up into two portions y_1 and y_2. In case, the cutoff frequency of G(s) is less than $w_d - kw_s$, y_2 makes no contribution to the output. Since y_1 contains frequency components derived from those of signal W, y could be obtained by passing W through some suitable nonlinear element $f_{eq}(.)$, which is termed as 'equivalent nonlinearity'.

Fig.(5.5) shows some usual nonlinearities and their equivalents as observed by Simpson using triangular and square wave dithers. As seen, whereas the triangular dither linearises the nonlinearity, the square wave dither eliminates the nonlinear channel.

Once the linearisation is achieved, the identification problem then becomes merely an impulse response measurement, which may be obtained using a pseudorandom binary signal as test perturbation and crosscorrelating the corresponding response with delayed versions of the perturbation.

The method is simple and easy to implement and obtain good measurements. However, all nonlinearities cannot be by injecting a high frequency signal.

5.3.3 Some remarks on the presently available crosscorrelation schemes for nonlinear system identification

The development of crosscorrelation art for the measurement of Volterra kernels of nonlinear systems is discussed in the preceding sub-section and the salient features of these schemes are pointed out. Concerning these schemes some points worthy of mention at this juncture are in order;

So far, mainly the identification problem has been considered only of those nonlinear systems which can be represented by the Volterra functional series expansion as stated in eqn.(5.2). However, the Volterra expansion was taken to objection on the basis that, with most systems, a differential equation description has more physical meaning than an integral equation. But, against this there is the fact that systems are rarely described accurately by low order differential equation, linear or nonlinear, and that the natural expression of correlation analysis is in integral form. Hence, there seems to be some justification for using Volterra series as a means of describing nonlinear systems. Indeed, Sandberg (1966) and Stark (1969) have presented the measurement of the first - and second order Volterra kernels for the nonlinear control system of the pupil of the eye. The discussion in these papers helps understanding on how to measure the Volterra kernels. Furthermore, the convergence of the Volterra series has been justified in the studies of Barrett (1965) and of Bansal (1969).

The test input signals considered in the currently used correlation schemes belong to one of the following groups -

(a) Pseudorandom binary signals based on m-sequences (PRBS)

(b) Pseudorandom ternary signals based on m-sequences (PRTS)

(c) Pseudorandom inverse repeat - or n-sequences.

(d) Four level signals derived from two level signals.

(e) Pseudorandom binary signals in association with high frequency signals (dither).

The relative advantages and drawbacks of the use of these signals in correlation method of nonlinear system identification may be stated as under :

(a) Pseudorandom binary signals (PRBS) have proven value for impulse response testing of linearized systems. They are quite easy to generate and process through the system. But their higher order autocorrelation function properties are not in close agreement with those of Gaussian white noise. These signals do not possess the antisymmetric property and hence cannot effectively discriminate against nonlinear contamination in the measurement of the linear kernel of a nonlinear system.

(b) The pseudorandom ternary signals (PRTS) have the chief advantage that they are antisymmetric and therefore discriminate against nonlinear contamination. Good measurements of the non-linear system dynamics are obtained using the signal as test perturbation. Indeed, the linear approximant of a nonlinear system can be well identified with the ternary signals.

The ternary chains have their second - ,third - and fourth order autocorrelation function approximating to those of Gaussian white noise and hence can be used to identify the first and second order Volterra kernels of a nonlinear system with reasonable accuracy.

However, anamolous regions exist for the fourth order autocorrelation function of the ternary m-sequence signal and so to obtain good measurements, it is necessary that this ternary chain period be about a hundred times more than the second order kernel settling time :(i.e.) $T = 100\ T_{s_2}$).

Another serious disadvantage is the difficulty of achieving three repeatable input levels with normal input hardware.

(c) Pseudorandom inverse repeat - or n-sequences can be generated by reversing every other bit of a PRBS. These sequences have twice the number of bits as the original PRBS. These sequences offer definite advantages over both the pseudorandom binary - and ternary sequences because they are antisymmetric and relatively easy to generate since only two signal levels are required. Thus n-sequences tend to circumvent some of the problems encountered in using pseudorandom signals in system identification.

(d) The 4-level signal derived from a 2-level binary signal in association with another periodic signal has proved to be of considerable importance in situations where drift is a problem because of its ability in making the correlator output insensitive to a nonlinear input-output path, formed from a zero memory nonlinearity followed by linear dynamics, in just one crosscorrelation only as against a minimum of two with other pseudorandom signals.

But injecting as well as correctly generating the 4-level signal is not so easy as with binary signal. Further use of this 4-level signal has so far been considered to identify only the linear channel of a nonlinear system and for its use in the measurement of higher order kernels, the higher order autocorrelation function properties of this signal are yet to be examined.

(e) The application of a high frequency signal (dither) to certain nonlinearities can change the effective characteristic of the nonlinear element in the system. The dither may either permit linearization or elimination of the nonlinear channels in certain open and closed-loop systems. Under these conditions the linear portions can be identified by impulse response techniques employing pseudorandom binary sequence as test perturbation. Hence this technique of dither injection can serve as a useful tool in system identification.

However, it should be noted that only certain types of nonlinearities are effectively linearised by dither injection, but not all. For instance, the cubic and dead-zone type of nonlinearities are not completely linearised by dither injection, although some linearisation is achieved.

It is, therefore, evident that there exists need for further work to obtain quick as well as better estimates of nonlinear system dynamics. The further work may be undertaken along the following directions :

(1) By effecting suitable transformations on the known test signals, a thorough search may be made for suitable system perturbations with desired correlation properties.

(2) The very pattern of crosscorrelation technique of system testing may be changed so that signals, which have proven value in system testing but for their lack of higher order correlation properties, may be effectively used to obtain reasonable estimates of the dynamics.

(3) It is also worthwhile considering the use of two or more principles exposed together rather than trying to find solution using the ideas independently.

(4) So far possible, interest may be confined to developing 2-level signals to retain the many advantages they offer in the correlation analysis of system dynamics.

In accordance with the above lines, some new correlation schemes are presented for nonlinear system identification in the next section.

5.4 SOME NEW CROSSCORRELATION SCHEMES FOR NONLINEAR SYSTEM IDENTIFICATION

In this section, some crosscorrelation schemes are presented for effective identification of a nonlinear system characterized by the first and the second Volterra kernels. It is shown that by means of this new correlation patterns, it is possible to effectively identify the first and second Volterra kernels using binary test inputs. As will be seen, the correlation time for the second kernel measurement is considerably reduced by this approach as compared to the methods in current use.

5.4.1 A new correlation method of system dynamic testing

5.4.1.1 The Theoretical considerations

According to Volterra (1920, 1958), the response y(t) of a time invariant system of a very general type to an input x(t) may be represented by the following series of functionals.o

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n) \prod_{k=1}^{\infty} x(t-\tau_k)\, d\tau_k \ \ldots \quad (5.28)$$

where $h_n(\tau_1,\tau_2,\ldots,\tau_n)$ is the nth order Volterra kernel of the system. Each of the kernels $h_1,h_2,h_3\ldots$ is the characteristics of the system. Hence the set of kernels h_n characterises the nonlinear system. Thus the identification problem of the nonlinear system consists of extracting these kernels from the input-output data.

In the conventional crosscorrelation schemes, the method adopted for obtaining the nth order system kernel is to perturb the system with a test signal x(t) and cross-correlate the corresponding response signal y(t) with n delayed versions of the input signal x(t). The correlator output will then yield the system kernel h_n provided the input signal x(t) has its autocorrelation function properties approximating closely to those of Gaussian white noise. The less is the approximation of correlation properties of x(t) to Gaussian noise, the more is the error in the kernel measurement. Thus, the key to the quality of measurement of the kernel by conventional approach is the closeness in approximation of the test signal autocorrelation properties to those of Gaussian white noise.

The above conventional correlation methods is, however, not preferable in situations where a test signal has all the desired properties for its use as system perturbation except that its higher order autocorrelations do not effectively agree with those of Gaussian white noise. To open the seal on the use of such test signals, it is necessary to adopt a new correlation pattern of system dynamic testing.

Furthermore, the conventional correlation method is not the only way for measuring the system dynamics.

When the system test perturbation x(t) does not possess the desired autocorrelation function properties so as to permit the use of conventional approach effectively, the system dynamics can still be measured provided the response signal y(t) is crosscorrelated with the delayed version of another signal, say $x_1(t)$, provided the crosscorrelation function properties between x(t) and $x_1(t)$ closely approximate to the autocorrelation properties of Gaussian white noise.

This new correlation pattern of nonlinear system testing is depicted in Fig.(5.6). As shown, the system is perturbed with the signal x(t) but the corresponding response y(t) is crosscorrelated with the delayed versions of another signal $x_1(t)$ instead of with the delayed lines of the perturbation x(t).

By introducing this change into the pattern of cross-correlation testing, what is effectively done is that the correlator output $\phi_{x_1 y}(\tau)$ is made to depend not solely on the correlation properties of perturbation signal x(t), but on the correlation properties of both the perturbation signal x(t) and the signal $x_1(t)$, used for crosscorrelation with y(t). Specifically, in this new method the correlator output depends on the crosscorrelation function properties between the signals x(t) and $x_1(t)$. So much so the quality of measured system kernels is based on how closely the crosscorrelation

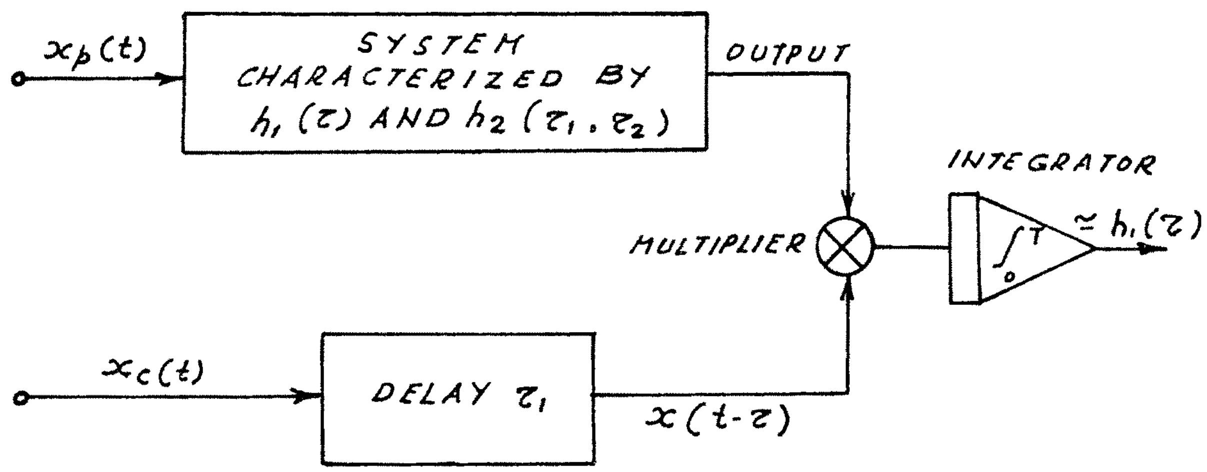

FIG. 5·6 (a) NEW CORRELATION PATTERN FOR THE MEASUREMENT OF THE FIRST-ORDER KERNEL $h_1(\tau)$

$x_p(t)$: PERTURATION SIGNAL

$x_c(t)$: SIGNAL USED FOR CROSS-CORRELATION WITH THE RESPONSE-SIGNAL $y(t)$ (i.e. CORRELATOR-SIGNAL)

