Chapter 1

Introduction to Random Matrix Theory (RMT)

1.1 About the Random Matrix Theory : A compendium

Random Matrix Theory (RMT) is a vast subject in itself with applications in many disciplines of science, mathematics, economics etc, and particularly in physics it deals with the study of complex, stochastic systems which are not very well understood or whose overall longtime behavior is unpredictable. RMT has achieved immense interest due to its applicability in many branches of physics and mathematics. It connects with Statistical Mechanics. In fact, the most significant feature of the Random matrix theory is its universality. It describes the universal properties of the system which are independent of the probability distribution used to construct random matrices. One thing the RMT depends upon is the type of symmetry imposed on the system, viz. time reversal, rotational, etc. which we shall discuss in detail further. The matrix elements here are drawn from some probability distribution, usually Gaussian. The spectra of above mentioned finite systems could be quite complicated and irregular. Here, one doesn't need the exact knowledge of the energy level positions nor it is

achievable. In the same manner, one does not need the exact knowledge of the Hamiltonian of the system if one is thorough with the statistics of eigenvalues of large $N \times N$ matrices with independent random elements. This idea was suggested by Wigner, Dyson and others who are the co-founders of this Random matrix theory. With these (and many more) features, RMT thus encompasses the study of compound nuclei, quantum dots, chaotic quantum billiards and many such stochastic systems.

The development of Random matrix theory is associated with different statisticians and physicists. RMT was first introduced in the field of mathematical statistics by Wishart in 1928. But then after a couple of decades in 1950s E. P. Wigner emerged with the applications of RMT in the branch of nuclear physics, which was then extended further in 1960s-70s by Dyson, Mehta, Gaudin, Porter, Rosenzweirg, French, Pandey, Bohigas and many other research groups. Wigner studied the neutron excitation spectra of heavy nuclei obtained using neutron resonances. Thereby he mentioned that for many body systems whose interaction is complex enough, the Hamiltonian representing the system should behave like a large random matrix. Moreover, he also theorized that the local statistical behavior of energy levels in a simple sequence is identical with the statistical behavior of eigen values of a random matrix. Simple sequence here means a sequence of levels possessing identical spin, parity and other conserved quantities as per the symmetries imposed on the system. We shall now begin with the preview of these symmetries which are to be imposed on various systems of fermions and bosons.

1.2 Space-Time symmetries and introduction to classical Gaussian Ensembles

Symmetries in physics are important to understand and simplify the solutions of many problems. If quantum systems have some specific space-time symmetry properties then these reflect in their respective Hamiltonians. Using the concepts of group

theory, one can represent the type of symmetry in Hamiltonian matrix by an equivalent matrix (which may be Hermition/real/quaternion as per the symmetry) and is invariant under a particular transformation. They are as follows :
For any quantum system,

- If the Hamiltonian matrix does not have time reversal symmetry, then it can be represented by a Hermition matrix invariant under *unitary transformation*.
- If the Hamiltonian matrix possess a time reversal symmetry as well as a good rotational symmetry with any spin interaction, then it can be represented by a real matrix invariant under *orthogonal transformation*.
- If the Hamiltonian matrix exhibits a time reversal symmetry and has a half-integral spin interaction, then it can be represented by a quaternion real matrix invariant under *symplectic transformation*.

A collection of such Hamiltonians with independent matrix elements as gaussian random numbers is a random matrix ensemble. Following are the ensembles usually used in random matrix theories.

1. **Gaussian Unitary Ensemble (GUE):**
   They are invariant under Unitary transformation $\Rightarrow UU^{\dagger} = I$
   - Symmetry : No Time Reversal symmetry
   - Angular momentum (spin interaction) : Any type (integer / half-integar)
   - Hamiltonian type : Hermitian
2. **Gaussian Orthogonal Ensemble (GOE):**
   They are invariant under Orthogonal transformation $\Rightarrow O\widetilde{O} = I$
   (a) Symmetry : Time Reversal symmetry present, Rotational symmetry absent
      - Angular momentum (spin interaction) : Integer type

- Hamiltonian type : Real symmetric

(b) Symmetry : Both Time Reversal and Rotational symmetry present

- Angular momentum (spin interaction) : Any type
- Hamiltonian type : Real symmetric

3. **Gaussian Symplectic Ensemble (GSE):**
   They are invariant under Symplectic transformation $\Rightarrow SZ\widetilde{S} = Z, SS^{\dagger} = I$

   - Symmetry : Time Reversal symmetry present, Rotational symmetry absent
   - Angular momentum (spin interaction) : Half-integer type
   - Hamiltonian type : Quaternion real

We are now acquainted to the types of random matrix ensembles and which symmetry sector they belong to. It is to be noted that the values of matrix elements here are drawn at random using some probability distribution. The nature of this probability distribution and other properties are discussed below, for any (GUE/GOE/GSE) Gaussian ensemble say of matrix size $N \times N$.
The probability $p(H)d[H]$ has to be invariant under any transformation $H' \rightarrow U^{-1}HU$ where U may be unitary, orthogonal or symplectic. Thus one can say $p(H)d[H] = p(H')d[H']$. Moreover, it is also required that the matrix elements are statistically independent which simplifies the problem mathematically. Here, the probability $p(H)$ should be independent of the set of basis states that are being used. *Mehta* in his book [1] defined the probability distribution which satisfy the above conditions and gives a generalized way to express the three Gaussian random matrix ensembles.