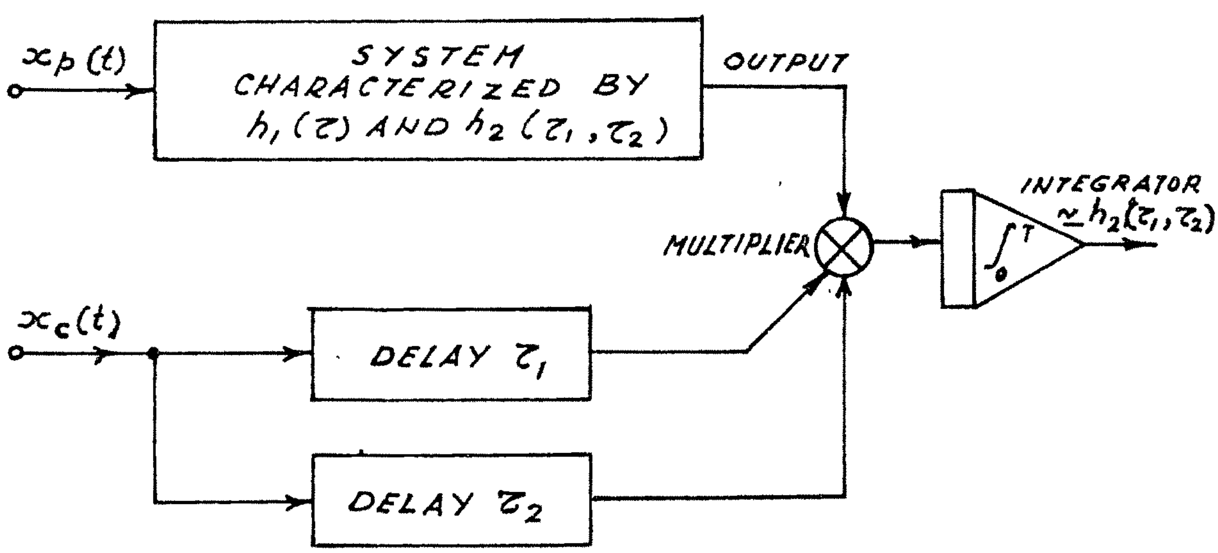

FIG. 5·6 (b) NEW-CORRELATION PATTERN FOR THE MEASUREMENT OF THE SECOND-ORDER KERNEL $h_2(\tau_1, \tau_2)$.

function properties between $x(t)$ and $x_1(t)$ approximate to the corresponding autocorrelation properties of Gaussian white noise.

Thus, in the new approach the nonlinear identification problem is one of finding suitable pairs of signals $x(t)$ and $x_1(t)$ with desired crosscorrelation between them. Herein lies a potential advantage of the new correlation pattern over the conventional counterpart. In this new approach, since two signals are involved ($x(t)$ and $x_1(t)$) it is possible to compensate for any drawbacks in the properties of one signal with the proper choice of the other signal. Such a possibility does not exist in the conventional correlation method. Furthermore, in case a pair of binary signals can be found with required crosscorrelations, all the advantages associated with the use of binary signals can be retained. Some striking advantages that can be gained by employing binary signals in dynamic testing are listed below.

(i) Two-level signals offer the possibility of easing or eliminating the equipment problems and of reducing the test duration. The delay of such signals is considerably easier and can be achieved with one of the available digital storage devices.

2. In many systems, it is possible for normal system hardware to introduce suitable binary perturbations for the test with ease.

3. Furthermore, a binary signal contains the maximum energy for a given peak value. Thus it is possible to obtain the greatest output signal to noise ratio for a given degree of system disturbance. This is important as the normal operation of the system under test is not to be disturbed.

4. The multiplication operation is much easier to mechanise because the binary signal reduces it to a simple gating operation.

5. Experiences with binary signal in the measurement of transfer functions of linearised systems suggest that the effort and cost is small enough to justify routine dynamic tests in many industrial processes.

Now, recalling the general Volterra expansion of a nonlinear system, the output y(t) to the test input x(t) is written as -

$$y(t) = \int_0^{T_{s_1}} h_1(u)\ x(t-u)\ du + \int_0^{T_{s_2}} \int_0^{T_{s_2}} h_2(u_1, u_2).$$

$$x(t-u_1)\ x(t-u_2)\,du_1 du_2$$

$$+\text{higher order functionals.. } (5.29)$$

where T_{s_1} and T_{s_2} are, respectively, the settling times of the first - and the second Volterra kernels h_1 and h_2.

In situations where the nonlinear element of the system permits to truncate the functional series at the 2nd term of where a higher frequency signal (dither) is injected into the system to effectively eliminate the higher order functionals, the input-output Volterra relation may be simplified to read -

$$y(t) = \int_o^{T_{s_1}} h_1(u)\ x(t-u)\ du + \int_o^{T_{s_2}} \int_o^{T_{s_2}} h_2(u_1,u_2)\ x(t-u_1)\ x(t-u_2) du_1 du_2 \quad \dots (5.30)$$

The new crosscorrelation scheme for the measurement of the kernels h_1 and h_2 are shown in Fig. (5.6). The correlator outputs in the respective schemes may be written as -

$$\phi_{x_c y}(\tau) = \int_o^{T_{s_1}} h_1(u)\ \phi_{x_c x_p}(c,u) du + \int_o^{T_{s_2}} \int_o^{T_{s_2}} h_2(u_1,u_2)\ \phi_{x_c x_p x_p}(\tau,u_1,u_2) du_1 du_2 \dots (5.31)$$

and

$$\phi_{x_c x_c y}(\tau_1,\tau_2) = \int_o^{T_{s_1}} h_1(u)\ \phi_{x_c x_c x_p}(\tau_1,\tau_2,u) du + \int_o^{T_{s_2}} \int_o^{T_{s_2}} h_2(u_1,u_2)\ \phi_{x_c x_c x_p x_p}(\tau_1,\tau_2,u_1,u_2) du_1 du_2 \quad \dots (5.32)$$

whereby

$$\phi_{\underbrace{x_c \dots x_c}_{j\ \text{times}},\underbrace{x_p \dots x_p}_{k\ \text{times}}} = \frac{1}{T} \int_o^T x_c(t-\tau_1) \dots x_c(t-\tau_j) . x_p(t-u_1) \dots x_p(t-u_k) dt$$

is the higher-order crosscorrelation function between $x_c(t)$ and $x_p(t)$.

As eqns. (5.31) and (5.32) indicate, it is the crosscorrelation function properties between the signals $x_{\rho}(t)$ and $x_{\xi}(t)$ that should facilitate the possibility of identifying the system kernels. In what follows, it is shown that a pseudorandom binary signal and a transformed version of it possess the desired crosscorrelation and hence permit accurate measurement of the system kernels. The correlation properties between a PRBS and its transformed version are derived in Section(5.4.1.2). The kernel measurement procedures are presented in Section (5.4.1.3).

5.4.1.2 Auto- and crosscorrelation properties between a PRBS and its transformed version

Let $\{a_i\}$ be an m-sequence of +1s and -1s and $\{b_i\}$ be the same sequence in which -1s have been replaced by 0s. Let the continuous version of the sequence $\{a_i\}$ be represented by x(t), and that of the sequence $\{b_i\}$ be represented by $x_1(t)$. Hence, x(t) is a pseudorandom binary signal and $x_1(t)$ is its transformed version. It is of present interest to consider the correlation properties between these two signals.

The crosscorrelation function of order 'jk' between the signals $x_1(t)$ and x(t) is defined by -

$$\phi(\underbrace{x_1 \ldots x_1}_{j \text{ times}};\ \underbrace{x \ldots x}_{k \text{ times}}) = \frac{1}{T}\int_0^T (x_1-\tau_1)\ldots(x_1-\tau_j)(x-u_1)(x-u_k)\,dt \qquad \ldots (5.33)$$

Since, x(t) and $x_1(t)$ are constant in any basic interval (t_o), $\phi_{x_1 \ldots x_1;\ x \ldots x}$ is linear in any τ_j or u_k in any interval $rt_o \leq \tau$ or $u \leq (r+1)t_o$, and the sampled function $\phi_{b \ldots b; a \ldots a}$ is therefore sufficient to define the continuous function $\phi_{x_1 \ldots x_1\ x \ldots x}$.

Moreover,

$$\phi_{\underbrace{x_1\ldots\ldots x_1}_{j\text{ times}};\underbrace{x\ldots\ldots x}_{k\text{ times}}}(r_1,\ldots r_j,s_1,\ldots s_k)$$

$$= \frac{1}{T}\int_o^T (x_1-r_1t_o)(x_1-r_2t_o)\ldots(x_1-r_jt_o)$$

$$\cdot\ (x-s_1t_o)(x-s_2t_o)\ldots(x-s_kt_o) \quad dt$$

$$= \frac{1}{N}\sum_{i=0}^{N-1}(b_{i-r_1}\cdot b_{i-r_2}\ldots b_{i-r_j})(a_{i-s_1}\cdot a_{i-s_2}\ldots a_{i-s_k})$$

$$= \phi_{\underbrace{b\ldots\ldots b}_{j\text{ times}};\ \underbrace{a\ \ldots\ldots a}_{k\text{ times}}}(r_1,\ldots r_j,s_1\ldots s_k) \quad \ldots \quad (5.33)$$

Whereby $\phi_{b\ldots b;a\ldots a}$ is the crosscorrelation function of order 'jk' between the sequences $\{b_i\}$ and $\{a_i\}$, which is therefore sufficient to define $\phi_{x_1\ldots x_1;x\ldots x}$ everywhere.

Similar reasoning is valid for the higher order autocorrelation functions of the signals x(t) and $x_1(t)$. For instance, the jth order autocorrelation function of $x_1(t)$ is written as

$$\phi_{\underbrace{x_1\ldots x_1}_{j\text{ times}}}(r_1,\ldots,r_j) = \frac{1}{T}\int_o^T x_1(t-r_1t_o)\ x(t-r_2t_o)\ldots x_1(t-r_jt_o)\ dt$$

$$= \frac{1}{N}\sum_{i=0}^{N-1}(b_{i-r_1}\ b_{i-r_2}\ldots b_{i-r_j})$$

$$= \phi_{\underbrace{b\ \ldots\ldots b}_{j\text{ times}}}(r_1,\ldots r_j) \quad \ldots \quad \ldots \quad (5.34)$$

Likewise, the k-ith order autocorrelation function of x(t) is written as -

$$\underset{k\ \text{times}}{\phi_{x\ldots\ldots x}}(s_1,\ldots,s_k) = \frac{1}{T}\int_o^T x(t-s_1t_o)\ x(t-s_2t_o)\ldots x(t-s_kt_o)\ dt$$

$$= \frac{1}{N}\sum_{i=0}^{N-1}(a_{i-s_1}\cdot a_{i-s_2}\ldots a_{i-s_k})$$

$$= \underset{k\ \text{times}}{\phi_{a\ldots\ldots a}}(s_1,\ldots,s_k) \quad \ldots \quad \ldots \quad (5.35)$$

The auto- and crosscorrelation functions of the sequences $\{a_i\}$ and $\{b_i\}$ (equivalently of the signals x(t) and $x_1(t)$), that are of present interest in the measurement of the first - and second kernels, are derived below.

(A) Second order autocorrelation function of $\{a_i\}$

$\{a_i\}$ is an m-sequence of period N digits with elements ± 1. It has the shift and add property over modulo-2, i.e.

$$\{a_i\} \oplus \{a_{i+s}\} = \{a_{i+t}\} \quad \ldots \quad \text{(mod-2 addition)}$$

where ál $\leq$ s, t $\leq$ N - 1.

The second order autocorrelation function of the sequence $\{a_i\}$ over one period (in accordance with the notation stated earlier) is defined by -

$$\phi_{aa}(s_1,s_2) = \frac{1}{N}\sum_{i=0}^{N-1} a_{i-s_1}\cdot a_{i-s_2} \quad \ldots \quad 1 \leq s_1, s_2 \leq N-1$$

$$= 1, \text{ For } s_1 - s_2 = 0$$

$$= -1/N \text{ for } s_1 - s_2 \neq 0. \quad \ldots \quad \ldots . \quad (5.36)$$

(B) The third-order autocorrelation function of $\{a_i\}$

By definition, the third order autocorrelation function of $\{a_i\}$ is written as -

$$\phi_{aaa}(s_1, s_2, s_3) = \frac{1}{N}\sum_{i=0}^{N-1} (a_{i-s_1} \cdot a_{i-s_2} \cdot a_{i-s_3}), \qquad 1 \le s_1, s_2, s_3 \le N-1$$

$$= 1, \quad \text{Provided any two shifted versions of } \{a_i\} \text{ exactly produce the third shifted version.}$$

Excluding the above condition,

$$\phi_{aaa}(s_1, s_2, s_3) = \frac{1}{N}\sum_{i=0}^{N-1} (a^{*}_{i-s_q} \cdot a_{i-s_3}),$$

Where $\{a^{*}_i\}$ is the binary conjugate of $\{a_i\}$; i.e. replacing ± 1s in $\{a_i\}$ by ∓ 1s gives $\{a^{*}_i\}$, and $0 \le q \le N-1$

Hence,

$$\begin{aligned} \phi_{aaa}(s_1, s_2, s_3) &= \frac{1}{N}, \quad \text{for } q \ne s_3 \\ &= -1, \quad \text{for } q = s_3. \end{aligned} \qquad \ldots \quad \ldots \quad (5.37)$$

(C) Second order autocorrelation function of transformed pseudorandom binary sequence $\{b_i\}$

The sequence $\{b_i\}$ is obtained by replacing -1s in $\{a_i\}$ by 0s. Therefore, we may write -

$$\{b_i\} = \frac{1}{2}\left[\{a_i\} + \{e_i\}\right] \quad \ldots \quad \ldots \quad (5.38)$$

where $\{e_i\}$ is an 'all-ONE' sequence.

The second order autocorrelation function of $\{b_i\}$ (in accordance with notation stated earlier) is given by -

$$\phi_{bb}(r_1, r_2) = \frac{1}{N} \sum_{i=0}^{N-1} (b_{i-r_1} . b_{i-r_2})$$

Utilizing equ.(5.38) leads to -

$$\phi_{bb}(r_1,r_2) = \frac{1}{4N} \sum_{i=0}^{N-1} (a_{i-r_1} + e_{i-r_1})(a_{i-r_2}+e_{i-r_2})$$

$$= \frac{1}{4N} \sum_{i=0}^{N-1} \left[(a_{i-r_1} . a_{i-r_2}) + (a_{i-r_1}) + (a_{i-r_2}) + (e_i) \right]$$

$$= \frac{1}{4} \left[\phi_{aa}(r_1, r_2) + \frac{N+2}{N} \right] \quad \ldots \quad \ldots \quad (5.39)$$

since $\{e_i\}$ is merely an 'all-one' sequence.

Hence,

$$\left.\begin{aligned} \phi_{bb}(r_1,r_2) &= \frac{N+1}{2N}, \quad \text{for } r_1 = r_2 \\ &= \frac{N+1}{4N}, \quad \text{for } r_1 \neq r_2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (5.40)$$

Typical second order autocorrelation function of sequences $\{a_i\}$ and $\{b_i\}$ are depicted in Fig.(5.7). As the figure shows, the autocorrelation function of the transformed sequence has a positive bias as against the negative bias present in that of original pseudorandom sequence $\{a_i\}$. Further , the peak value in case of $\{b_i\}$ is reduced as compared to the value 1 in case of $\{a_i\}$.

(D) The third order autocorrelation function of $\{b_i\}$

By definition, the third order autocorrelation function of $\{b_i\}$ is written as - (over one period)

$$\begin{aligned}
\phi_{bbb}(r_1, r_2, r_3) &= \frac{1}{N}\sum_{i=0}^{N-1} (b_{i-r_1} \cdot b_{i-r_2} \cdot b_{i-r_3}) \\
&= \frac{1}{N}\sum_{i=0}^{N-1} (b_{i-r_1}) \quad \text{for } r_1 = r_2 = r_3 \\
& \qquad\qquad\qquad\qquad 1 \leq r_1, r_2, r_3 \leq N-1 \\
&= \frac{N+1}{2N} \\
&= \frac{1}{N}\sum_{i=0}^{N-1} (b_{i-r_1} \cdot b_{i-r_3}) \quad \text{for } r_1 = r_2 \neq r_3 \\
&= \frac{N+1}{4N} \\
&= \frac{1}{N}\sum_{i=0}^{N-1} (b_{i-r_1} \cdot b_{i-r_2}) \quad \text{for } r_1 \neq r_2 \neq r_3 \\
& \qquad\qquad\qquad\qquad \ldots (5.41) \\
&= N + 1 / 4N
\end{aligned}$$

Whereby eqns. (5.38) and (5.40) are made use of.

(E) The crosscorrelation functions between $\{b_i\}$ and $\{a_i\}$

By definition, the crosscorrelation function between $\{b_i\}$ and $\{a_i\}$ of order $j = 1$, $k = 1$ is written as -

$$\begin{aligned}
\phi_{ba}(r_1, s_1) &= \frac{1}{N}\sum_{i=0}^{N-1} (b_{i-r_1} \cdot a_{i-s_1}) \qquad 1 \leq r_1, s_1 \leq N-1 \\
&= \frac{1}{2N}\sum_{i=0}^{N-1} (a_{i-r_1} + e_{i-r_1}) \cdot a_{i-s_1}
\end{aligned}$$

Hence,

$$\phi_{ba}(r_1, s_1) = \frac{1}{2N} \sum_{i=0}^{N-1} (a_{i-r_1} \cdot a_{i-s_1} + a_{i-s_1})$$

$$= \frac{1}{2}\left[\phi_{aa}(r_1, s_1) + \frac{1}{N}\right]$$

$$= \begin{cases} (N+1)/2N & \text{for } r_1 = s_1 \\ 0 & \text{for } r_1 \neq s_1 \end{cases} \quad \ldots \quad \ldots \qquad (5.42)$$

whereby the values of ϕ_{aa} are substituted from eqn. (5.34).
A typical crosscorrelation function ϕ_{ba} is shown in Fig. (5.7)

(F) The crosscorrelation function $\phi_{baa}(r_1, s_1, s_2)$

As will be seen in the following sub-section, this crosscorrelation function ϕ_{baa} is encountered in the measurement of both the first order - and second order Volterra kernels. Specifically, of particular interest in the first order kernel measurement are the values of the function $\phi_{baa}(r_1, s_1, s_2)$ as a function of s_1 and s_2 over the grid (s_1, s_2) for fixed value of r_1. Similarly, in the measurement of the 2nd order Volterra kernel, values of $\phi_{baa}(r_1, s_1, s_2)$ as a function of r_1 for fixed values of (s_1, s_2) is of main consideration. Because of its importance, this function will be considered in some detail as follows -

By definition, the crosscorrelation function between $\{a_i\}$ and $\{b_i\}$ of order 'jk' with j=1 and k=2 is given by -

$$\phi_{baa}(r_1, s_1, s_2) = \frac{1}{N} \sum_{i=0}^{N-1} (b_{i-r_1} \cdot a_{i-s_1} \cdot a_{i-s_2})$$

$$= \frac{1}{2N} \sum_{i=0}^{N-1} (a_{i-r_1} + e_{i-r_1}).(a_{i-s_1} \cdot a_{i-s_2})$$

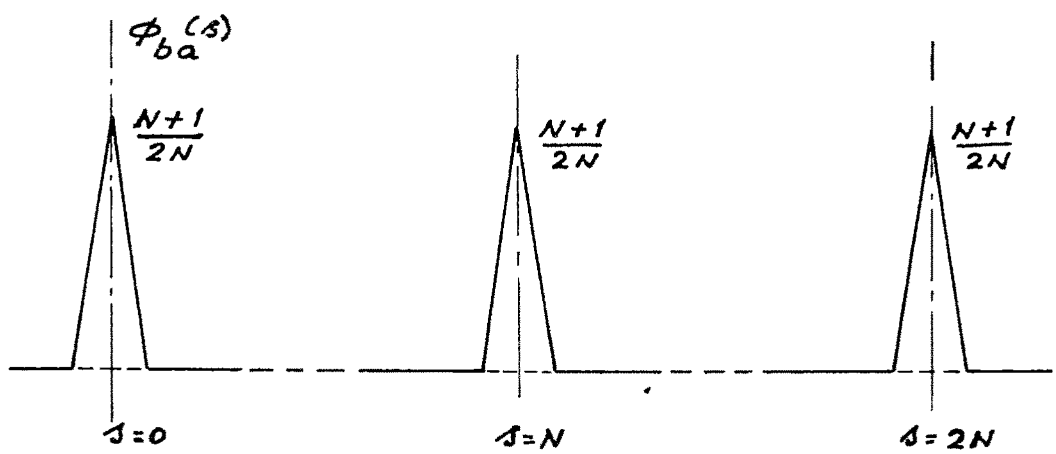

FIG. 5·7 CROSS-CORRELATION FUNCTION

$\phi_{ba}(s=r_1-s_1)$

Hence,

$$\phi_{baa}(r_1, s_1, s_2) = \frac{1}{2}\left[\phi_{aaa}(r_1, s_1, s_2) + \phi_{aa}(s_1, s_2)\right]$$

$$= \frac{1}{2}\left(\frac{1}{N} + 1\right) \qquad \text{for } s_1 = s_2, \text{ any } r_1$$

$$= \frac{1}{2}\left(\frac{1}{N} - \frac{1}{N}\right) = 0, \qquad \text{for } r_1 = s_1 \text{ or } s_2 \text{ but } s_1 \neq s_2$$

$$\left.\begin{array}{l} = \frac{1}{2}\left(-1 - \frac{1}{N}\right), \\ \quad \text{OR} \\ 0 \end{array}\right\} \quad \text{for } r_1 \neq s_1 \neq s_2 \qquad \ldots (5.43)$$

It is of interest to find the values of (s_1, s_2) for $s_1 \neq s_2$ over which the function ϕ_{baa} assumes the value $\frac{1}{2}(-1 - \frac{1}{N})$ or equivalently over which the function ϕ_{aaa} is assigned the value -1.

Clearly, this happens when the product of the two sequences with shifts s_1 and s_2 produces the binary conjugate of the remaining shifted version of the m-sequence $\{a_i\}$, i.e. when

$$\{a_{i-s_1}\} \cdot \{a_{i-s_2}\} = \{a^*_{i-r_1}\} \quad \text{(Star represents conjugate operation)} \quad \ldots (5.44)$$

This relation when translated into a delay equation appears as -

$$(D^{s_1} \cdot D^{s_2}) = (D^{r_1})^*$$

Since the binary multiplication is just the negative of modulo-2 operation, the above relation may be expressed as -

$$D^{s_1} \oplus D^{s_2} = D^{r_1}$$

Thus the problem is to find the set of values of (s_1, s_2) for which the above equation is valid, over the region $0 \leq r_1, s_1, s_2 \leq N - 1$ for $r_1 \neq s_1 \neq s_2$.

To begin with, r_1 can be assigned any value in the region 0 to N-1. And for a chosen r_1, s_1 assumes any of the values in the same region, excepting the selected value of r_1. Similarly, for given r_1 and s_1, s_2 can have any of the possible values in the same region, excluding those of r_1 and s_1.

The total number of the possible sets of r_1, s_1, s_2 without any constraint in the said region equals N^3. Of these N^3 sets, from the statement of the above paragraph, only $N(N-1)(N-2)$ number of sets correspond to the case of $r_1 \neq s_1 \neq s_2$.