Adding to the popularity of RMT initiated by Wigner and many others in 1950-60s, Bohigas, Giaonni and Schmit [2] conjectured that RMT should be equally applicable to the spectra of all chaotic systems. This encouraged the research interest in RMT and hence it's utility in quantum chaotic domains, quantum billiards, metallic grains, multi-qubit systems etc. After several years of research on quantum systems

and particularly those systems which are classically chaotic, it was understood that all the generic quantum systems whose classical analogue is chaotic follow random matrix statistics. This fact has been theorized well in the books authored by Haake [3] and Stockmann [4]

Gaussian Random matrix ensembles for m-particle system involve interaction up to m-body in character and are dominated by m-body interaction. Whereas for realistic systems such as atoms, nuclei, quantum dots etc. the inter-particle interaction is known to be only one body and two body in character. Talking in terms of ensemble averaged state density for random matrix ensemble, it is semicircle which is radically different from Gaussian form as observed for finite interacting many particle system. This fact motivated French & Wong [5] , Bohigas & Flores [6] and they introduced the Two Body Random matrix Ensemble (TBRE). The two-body interaction here is taken as GOE and then the m particle matrix is constructed using the m-particle basis states using the direct product structure. As the GOE in 2-particle space is embedded in the $m(m > 2)$ particle space, it is called embedded GOE of two-body interactions [EGOE(2)]. One can also define in the same manner the EGOE(k) which considers the k-body interactions. This was done for fermions. For finite interacting bosons, some embedded ensembles (EE) were investigated by like Borgonovi et al [7, 8] using simple symmetrized two coupled rotor models, Alhassid and others [9, 10, 11, 12] employed the interacting boson model of atomic nuclei to investigate nature of level fluctuations, occupation number, etc. Also, Bijker and Frank [13, 14] studied the probability of occurrence of regular spectra in embedded ensembles (EE) for (sp) and (sd) boson models for diatomic molecules and atomic nuclei respectively. Moreover, Kota & Potbhare [15] and Manfredi [16]. studied the nature of the state density for the dense boson system where the interaction was restricted only up to two-body. It was observed that state densities are Gaussian in nature. Also these systems exhibit an average-fluctuation separation with smoothed density being corrected Gaussian whereas the fluctuations are of GOE type, as stated in [17]. In addition to it, after the experimental study of the famous Bose Einstein Condensates, recently Asaga et

al [18] confirmed that " in an atomic trap, bosonic atoms occupy partly degenerate single particle states - the interaction will lift the degeneracy and a random matrix approach should reveal the generic features of the resulting system". This gives us motivation to study the other aspects of different bosonic embedded ensembles.

In the present work, initially the BEGOE(k) model of spinless bosons with k body interaction is defined in Chapter 2 ('B' stands for Bosons) where we deal with the dense limit of bosons. Earlier mentioned was about the nature of densities for finite interacting boson systems in two-particle space, also the results for the EGOE(2) fermionic ensemble were carried out in the respective dilute limit. But what would be the case for bosons under dense limit, where the interaction propagates from 2-particle space to k-particle space, was still a puzzle to crack. The second chapter of this thesis introduces an embedded Gaussian orthogonal ensemble of spinless bosons termed as BEGOE(k) and the ensemble averaged state density is hence measured using an interpolating parameter. As the rank of the body interaction k increases, one can observe a transition from Gaussian to semicircle and the transition point coming out to be $k = m/2$. The method of Nearest Neighbor Spacing Distribution (NNSD) is introduced and an analysis of NNSD is done by varying the body rank k. It is interesting to observe that the NNSDs are consistent with their Wigner (GOE) nature irrespective of the body rank k. Thus Chapter 2 ends with a thorough discussion on energy eigenvalue spectrum and the spectral properties for spinless system of bosons.

Embedded ensembles are usually studied using only two-body interactions i.e. by allowing the single particle states to be degenerate. However the Hamiltonian for realistic systems consists of a mean field represented by one-body interaction plus a complexity (or chaos) generating two-body interaction. When one studies the lower order moments of the eigenvalue distribution [15] it reveals that for dense boson systems, the two-body part dominates over the one body part. In the present work, for two bosonic models BEGOE(1+2)-F (Chapter 3) and BEGOE(1+2)-$S1$ (Chapter 4), we consider the interaction to be $H{=}H_1 + \lambda V_2$, where λ is the strength of the

two-body which can be varied. Similarly, for a fermionic ensemble EGOE(1+3)-s (Chapter 5) we consider three body interaction $H=H_1 + \lambda V_3$ and spectral properties are analyzed.