Now, a binary m-sequence of period N equal to 2^n-1, n = integer, generates all possible non-zero n-digit numbers in the range 1 to 2^n-1. Consequently, in the m-sequence, there are $\frac{N-1}{2}$ pairs of n-bit numbers that are conjugate of each other.

As a result of the above property, in one period of $\{a_i\}$, with the value of r_1 fixed, for each value of s_1 there is exactly one value of s_2 for which -

$$D^{s_1} \oplus D^{s_2} = D^{r_1}, \qquad (r_1 \neq s_1 \neq s_2)$$

Since, r_1 can have N possible values, and for a given r_1, s_1 can have only (N-1) values, there are only N(N-1) number of sets of r_1, s_1, s_2 which satisfy the above delay equation and thus assign the value -1 to ϕ_{aaa} or equivalently the value $\frac{1}{2}(-1-\frac{1}{N})$ to the desired crosscorrelation function $\phi_{baa}(r_1, s_1, s_2)$.

To find these $N(N-1)$ sets of (r_1, s_1, s_2) referred to before, the following procedure is adopted -

Step 1 : Write all the N phase shifted versionsof the m-sequence $\{a_i\}$ starting with the N distinct starting phases, and consider the first sequence $\{a_i\}$ as the reference sequence.

Step 2 : Select any one of these N shifted versions of the m-sequence, say $\{a_{i-r_1}\}$, and thus fix the value of r_1.

Step 3 : Excluding the sequence $\{a_{i-r_1}\}$, from the remaining set of N-1 phase shifted versions of $\{a_i\}$, find the pairs of sequences $\{a_{i-s_1}\}$ and $\{a_{i-s_2}\}$, which add up in the modulo-2 sense to that of $\{a_{i-r_1}\}$. This can be done with ease by mod-2 adding the first n digits of any two sequences in the set of N-1 sequences, and comparing the resulting n-digit number with the first n digits of the sequence $\{a_{i-r_1}\}$. For each chosen value of r_1, ignoring the order of occurance of s_1 and s_2, there are $(N-1)/2$ pairs of (s_1, s_2).

It is these distinct pairs of s_1, s_2, in the case of $r_1 \neq s_1 \neq s_2$, that make the function ϕ_{baa} to assume the value $\frac{1}{2}\left(- 1 - \frac{1}{N} \right)$.

From eqn.(5.44), it is evident that the values of (s_1, s_2) for given r_1 depend on the characteristic delay polynomial governing the m-sequence. (See example 5.1 and 5.2)

In finding the pairs (s_1, s_2), it is necessary to follow the step No.3 for only one value of r_1, say for $r_1=0$. And, for any other value of $r_1 = R_i$(say), the values of (s_1, s_2) are obtained by simply adding the value R_i to the values of (s_1, s_2) obtained in case of $r_1=0$ and reducing the resulting values over mod-N.

Some remarks on the values of $\phi_{baa}(r_1, s_1, s_2)$

The correlation function $\phi_{baa}(r_1, s_1, s_2)$ over the region 0 to N-1 (i.e. over one period of the m-sequence $\{a_i\}$) is seen to assume values as under :

(i) ϕ_{baa} exhibits its positive spikes for $s_1=s_2$, independent of all r_1.

The amplitude of the positive spike = $\frac{1}{2}(1 + \frac{1}{N})$

The number of the positive spikes = N^2 ... ... (5.45)

(ii) ϕ_{baa} exhibits its negative spikes for some sets of values of $(r_1 \neq s_1 \neq s_2)$

The amplitude of the negative spike = $-\frac{1}{2}(1 + \frac{1}{N})$

The number of the negative spikes = N(N-1) ... ... (5.46)

The exact location of the negative spikes can be determined in accordance with the steps outlined on page 454. Further, the location of these negative spikes depends on the particular characteristic delay polynomial generating the m-sequence.

(iii) ϕ_{baa} assumes the value 'zero' in two cases :

(i) For $r_1 = s_1$ or s_2, $s_1 \neq s_2$

(ii) For some values of the sets $(r_1 \neq s_1 \neq s_2)$

The number of ($r_1 = s_1$ or s_2) - sets for $s_1 \neq s_2 = 2N(N-1)$ ... (5.47)

The number of $(r_1 \neq s_1 \neq s_2)$ sets over which ϕ_{baa} is zero

= [(Total No. of $r_1 \neq s_1 \neq s_2$ sets)
- (No. of $r_1 \neq s_1 = s_2$ sets which make the function ϕ_{baa} assume its negative spike)]

= [N(N-1)(N-2) - N(N-1)] = N(N-1)(N-3) ... (5.48)

To clear the course of analysis and to show the dependence of the location of negative spikes of the function ϕ_{baa} on the characteristic delay polynomial of the m-sequence $\{a_i\}$, the correlation function is illustrated by two examples below. These examples will be particularly referred in the following sub-section, where the measurement procedures for obtaining the Volterra kernels by means of the new correlation pattern of system testing is presented.

Example 5.1

Here, the expression for the function ϕ_{baa} stated in eqn. (5.43) is applied to the binary m-sequence $\{a_i\}$ with the characteristic delay polynomial

$F(D) = I \oplus D^2 \oplus D^5$ ... ... (mod-2 addition)

of order

$n = 5.$

The m-sequence $\{a_i\}$ generated by the polynomial with period

$N = 2^5 - 1 = 31$

with levels +1 and -1 (normalized values) reads as :

$\{a_i\}$: 11111-1-111-11-1-11-1-1-1-11-11-1111-111-1-1-1,.. repeats

Hence, the transformed sequence b_i reads from eqn. (5.35) as :

$\{b_i\}$: 1111100110100100001010111011000, ... repeats

From eqn.(5.43), $\phi_{baa}(r_1, s_1, s_2) = \frac{1}{2}(1 + \frac{1}{31})$ (+ spike)

$= 0$ ‖ For $0 \leq s_1 = s_2 \leq 30$
For $0 \leq r_1 = s_1$ or $s_2 \leq 30$
and
For $\phi_{aaa} = 1/N$

$= -\frac{1}{2}(1 + 1/31)$ (-ve spike)
For $\phi_{aaa} = -1$

The values of (s_1, s_2) for fixed r_1 at which ϕ_{baa} exhibits negative spikes will be obtained as per procedure stated on page 454.

For the m-sequence considered for illustration, in accordance with the steps stated earlier, the values of N(N-1)/2 sets of s_1, s_2 are obtained and given in Table 5.2. The step 1 in this procedure is illustrated in Table 5.1. In this Table 5.1, for purposes of convenience, the state '-1' in the m-sequence is represented by '2'.

Table 5.1 : Step 1 in finding pairs of s_1, s_2 for given r_1.

m-sequence	:	The N phase shifted versions of reference sequence		
a_i	:	1111122112122122221212111211222,	..	repeats
a_{i-1}	:	2111112211212212222121211121122,	..	"
a_{i-2}	:	2211111221121221222212121112112,	..	"
a_{i-3}	:	2221111122112122122221212111211,	..	"
a_{i-4}	:	1222111112211212212222121211121,	..	"
a_{i-5}	:	1122211111221121221222212121112,	..	"
a_{i-6}	:	2112221111122112122122221212111,	..	"
a_{i-7}	:	1211222111112211212212222121211,	..	"
a_{i-8}	:	1121122211111221121221222212121,	..	"
a_{i-9}	:	1112112221111122112122122221212,	..	"
a_{i-10}	:	2111211222111112211212212222121,	..	"
a_{i-11}	:	1211121122211111221121221222212,	..	"
a_{i-12}	:	2121112112221111122112122122221,	..	"
a_{i-13}	:	1212111211222111112211212212222,	..	"
a_{i-14}	:	2121211121122211111221121221222,	..	"
a_{i-15}	:	2212121112112221111122112122122,	..	"
a_{i-16}	:	2221212111211222111112211212212,	..	"
a_{i-17}	:	2222121211121122211111221121221,	..	"
a_{i-18}	:	1222212121112112221111122112122,	..	"
a_{i-19}	:	2122221212111211222111112211212,	..	"
a_{i-20}	:	2212222121211121122211111221121,	..	"
a_{i-21}	:	1221222121211121122211111122112,	..	"
a_{i-22}	:	2122122221212111211222111112211,	..	"
a_{i-23}	:	1212212222121211121122211111221,	..	"
a_{i-24}	:	1121221222212121112112221111122,	..	"
a_{i-25}	:	2112122122221212111211222111112,	..	"
a_{i-26}	:	2211212212222121211121122211111,	..	"
a_{i-27}	:	1221121221222212121112112221111,	..	"
a_{i-28}	:	1122112122122221212111211222111,	..	"
a_{i-29}	:	1112211212212222121211121122211,	..	"
a_{i-30}	:	1111221121221222212121112112221,	..	"

Table 5.2: For a fixed value of r_1, the distinct possible values of (s_1, s_2) for which $\phi_{baa} = \frac{1}{2}(-1 - \frac{1}{N})$