When spin of any particle is into the picture, we very well know that fermions have half-integral spin whereas the bosons have a zero or integral spin. However in chapter 3 introduced is an embedded gaussian orthogonal ensemble of bosons, but bosons here occupy a fictitious spin half degree of freedom BEGOE(1+2)-F. The idealogy comes from the two species boson system, here the mean field is defined using random single particle energies. Our system is a finite system where $m = 10$ number of bosons are to be distributed in $\Omega = 4$ single particle states which are doubly degenerate. The method to construct the system Hamiltonian is also given for this ensemble. The theory also includes a brief explanation about the Dyson Mehta (Δ_3)statistics which unlike the NNSD is a long range statistics. Thus for a fixed (m, F) the above mentioned spectral properties viz. NNSD and Δ_3 statistics are analyzed by using numerical and analytical method. All these are calculated as a function of the two body interaction strength λ. Moreover power spectrum analysis is also carried out for each spin F and thus verified is the $1/f$-noise behavior of the BEGOE(1+2)-F spectra.

After having studied the spinless systems and those with a fictitious spin, the work further moves ahead with the introduction of a bosonic embedded Gaussian orthogonal ensemble with spin one degree of freedom which is modeled in chapter 4. This is denoted by BEGOE(1+2)-$S1$ where mean field part is added with the two body part, and the strength of the two body is varied by the interaction strength λ. Here a finite number of $m = 10$ bosons are distributed in Ω number of single particle states which are triply degenerate. After the construction of Hamiltonian, the embedding algebra is also shown in detail. State density for fixed (m, S) is calculated for the dense limit and it comes out to be close to Gaussian. Then, using the method of Nearest Neighbor Spacing Distribution (NNSD) a transition is shown from Poisson to Wigner (GOE) as the interaction strength λ increases. In the strong interaction

limit, the level fluctuations follow GOE statistics. Moreover, covariances in energy centroids and spectral variances are studied which show that there are fluctuations in the energy centroids whereas spectral widths are nearly constant with respect to S for $S < S_{max}/2$. Along with these results, two different pairing symmetry algebras (i)$SO(\Omega) - SU(3)$ pairing and (ii) $SO(3\Omega)$ pairing are also studied and numerical results show that random interactions generate ground states with maximal value for the pair expectation value. All the results are organized and graphically depicted at the end of the chapter.

Till now the work was all about bosonic systems with and without spins but in the next Chapter 5 a typical embedded ensemble of fermions with spin half degree of freedom where the number of fermions are $m > 2$. The inter-particle interaction is known to be mainly one-body and two-body in character in real many particle systems, though it is not possible to rule out three-body and higher particle interactions in the many particle systems. Also, previously the EGOE(1+2)-s ensemble was studied for multi-qubit systems hence this gives us a motivation to study the EGOE(1+3)-s model. This chapter discusses three body interaction in the presence of mean field using the EGOE(1+3)-s model which plays within the dilute limit of fermions. The three body force although being very weak can be expected to give a fine contribution to the Hamiltonian when the two body force is assumed to be absent. For this EGOE(1+3)-s ensemble, a method of constructing the system Hamiltonian is explained. From the eigenvalue spectrum of this Hamiltonian matrix the eigenvalue density of the system is calculated for fixed value of three body interaction strength. Whether the ensemble exhibits the average-fluctuation separation or not is also checked here. Moreover, the two spacing statistics short range (NNSD) and long range (Δ_3) statistics are studied here in the same strong interaction limit.

In Chapter 6 a transition is observed from Poisson to GOE using a method which is recently introduced called the distribution of ratio of consecutive level spacings $[P(r)dr]$. Particularly for short range spacing statistics, we are very well aware about other methods like NNSD to check the chaoticity of the system. But this method of

$P(r)dr$ can be an alternative to the traditional NNSDs with its different approach of taking the ratio of spacings between two consecutive levels. It was introduced by Ognaysen and Huse in 2007. Starting with the methodology of this method, in this chapter we shall apply it to an interpolating matrix ensemble with a parameter λ giving Poisson ($\lambda = 0$) to GOE ($\lambda \to \infty$) transition. The average of ratio (r) of consecutive level spacings has been analyzed for different matrix dimensions ($d \leq 1000$). Also, we introduce a new transition parameter which helps in making the results universal, which is verified by using the concept of mapping. The chapter also briefs about two realistic non-random models of a finite lattice chain of interacting spins based on the Heisenberg's XXX and XXZ model. For these models also we study the integrability to chaos transition using $P(r)dr$ analysis.

To end, Chapter 7 compiles chapter-wise the work presented in this thesis. It also discusses the possibilities of future work in connection with the systems studied here.

Finally, we have two Appendices A and B which may explain some important concepts along with derivations. These appendices correspond to Chapter 4 and Chapter 6 respectively.