r_1	(s_1, s_2)														
0:	(1,18)	(2,5)	(3,29)	(4,10)	(6,27)	(7,22)	(8,20)	(9,16)	(11,19)	(12,23)	(13,14)	(15,24)	(17,30)	(21,25)	(26,28)
1:	(2,19)	(3,6)	(4,30)	(5,11)	(7,28)	(8,23)	(9,21)	(10,17)	(12,20)	(13,24)	(14,15)	(16,25)	(18,0)	(22,26)	(27,29)
2:	(3,20)	(4,7)	(5,0)	(6,12)	(8,29)	(9,24)	(10,22)	(11,18)	(13,21)	(14,25)	(15,16)	(17,26)	(19,1)	(23,27)	(28,30)
3:	(4,21)	(5,8)	(6,1)	(7,13)	(9,30)	(10,25)	(11,23)	(12,19)	(14,22)	(15,26)	(16,17)	(18,27)	(20,2)	(24,28)	(29,0)
4:	(5,22)	(6,9)	(7,2)	(8,14)	(10,0)	(11,26)	(12,24)	(13,20)	(15,23)	(16,27)	(17,18)	(19,28)	(21,3)	(25,29)	(30,1)
5:	(6,23)	(7,10)	(8,3)	(9,15)	(11,1)	(12,27)	(13,25)	(14,21)	(16,24)	(17,28)	(18,19)	(20,29)	(22,4)	(26,30)	(0,2)
6:	(7,24)	(8,11)	(9,4)	(10,16)	(12,2)	(13,28)	(14,26)	(15,22)	(17,25)	(18,29)	(19,20)	(21,30)	(23,5)	(27,0)	(1,3)
7:	(8,25)	(9,12)	(10,5)	(11,17)	(13,3)	(14,29)	(15,27)	(16,23)	(18,26)	(19,30)	(20,21)	(22,0)	(24,6)	(28,1)	(2,4)
8:	(9,26)	(10,13)	(11,6)	(12,18)	(14,4)	(15,30)	(16,28)	(17,24)	(19,27)	(20,0)	(21,22)	(23,1)	(25,7)	(29,2)	(3,5)
9:	(10,27)	(11,14)	(12,7)	(13,19)	(15,5)	(16,0)	(17,29)	(18,25)	(20,28)	(21,1)	(22,23)	(24,2)	(26,8)	(30,3)	(4,6)
10:	(11,28)	(12,15)	(13,8)	(14,20)	(16,6)	(17,1)	(18,30)	(19,26)	(21,29)	(22,2)	(23,24)	(25,3)	(27,9)	(0,4)	(5,7)
11:	(12,29)	(13,16)	(14,9)	(15,21)	(17,7)	(18,2)	(19,0)	(20,27)	(22,30)	(23,3)	(24,25)	(26,4)	(28,10)	(1,5)	(6,8)
12:	(13,30)	(14,17)	(15,10)	(16,22)	(18,8)	(19,3)	(20,1)	(21,28)	(23,0)	(24,4)	(25,26)	(27,5)	(29,11)	(2,6)	(7,9)
13:	(14,0)	(15,18)	(16,11)	(17,23)	(19,9)	(20,4)	(21,2)	(22,29)	(24,1)	(25,5)	(26,27)	(28,6)	(30,12)	(3,7)	(8,10)
14:	(15,1)	(16,19)	(17,12)	(18,24)	(20,10)	(21,5)	(22,3)	(23,30)	(25,2)	(26,6)	(27,28)	(29,7)	(0,13)	(4,8)	(9,11)
15:	(16,2)	(17,20)	(18,13)	(19,25)	(21,11)	(22,6)	(23,4)	(24,0)	(26,3)	(27,7)	(28,29)	(30,8)	(1,14)	(5,9)	(10,12)
16:	(17,3)	(18,21)	(19,14)	(20,26)	(22,12)	(23,7)	(24,5)	(25,1)	(27,4)	(28,8)	(29,30)	(0,9)	(2,15)	(6,10)	(11,13)
17:	(18,4)	(19,22)	(20,15)	(21,27)	(23,13)	(24,8)	(25,6)	(26,2)	(28,5)	(29,9)	(30,0)	(1,10)	(3,16)	(7,11)	(12,14)
18:	(19,5)	(20,23)	(21,16)	(22,28)	(24,14)	(25,9)	(26,7)	(27,3)	(29,6)	(30,10)	(0,1)	(2,11)	(4,17)	(8,12)	(13,15)
19:	(20,6)	(21,24)	(22,17)	(23,29)	(25,15)	(26,10)	(27,8)	(28,4)	(30,7)	(0,11)	(1,2)	(3,12)	(5,18)	(9,13)	(14,16)
20:	(21,7)	(22,25)	(23,18)	(24,30)	(26,16)	(27,11)	(28,9)	(29,5)	(0,8)	(1,12)	(2,3)	(4,13)	(6,19)	(10,14)	(15,17)
21:	(22,8)	(23,26)	(24,19)	(25,0)	(27,17)	(28,12)	(29,10)	(30,6)	(1,9)	(2,13)	(3,4)	(5,14)	(7,20)	(11,15)	(16,18)
22:	(23,9)	(24,27)	(25,20)	(26,1)	(28,18)	(29,13)	(30,11)	(0,7)	(2,10)	(3,14)	(4,5)	(6,15)	(8,21)	(12,16)	(17,19)
23:	(24,10)	(25,28)	(26,21)	(27,2)	(29,19)	(30,14)	(0,12)	(1,8)	(3,11)	(4,15)	(5,6)	(7,16)	(9,22)	(13,17)	(18,20)
24:	(25,11)	(26,29)	(27,22)	(28,3)	(30,20)	(0,15)	(1,13)	(2,9)	(4,12)	(5,16)	(6,7)	(8,17	(10,23)	(14,18)	(19,21)
25:	(26,12)	(27,30)	(28,23)	(29,4)	(0,21)	(1,16)	(2,14)	(3,10)	(5,13)	(6,17)	(7,8)	(9,18)	(11,24)	(15,19)	(20,22)
26:	(27,13)	(28,0)	(29,24)	(30,5)	(1,22)	(2,17)	(3,15)	(4,11)	(6,14)	(7,18)	(8,9)	(10,19)	(12,25)	(16,20)	(21,23)
27:	(28,14)	(29,1)	(30,25)	(0,6)	(2,23)	(3,18)	(4,16)	(5,12)	(7,15)	(8,19)	(9,10)	(11,20)	(13,26)	(17,21)	(22,24)
28:	(29,15)	(30,2)	(0,26)	(1,7)	(3,24)	(4,19)	(5,17)	(6,13)	(8,16)	(9,20)	(10,11)	(12,21)	(14,27)	(18,22)	(23,25)
29:	(30,16)	(0,3)	(1,27)	(2,8)	(4,25)	(5,20)	(6,18)	(7,14)	(9,17)	(10,21)	(11,12)	(13,22)	(15,28)	(19,23)	(24,26)
30:	(0,17)	(1,4)	(2,28)	(3,9)	(5,26)	(6,21)	(7,19)	(8,15)	(10,18)	(11,22)	(12,13)	(14,23)	(16,29)	(20,24)	(25,27)

To show the dependence of the location of negative spikes of the function ϕ_{baa} on the characteristic delay polynomial governing the m-sequence, another example is considered below for illustration -

<u>Example 5.2</u> :

Here the expression for ϕ_{baa} stated in eqn.(5.43) is applied to the binary m-sequence $\{a_i\}$ with the characteristic delay polynomial

$$F(D) = I \oplus D \oplus D^2 \oplus D^4 \oplus D^5 \quad \text{...(mod-2 addition)}$$

of order

$n = 5.$

The m-sequence $\{a_i\}$ with period

$N = 2^5 - 11 = 31$

with levels +1 and -1 normalized values reads as :

$\{a_i\}$: 11111-111-1-1111-1-1-1-111-11-11-1-11-1-1-11-1,..repeats

Hence, the transformed sequence $\{b_i\}$ from eqn.(5.35) is :

$\{b_i\}$: 1111101100111000011010100100000, ... repeats

From eqn.(5.43) the values of $\phi_{baa}(r_1, s_1, s_2)$ are :

$$\phi_{baa}(r_1, s_1, s_2) = \frac{1}{2}\left(1 + \frac{1}{31}\right) \quad \text{For } 0 \leq s_1 = s_2 \leq 30$$

$$= 0 \quad \text{For } 0 \leq r_1 = s_1 \text{ or } s_2 \leq 30 \text{ and For } \phi_{aaa} = 1/31$$

$$= -\frac{1}{2}\left(1 + \frac{1}{31}\right), \text{ For } \phi_{aaa} = -1$$

Values of (s_1, s_2) for given r_1 that make ϕ_{baa} assume the negative spike are determined as per procedure on Page 454.

Following the steps 1, 2, and 3 stated on Page 454, the values of (s_1, s_2) for fixed r_1 which make the function ϕ_{baa} assume the negative maximum are obtained and given in Table (5.4). The step 1 in this connection is shown in Table 5.3. As in the previous example, in this Table 5.3 the state -1 in the m-sequence $\{a_i\}$ is represented by '2'.

Table 5.3 : Step 1 in finding values of (s_1, s_2) of example 5.2

m-sequence		The N phase-shifted versions of the m-sequence	
a_i	:	1111121122111222211212122122212, ...	repeats
a_{i-1}	:	2111112112211122221121212212221, ...	"
a_{i-2}	:	1211111211221112222112121221222, ...	"
a_{i-3}	:	2121111121122111222211212122122, ...	"
a_{i-4}	:	2212111112112211122221121212212, ...	"
a_{i-5}	:	2221211111211221112222112121221, ...	"
a_{i-6}	:	1222121111121122111222211212122, ...	"
a_{i-7}	:	2122212111112112211122221121212, ...	"
a_{i-8}	:	2212221211111211221112222112121, ...	"
a_{i-9}	:	1221222121111121122111222211212, ...	"
a_{i-10}	:	2122122212111112112211122221121, ...	"
a_{i-11}	:	1212212221211111211221112222112, ...	"
a_{i-12}	:	2121221222121111121122111222211, ...	"
a_{i-13}	:	1212122122212111112112211122221, ...	"
a_{i-14}	:	1121212212221211111211221112222, ...	"
a_{i-15}	:	2112121221222121111121122111222, ...	"
a_{i-16}	:	2211212122122212111112112211122, ...	"
a_{i-17}	:	2221121212212221211111211221112, ...	"
a_{i-18}	:	2222112121221222121111121122111, ...	"
a_{i-19}	:	1222211212122122212111112112211, ...	"
a_{i-20}	:	1122221121212212221211111211221, ...	"
a_{i-21}	:	1112222112121221222121111121122, ...	"
a_{i-22}	:	2111222211212122122212111112112, ...	"
a_{i-23}	:	2211122221121212212221211111211, ...	"
a_{i-24}	:	1221112222112121221222121111121, ...	"
a_{i-25}	:	1122111222211212122122212111112, ...	"
a_{i-26}	:	2112211122221121212212221211111, ...	"
a_{i-27}	:	1211221112222112121221222121111, ...	"
a_{i-28}	:	1121122111222211212122122212111, ...	"
a_{i-29}	:	1112112211122221121212212221211, ...	"
a_{i-30}	:	1111211221112222112121221222121, ...	"

Table 5.4: Values of (s_1, s_2) for fixed value of r_1 for which the function $\phi_{baa}(r_1, s_1, s_2) = -\frac{1}{2}(1 + \frac{1}{N})$

r_1	(s_1, s_2)														
0:	(1,13)	(2,26)	(3,23)	(4,21)	(5,7)	(6,15)	(8,11)	(9,25)	(10,14)	(12,30)	(16,22)	(17,27)	(18,19)	(20,28)	(24,29)
1:	(2,14)	(3,27)	(4,24)	(5,22)	(6,8)	(7,16)	(9,12)	(10,26)	(11,15)	(13,0)	(17,23)	(18,28)	(19,20)	(21,29)	(25,30)
2:	(3,15)	(4,28)	(5,25)	(6,23)	(7,9)	(8,17)	(10,13)	(11,27)	(12,16)	(14,1)	(18,24)	(19,29)	(20,21)	(22,30)	(26,0)
3:	(4,16)	(5,29)	(6,26)	(7,24)	(8,10)	(9,18)	(11,14)	(12,28)	(13,17)	(15,2)	(19,25)	(20,30)	(21,22)	(23,0)	(27,1)
4:	(5,17)	(6,30)	(7,27)	(8,25)	(9,11)	(10,19)	(12,15)	(13,29)	(14,18)	(16,3)	(20,26)	(21,0)	(22,23)	(24,1)	(28,2)
5:	(6,18)	(7,0)	(8,28)	(9,26)	(10,12)	(11,20)	(13,16)	(14,30)	(15,19)	(17,4)	(21,27)	(22,1)	(23,24)	(25,2)	(29,3)
6:	(7,19)	(8,1)	(9,29)	(10,27)	(11,13)	(12,21)	(14,17)	(15,0)	(16,20)	(18,5)	(22,28)	(23,2)	(24,25)	(26,3)	(30,4)
7:	(8,20)	(9,2)	(10,30)	(11,28)	(12,14)	(13,22)	(15,18)	(16,1)	(17,21)	(19,6)	(23,29)	(24,3)	(25,26)	(27,4)	(0,5)
8:	(9,21)	(10,3)	(11,0)	(12,29)	(13,15)	(14,23)	(16,19)	(17,2)	(18,22)	(20,7)	(24,30)	(25,4)	(26,27)	(28,5)	(1,6)
9:	(10,22)	(11,4)	(12,1)	(13,30)	(14,16)	(15,24)	(17,20)	(18,3)	(19,23)	(21,8)	(25,0)	(26,5)	(27,28)	(29,6)	(2,7)
10:	(11,23)	(12,5)	(13,2)	(14,0)	(15,17)	(16,25)	(18,21)	(19,4)	(20,24)	(22,9)	(26,1)	(27,6)	(28,29)	(30,7)	(3,8)
11:	(12,24)	(13,6)	(14,3)	(15,1)	(16,18)	(17,26)	(19,22)	(20,5)	(21,25)	(23,10)	(27,2)	(28,7)	(29,30)	(0,8)	(4,9)
12:	(13,25)	(14,7)	(15,4)	(16,2)	(17,19)	(18,27)	(20,23)	(21,6)	(22,26)	(24,11)	(28,3)	(29,8)	(30,0)	(1,9)	(5,10)
13:	(14,26)	(15,8)	(16,5)	(17,3)	(18,20)	(19,28)	(21,24)	(22,7)	(23,27)	(25,12)	(29,4)	(30,9)	(0,1)	(2,10)	(6,11)
14:	(15,27)	(16,9)	(17,6)	(18,4)	(19,21)	(20,29)	(22,25)	(23,8)	(24,28)	(26,13)	(30,5)	(0,10)	(1,2)	(3,11)	(7,12)
15:	(16,28)	(17,10	(18,7)	(19,5)	(20,22)	(21,30)	(23,26)	(24,9)	(25,29)	(27,14)	(0,6)	(1,11)	(2,3)	(4,12)	(8,13)
16:	(17,29)	(18,11)	(19,8)	(20,6)	(21,23)	(22,0)	(24,27)	(25,10)	(26,30)	(28,15)	(1,7)	(2,12)	(3,4)	(5,13)	(9,14)
17:	(18,30)	(19,12)	(20,9)	(21,7)	(22,24)	(23,1)	(25,28)	(26,11)	(27,0)	(29,16)	(2,8)	(3,13)	(4,5)	(6,14)	(10,15)
18:	(19,0)	(20,13)	(21,10)	(22,8)	(23,25)	(24,2)	(26,29)	(27,12)	(28,1)	(30,17)	(3,9)	(4,14)	(5,6)	(7,15)	(11,16)
19:	(20,1)	(21,14)	(22,11)	(23,9)	(24,26)	(25,3)	(27,30)	(28,13)	(29,2)	(0,18)	(4,10)	(5,15)	(6,7)	(8,16)	(12,17)
20:	(21,2)	(22,15)	(23,12)	(24,10)	(25,27)	(26,4)	(28,0)	(29,14)	(30,3)	(1,19)	(5,11)	(6,16)	(7,8)	(9,17)	(13,18)
21:	(22,3)	(23,16)	(24,13)	(25,11)	(26,28)	(27,5)	(29,1)	(30,15)	(0,4)	(2,20)	(6,12)	(7,17)	(8,9)	(10,18)	(14,19)
22:	(23,4)	(24,17)	(25,14)	(26,12)	(27,29)	(28,6)	(30,2)	(0,16)	(1,5)	(3,21)	(7,13)	(8,18)	(9,10)	(11,19)	(15,20)
23:	(24,5)	(25,18)	(26,15)	(27,13)	(28,30)	(29,7)	(0,3)	(1,17)	(2,6)	(4,22)	(8,14)	(9,19)	(10,11)	(12,20)	(16,21)
24:	(25,6)	(26,19)	(27,16)	(28,14)	(29,0)	(30,8)	(1,4)	(2,18)	(3,7)	(5,23)	(9,15)	(10,20)	(11,12)	(13,21)	(17,22)
25:	(26,7)	(27,20)	(28,17)	(29,15)	(30,1)	(0,9)	(2,5)	(3,19)	(4,8)	(6,24)	(10,16)	(11,21)	(12,13)	(14,22)	(18,23)
26:	(27,8)	(28,21)	(29,18)	(30,16)	(0,2)	(1,10)	(3,6)	(4,20)	(5,9)	(7,25)	(11,17)	(12,22)	(13,14)	(15,23)	(19,24)
27:	(28,9)	(29,22)	(30,19)	(0,17)	(1,3)	(2,11)	(4,7)	(5,21)	(6,10)	(8,26)	(12,18)	(13,23)	(14,15)	(16,24)	(20,25)
28:	(29,10)	(30,23)	(0,20)	(1,18)	(2,4)	(2,12)	(5,8)	(6,22)	(7,11)	(9,27)	(13,19)	(14,24)	(15,16)	(17,25)	(21,26)
29:	(30,11)	(0,24)	(1,21)	(2,19)	(3,5)	(4,13)	(6,9)	(7,23)	(8,12)	(10,28)	(14,20)	(15,25)	(16,17)	(18,26)	(22,27)
30:	(0,12)	(1,25)	(2,22)	(3,20)	(4,6)	(5,14)	(7,10)	(8,24)	(9,13)	(11,29)	(15,21)	(16,26)	(17,18)	(19,27)	(23,28)

From the two examples (5.1) and (5.2) considered for illustration of the correlation function $\phi_{baa}(r_1, s_1, s_2)$, the following may be noted :

(i) ϕ_{baa} exhibits its negative spikes only for certain values of (s_1, s_2) for given r_1 in the $(r_1 \neq s_1 \neq s_2)$ - space.

(ii) The location of these negative spikes depends on the characteristic polynomial governing the m-sequence $\{a_i\}$.

(iii) For any m-sequence $\{a_i\}$, there is a definite region of $(r_1 \neq s_1 \neq s_2)$ - space over which the function ϕ_{baa} is identically equal to zero.

For instance, for the m-sequence of Example (5.2), for $0 \leq r_1 \leq 12$, and $0 \leq s_1, s_2 < 5$, ϕ_{baa} is always zero, as may be seen from Table 5.4.

Similarly, for the m-sequence of Example 5.1, for $0 \leq r_1 \leq 12$ and $0 \leq s_1, s_2 < 3$, ϕ_{baa} is always zero as may be seen from Table 5.2.

Consequently, proper choice of the characteristic polynomial is necessary to make the function ϕ_{baa} assume zero values over a wider range of (s_1, s_2) for fixed value of r_1 in the $(r_1 \neq s_1 \neq s_2)$ - space. For an m-sequence of fixed length, there are quite a few number of characteristic polynomial for which, the range of s_1, s_2 over which $\phi_{baa} = 0$ is about 1/3 the range of r_1. For m-sequence of length N = 31, some such polynomials are given below -

Polynomial	Range over which ϕ_{baa} is zero
$I + D + D^2 + D^4 + D^5$	$0 \leq r_1 \leq 12$, $0 \leq s_1, s_2 < 5$
$I + D + D^2 + D^3 + D^5$	.. same ..
$I + D + D^3 + D^4 + D^5$	.. same ..
$I + D^2 + D^3 + D^4 + D^5$	.. same ..

(v) Referring to eqn.(5.43), rewritten below for convenience,

$$\phi_{baa}(r_1, s_1, s_2) = \frac{1}{2}[\phi_{aaa}(r_1, s_1, s_2) + \phi_{aa}(s_1, s_2)]$$

it is clear that the region of $r_1 \neq s_1 \neq s_2$ over which ϕ_{baa} is identically equal to zero is also the region over which the function ϕ_{aaa} (assumes the value '1/N' .

Finally, in the measurement of nonlinear system Volterra kernels, another crosscorrelation function that is of importance is the crosscorrelation between the sequences $\{a_i\}$ and $\{b_i\}$ of order 'jk' with j = 2 and k = 2, denoted by - $\phi_{bbaa}(r_1, r_2, s_1, s_2)$, an expression for which is derived below and some aspects relevant to the system testing are stated.

(G) The crosscorrelation function $\phi_{bbaa}(r_1, r_2, s_1, s_2)$

By definition,

$$\begin{aligned}\phi_{bbaa}(r_1, r_2, s_1, s_2) &= \frac{1}{N}\sum_{i=0}^{N-1}(b_{i-r_1} \cdot b_{i-r_2} \cdot a_{i-s_1} \cdot a_{i-s_2}) \\ &= \frac{1}{4N}\sum_{i=0}^{N-1}(a_{i-r_1} \cdot a_{i-r_2} + a_{i-r_1} + a_{i-r_2} + e_i) \\ &\qquad \cdot (a_{i-s_1} \cdot a_{i-s_2}) \\ &= \frac{1}{4}[\phi_{aaaa}(r_1, r_2, s_1, s_2) + \phi_{aaa}(r_1, s_1, s_2) \\ &\qquad + \phi_{aaa}(r_2, s_1, s_2) + \phi_{aa}(s_1, s_2)]\end{aligned}$$

Of present interest are the values of ϕ_{bbaa} as a function of r_1 and r_2 for fixed values of $(s_1, s_2 \neq s_1)$.

...(5.49)

In such a case, the following possibilities are to be considered -

(i) $s_1 \neq s_2$, $r_1 = r_2$

(ii) $s_1 \neq s_2$, $s_1 = r_1$, $s_2 \neq r_2$

(iii) $s_1 \neq s_2$, $s_1 = r_1$, and $s_2 = r_2$

(iv) $s_1 \neq s_2 \neq r_1 \neq r_2$

The values of the correlation function $\phi_{bbaa}(r_1, r_2, s_1, s_2)$ in the above mentioned situations are given below -

(i) For $s_1 \neq s_2$, $r_1 = r_2$

$$\phi_{bbaa}(r_1, r_2, s_1, s_2) = \frac{1}{4}\left[\left(-\frac{1}{N} \right) + \left(\frac{1}{N} \text{ or } -1 \right) + \left(\frac{1}{N} \text{ or } -1 \right) + \left(-\frac{1}{N} \right) \right]$$

$$= 0 , \quad \text{for } \left.\begin{matrix} \phi_{aaa}(r_1, s_1, s_2) \\ \text{and} \\ \phi_{aaa}(r_2, s_1, s_2) \end{matrix}\right\} = \frac{1}{N}$$

$$= -\left(\frac{N+1}{2N}\right) , \quad \text{for } \phi_{aaa} = -1 \qquad \ldots(5.50)$$

(ii) For $s_1 \neq s_2$, $s_1 = r_1$, $s_2 \neq r_2$

$$\phi_{bbaa}(r_1, r_2, s_1, s_2) = \frac{1}{4}\left[\left(-\frac{1}{N} \right) + \left(\frac{1}{N} \right) + \left(-1 \text{ or } \frac{1}{N} \right) + \left(-\frac{1}{N} \right) \right]$$

$$= \frac{1}{4}\left(-\frac{1}{N} - 1 \right), \text{for } \phi_{aaa}(r_2, s_1, s_2) = -1$$

$$= 0, \qquad \text{for } \phi_{aaa}(r_2, s_1, s_2) = \frac{1}{N} \qquad \ldots(5.51)$$

(iii) For $s_1 \neq s_2$, $s_1 = r_1$, and $s_2 \neq r_2$

$$\phi_{bbaa}(r_1, r_2, s_1, s_2) = \frac{1}{4}\left(1 + \frac{1}{N} + \frac{1}{N} - \frac{1}{N}\right)$$

$$= \frac{1}{4}\left(1 + \frac{1}{N}\right) \quad \ldots \quad \ldots \quad (5.52)$$

(iv) For $s_1 \neq s_2 \neq r_1 \neq r_2$

$$\phi_{bbaa}(r_1, r_2, s_1, s_2) = \frac{1}{4}\Big[\left(1 \text{ or } -\frac{1}{N}\right) + \left(-1 \text{ or } \frac{1}{N}\right)$$

$$+\left(-1 \text{ or } \frac{1}{N}\right) + \left(-\frac{1}{N}\right)\Big] \ldots \quad (5.53)$$

The possible values are : $0, \pm \frac{N+1}{4N}, -\frac{N+1}{2N}$

Now, when the region of (s_1, s_2) is considered as -

$0 \leq s_1, s_2 \neq s_1 \leq N/6$,

from eqn.(5.42) and from the property of ϕ_{aaa} stated on page 463, the following is written for 2nd and 3rd terms of eqn.(5.49) :

$$\left.\begin{array}{c} \phi_{aaa}(r_1, s_1, s_2) \\ \text{and} \\ \phi_{aaa}(r_2, s_1, s_2) \end{array}\right\} = \frac{1}{N} \quad \ldots \quad \ldots \quad (5.54a)$$

Further, in the said region for (s_1, s_2), the following equation is valid for the first term of eqn.(5.49) :

$$\phi_{aaaa}(r_1, r_2, s_1, s_2) = -\frac{1}{N} \quad \ldots \quad \ldots \quad (5.54b)$$

Summarizing the essential content on the values of ϕ_{bbaa} the following equations may now be written :

Over the region : $0 \leq s_1, s_2 \neq s_1 \leq N/6$,

$$\phi_{bbaa}(r_1, r_2, s_1, s_2) = \frac{1}{4}\left(1 + \frac{1}{N}\right), \text{for } s_1 = r_1 \text{ and } s_2 = r_2$$

$$= 0 \text{ otherwise} \quad \ldots \quad \ldots \quad (5.55)$$

An example is illustrated on the following page.

Some comments on eqns.(5.54a and b) are in order below :

Referring to eqns. (5.54b), the function $\phi_{aaaa}(r_1, r_2, s_1, s_2)$ can have only two values, namely +1 and - 1/N. And, for fixed values of (s_1, s_2), except when $s_1, s_2 = r_1, r_2$ (order is unimportant), the value of the function can be shown to be equal to - 1/N over the region $0 \leq r_1, r_2, s_1, s_2 \leq N/6$(approx), for many m-sequences of period N over mod-2, excluding the m-sequences that are generated by simple trinomial type characteristic delay polynomials.

For instance, consider the trinomial given by -

$$I \oplus D^2 \oplus D^5$$

generating an m-sequence $\{a_i\}$ of period 31 digits. Accordingly, the characteristic equation is -

$$(I \oplus D^2 \oplus D^5) \{a_i\} = 0$$

i.e. $\{a_{i-2}\} \oplus \{a_{i-5}\} = \{a_i\}$

squaring the above eqn. over mod-2, gives

$$\{a_{i-4}\} \oplus \{a_{i-10}\} = \{a_i\} \quad \text{(as per mod-2 addition property)}$$

Multiplying the original - and the squared - equation and averaging over N leads to -

$$\phi_{aaaa}(2, 5, 4, 10) = 1$$

Thus, eqn.(5.54b) becomes invalid when the m-sequence $\{a_i\}$ under test is associated with a trinomial over mod-2.

It is, therefore, assumed that the m-sequences under consideration are generated by characteristic delay polynomials which are not simple trinomials. Under the circumstances, eqn.(5.54b) is valid for many polynomials for given period of the m-sequence $\{a_i\}$.

Further, eqn. (5.54a), namely -

$$\left.\begin{array}{c} \phi_{aaa}(r_1, s_1, s_2) \\ \text{and} \\ \phi_{aaa}(r_2, s_1, s_2) \end{array}\right\} = 1/N \qquad \begin{array}{l} \text{for } s_1 \neq s_2 \text{ and} \\ 0 \leq r_1, r_2, s_1, s_2 \leq N/6 \end{array}$$

implies that

$$\{a_{i-s_1}\} \cdot \{a_{i-s_2}\} \neq \{a_{i-r_1}\} \text{ or } \{a_{i-r_2}\}$$

over the said period 0 to N/6, i.e.

$$\{a_{i-s_1}\} \oplus \{a_{i-s_2}\} \neq \{a_{i-r_1}\} \qquad \text{(mod-2 addition)}$$

Let $\{a_{i-s_1}\} \oplus \{a_{i-s_2}\} = \{a_{i-s_i}\}$

Accordingly, the required condition is

$$s_i \neq r_1$$

In accordance with the above steps, an example is illustrated on the following page.

It may be noted here that for a fixed period of the m-sequence, $N = 2^n - 1$, for each value of n(integer) there are quite a few characteristic delay polynomials whose m-sequences satisfy the conditions stated in eqn.(5.55).

Example 5.3 :

Here, the expression for $\phi_{bbaa}(r_1, r_2, s_1, s_2)$ stated in eqn.(5.55) is applied to the binary m-sequence $\{a_i\}$ governed by the characteristic polynomial -

$$F(D) = I \oplus D \oplus D^2 \oplus D^4 \oplus D^5 \quad \text{(mod-2 addition)}$$

The m-sequence $\{a_i\}$ with period N = 31 digits is given by-

$\{a_i\}$: 11111-111-1-1111-1-1-1-111-11-11-1-11-1-1-11-1,..repeats

Hence, the transformed sequence $\{b_i\}$ from eqn.(5.35) is :

$\{b_i\}$: 1111101100111000011010100100010, ... repeats

Here, the region of r_1, r_2, s_1, s_2 is given by -

$0 \leq r_1, r_2, s_1, s_2 \leq N/6 = 31/6 = 5$ (approximately)

From eqn.(5.55a), for $(s_1, s_2) = (r_1, r_2)$

$$\phi_{bbaa}(r_1, r_2, s_1, s_2) = \frac{1}{4}\left(1 + \frac{1}{31}\right)$$

Further, the functions $\phi_{aaa}(r_1, s_1, s_2)$ and $\phi_{aaa}(r_2, s_1, s_2)$ are seen to assume the value 1/N over the interval $0 \leq r_1, r_2, s_1, s_2 \leq N/6$, in view of the content of Table 5.5. Hence, the value of the function $\phi_{bbaa}(r_1, r_2, s_1, s_2)$ as stated in eqn.(5.55b) assumes zero value for $(s_1, s_2) \neq (r_1, r_2)$.

(Table 5.5) is given on the following page)

Table 5.5 : Values of ϕ_{bbaa} in the region $0 \leq r_1, r_2, s_1, s_2 \leq 5$ for $s_i \neq s_2$

(s_1, s_2)	S_i	Remarks	$\phi_{aaa}(r_1 \text{ or } r_2, s_1, s_2)$	$\phi_{bbaa}(r_1, r_2, s_1, s_2)$
0,1	19	$S_i \neq r_1$ or r_2	1/31	0
0,2	7	"	"	"
0,3	11	"	"	"
0,4	14	"	"	"
0,5	29	"	"	"
1,2	20	"	"	"
1,3	8	"	"	"
1,4	12	"	"	"
1,5	15	"	"	"
2,3	21	"	"	"
2,4	9	"	"	"
2,5	13	"	"	"
3,4	22	"	"	"
3,5	10	"	"	"
4,5	23	"	"	"

5.4.1.3 Use of a pseudurandom binary signal and its transformed version for nonlinear system identification

The correlation schemes employing the pseudorandom binary signal and its transformed version of period ξ T for the estimation of first and second order kernel of a nonlinear system are shown in Figs. (5.8a) and (5.8b) respectively. The PRBS x(t) and its transformed version $x_1(t)$ are related by -

$$x_1(t) = \frac{1}{2}\,[x(t) + 1]$$

(A) Measurement procedure for linear kernel

Consider the scheme of Fig. (5.8a). The system is perturbed with the PRBS x(t) and the corresponding response signal y(t) is crosscorrelated with delayed versions of transformed PRBS $x_1(t)$, i.e. $x_1(t-\tau)$ where τ is a fixed time shift expressed as an integral multiple of the signal basic interval (t_o).

The general expression for the correlator output in such a case is stated previously in eqn. (5.31), whereby $x_p(t)$ denotes the system perturbation and $x_c(t)$ is the correlator signal. In the present case,

$x_p(t) = x(t)$ (PRBS).

$x_c(t) = x_1(t)$ (Transformed PRBS)

Effecting the substitution, correlator output of eqn. (5.31) is given by -

$$\phi_{x_1y}(\tau) = \int_0^{T_{s_1}} h_1(u)\,\phi_{x_1x}(\tau,u)\,du + \int_0^{T_{s_2}}\int_0^{T_{s_2}} h_2(u_1,u_2)\,\phi_{x_1xx}(\tau,u_1,u_2)\,du_1du_2 \quad \ldots (5.56)$$

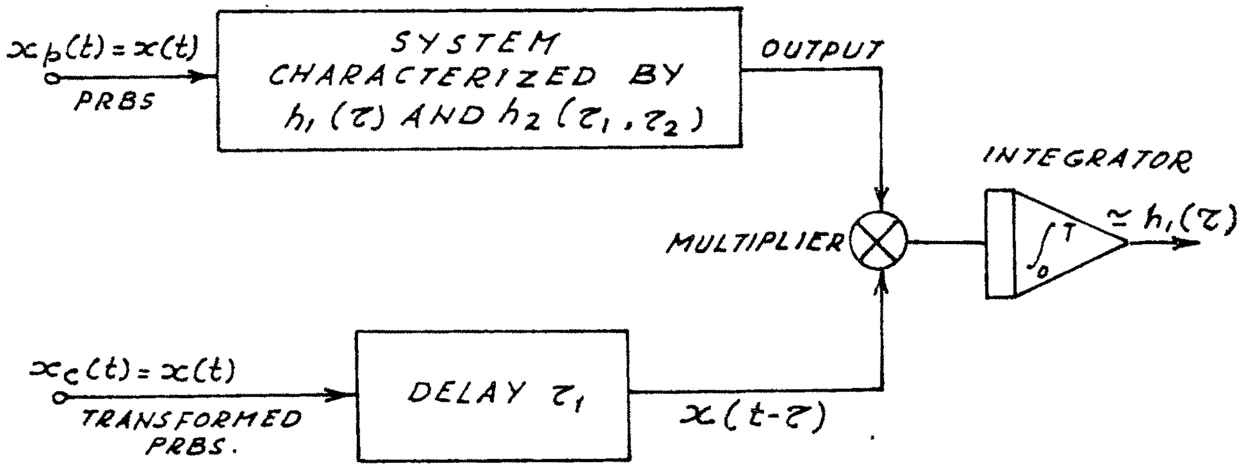

FIG. 5·8 (a) CORRELATION PATTERN FOR THE MEASUREMENT OF THE FIRST-ORDER KERNEL $h_1(\tau)$

$x_p(t)$: PERTURATION SIGNAL

$x_c(t)$: SIGNAL USED FOR CROSS-CORRELATION WITH THE RESPONSE-SIGNAL $y(t)$ (i.e. CORRELATOR-SIGNAL)

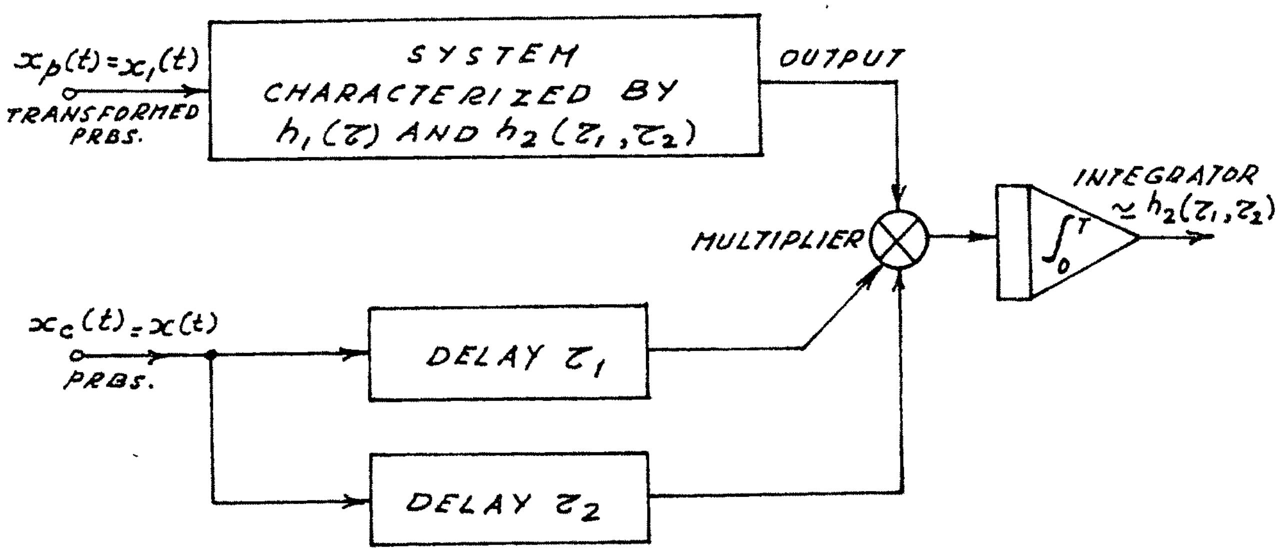

FIG. 5·8 (b) CORRELATION PATTERN FOR THE MEASUREMENT OF THE SECOND-ORDER KERNEL $h_2(\tau_1, \tau_2)$.

whereby T_{s_1} and T_{s_2} are the settling times of the first - and second order Volterra kernels. Some points to be noted in connection with these settling times are stated below -

In practice, the second kernel settles down much faster than the first order kernel. In fact, Gyftopoulos (1967) points out that the second order kernel settling time (T_{s_2}) may, in certain cases, be one-fiftyth of the linear kernel settling time (T_{s_1}). In view of this fast dying out nature of the second-order kernel, presuming the worst case, the second kernel settling time can be safely assumed to be less than $\frac{1}{3}$ or $\frac{1}{4}$ of the first kernel settling time, that is

$$T_{s_2} = = \frac{1}{3} T_{s_1} \text{(approx)} .. \quad ... \quad (5.57)$$

Now, in a crosscorrelation experiment, the period T of the perturbation signal should be greater than the maximum settling time of the system under test. To account for any possible error, and other disturbances, the period T is normally taken as 2 to 3 times the maximum settling time. Thus,

$$T = 2 \text{ to } 3 \text{ times } T_{s_1} \quad ... \quad ... \quad (5.58)$$

Utilizing eqns. (5.57) and (5.58), the expression for correlator output of eqn. (5.56) reads as - (with $T = 2.5T_{s_1}$)

$$\phi_{x_1y}(\tau) = \int_o^{0.4T} h_1(u)\, \phi_{x_1x}(\tau, u)\, du + \int_o^{0.15T} \int_o^{0.15T} h_2(u_1, u_2)\, \phi_{x_1xx}(\tau, u_1, u_2)\, du_1 du_2 \quad ... \quad (5.59)$$

The properties of the continuous crosscorrelation functions $\phi_{x_1x}(\tau,u)$ and $\phi_{x_1xx}(\tau,u_1,u_2)$ are already derived in the previous sub-section by considering their discrete versions ϕ_{ba} and ϕ_{baa} respectively.

Now, the interpretation of the one - and two - dimensional convolution integrals in eqn.(5.59) is as follows -

From the properties of ϕ_{ba} stated on page 451 (Eqn.5.42), during the integration time 0 to T_{s_1} (= 0.4T), the crosscorrelation function $\phi_{x_1x}(\tau,u)$ consists of a triangular spike of height $\frac{1}{2}(1 + \frac{1}{N})$ and width $2t_o$, thus giving an area of $\frac{1}{2}(1 + \frac{1}{N})t_o$ with zero bias.

From the properties of ϕ_{baa} stated in the previous sub-section (pages 457 and 462), over the integration time 0 to T_{s_2} (= 0.15T), the crosscorrelation function $\phi_{x_1xx}(\tau,u_1,u_2)$ is zero everywhere except when $u_1 = u_2$. Hence there is some contribution from the second integral to the correlator output in Eqn.(5.59).

Carrying out the multiplication and integration of eqn.(5.59) gives -

$$\phi_{x_1y}(\tau) = \frac{1}{2}\left(1 + \frac{1}{N}\right)t_o\, h_1(\tau) \quad + \text{(Contribution of second integral for } u_1 = u_2) \qquad \ldots (5.60)$$

Now, if the crosscorrelation is again performed using inverse of the signals x(t) and $x_1(t)$, the above analysis leads to the result (representing the response signal in this case by $y'(t)$)-

$$\phi_{x_1^* y'}(\tau) = \frac{1}{2}\left(1+\frac{1}{N}\right)t_o\, h_1(\tau) \quad \text{-(contribution of second integral for } u_1 = u_2) \qquad \ldots (5.61)$$

$$\text{Hence, } h_1(\tau) = \frac{Nt_o}{(N+1)t_o^2}\left[\phi_{x_1 y} + \phi_{x_1^* y'}\right] \ldots \qquad \ldots (5.62)$$

$$= \frac{T}{(T+t_o)t_o}\left[\phi_{x_1 y}(\tau) + \phi_{x_1^* y'}(\tau)\right]$$

Thus, it is possible to measure the linear approximant of a nonlinear system using the new correlation method and employing binary signals.

(B) Measurement procedure for second order kernel - $h_2(\tau_1,\tau_2)$

Consider the correlation scheme of Fig.(5.8b). The system is perturbed here with the transformed pseudorandom binary signal $x_1(t)$ and the corresponding system response, say, $y_1(t)$ is crosscorrelated with two delayed versions of the pseudorandom binary signal x(t). The reasons for perturbing the system with $x_1(t)$ in the second kernel measurement will be clarified later.

The general expression for the correlator output in such a case is stated previously in eqn.(5.32), whereby $x_p(t)$ denotes the system perturbation and $x_c(t)$ is the correlator signal. In the present case,

$$
\begin{aligned}
x_p(t) &= x_1(t) \quad \text{(Transformed PRBS)} \\
x_c(t) &= x(t) \quad \text{(PRBS)}
\end{aligned}
\qquad \ldots \qquad (5.63)
$$

Effecting the above substitution, the correlator output in Eqn.(5.32) is given by -

$$
\begin{aligned}
\phi_{xxy_1}(\tau_1,\tau_2) = &\int_0^{T_{s_1}} h_1(u)\ \phi_{xxx_1}(\tau_1,\tau_2,u)\,du \\
&+ \int_0^{T_{s_2}}\int_0^{T_{s_2}} h_2(u_1,u_2)\ \phi_{xxx_1x_1}(\tau_1,\tau_2,u_1,u_2)\,du_1du_2
\end{aligned}
$$

$$\ldots \quad (5.64)$$

Whereby T_{s_1} and T_{s_2} are the first and second kernel settling times.

Brief discussion on these settling times in a physical system is already given before. Utilizing Eqns.(5.57) and (5.58), the correlator output of Eqn. (5.64) becomes -

$$
\begin{aligned}
\phi_{xxy_1}(\tau_1,\tau_2) = &\int_0^{0.4T} h_1(u)\ \phi_{xxx_1}(\tau_1,\tau_2,u)\,du \\
&+ \int_0^{0.15T}\int_0^{0.15T} h_2(u_1,u_2)\ \phi_{xxx_1x_1}(\tau_1,\tau_2,u_1,u_2)\,du_1du_2
\end{aligned}
$$

$$\ldots \quad (5.65)$$

The properties of the continuous crosscorrelation functions $\phi_{xxx_1}(\tau_1,\tau_2,u)$ and $\phi_{xxx_1x_1}(\tau_1,\tau_2,u_1,u_2)$ are already derived in the previous sub-section by considering their discrete versions $\phi_{aab}(=\phi_{baa})$ and $\phi_{aabb}(=\phi_{bbaa})$.

Now, the interpretation of the convolution integrals on the right hand side of eqn. (5.65) is as follows :

From the properties of ϕ_{aab} stated in the previous sub-section (Pages 457 and 462), over the measurement interval 0 to $T_{s_1} = 0.4T$, for all values of $\tau_1 \neq \tau_2$, the crosscorrelation function $\phi_{xxx_1}(\tau_1,\tau_2,u)$ is identically equal to zero. Hence, the first integral on the right hand side of eqn. (5.65) contributes nothing to the correlator output.

From the properties of ϕ_{aabb} stated in the previous sub-section on page 465, over the measurement interval 0 to $T_{s_2} = 0.15T$, the crosscorrelation function $\phi_{xxx_1x_1}(\tau_1,\tau_2,u_1,u_2)$ assumes its maximum value at ($\tau_1 = u_1$ and $\tau_2 = u_2$) and has the property given by :

$$\phi_{xxx_1x_1}(\tau_1 = u_1, \tau_2 = u_2) = \frac{1}{4}\left(1 + \frac{1}{N}\right), \text{ for } \tau_1 \neq \tau_2$$
$$= 0 \text{ otherwise}$$

Hence, eqn. (5.65) under the circumstances settles down to :

$$\phi_{xxy_1}(\tau_1,\tau_2) = \frac{1}{2}\left(1 + \frac{1}{N}\right)t_o \cdot h_2(\tau_1,\tau_2) \qquad (\tau_1 \neq \tau_2)$$

From which ,
$$h_2(\tau_1,\tau_2) = \frac{2T}{(T + t_o)\, t_o}\, \phi_{xxy_1}(\tau_1,\tau_2)$$

Thus, it is possible by this new correlation pattern to effectively identify the second order Volterra kernel in a single crosscorrelation using considerably shorter averaging period.

Some advantages of the proposed method

1. The test-signals used in the new correlation scheme are the well-known PRBS and its transformed version, which facilitate ease of generation, and simple realization of delay and multiply circuits.

2. The crosscorrelation functions between the PRBS and the transformed PRBS are well-defined and no anamolous regions exist over the measurement interval. Hence, the proposed method gives good measurements of the system characteristic kernels.

3. Accurate estimate of the second order kernel is possible by this new approach using the test signal period of only about six times the kernel settling time, thus considerably reducing the averaging period.

4. The crosscorrelation function $\phi_{xx_1}(\tau)$ over one period of x(t) is a triangular spike at $\tau = 0$ and has zero off-peak value. So also, the correlation function $\phi_{xxx_1x_1}(\tau_1,\tau_2)$ has zero off-peak value over the measurement interval 0 to T_{s_2}. Hence bias estimation is not necessary.

5. The method uses an experimental configuration and procedure of high simplicity.

5.5 SUMMARY

In this chapter, the problem of nonlinear system identification by means of the crosscorrelation method using pseudorandom test signals has been considered in some detail.

The Volterra series expansion of the response of a nonlinear system is described, and a systematic exposition is made of the crosscorrelation art for identifying the kernel functions which occur in this expansion. It is pointed out that the antisymmetric pseudorandom signals are well-suited in the measurement of the linear approximant of a nonlinear system, because of their ability to discriminate against system nonlinearities. It is also noted that the higher order autocorrelation functions of these signals impose a limitation in using them to effectively identify the kernels of order greater than 2. Even to obtain proper estimates of the second kernel, test signal of period about 100 times the kernel settling time is necessitated.

A new crosscorrelation pattern of system dynamic testing is, therefore, proposed here and the possibility of employing a PRBS and a transformed PRBS as test signals in this new scheme is examined. The correlation pattern is new in the sense that two test-signals are employed, one for perturbing the system while the other for crosscorrelating with the system response to the perturbation. Hence, the quality of measured kernels is made to depend on the properties of the crosscorrelation functions between the perturbation signal and the signal used for crosscorrelation with the system output instead of on the autocorrelation properties of the perturbation signal alone. Such a scheme obviously removes the seal on the use of those test signals which are otherwise well-suited for system testing but for their lack in the desired higher order autocorrelation functions. As two signals are employed, any limitation posed by one may be compensated by the other.

By this new correlation scheme employing binary signals, the linear and second kernel functions can be effectively identified within reasonable averaging period. In fact, by this approach, the averaging period for the second kernel estimation is considerably shortened due to well-defined crosscorrelation between the PRBS and its transformed version. These signals, being binary, circumvent some of the practical difficulties and offer appreciable advantages with respect to the instrumentation of the measurements.

The nonlinear system considered is characterized by the first - and second order Volterra kernels. In case, the non-linearity is quite significant, the proposed method in association with the dither-injection principle should prove quite useful for effective system identification.