# Chapter 5

# EGOE(1+3)-$s$ : Embedded fermionic ensemble with three body interaction

## 5.1 Introduction

From previous chapters and past study, we are very well accustomed with the spectral properties and nature of correlations for the spinless systems as well as for systems with random two body interaction in the presence of mean field. We have results for BEGOE(1+2)-$F$ model with a fictitious spin and then for BEGOE(1+2)-$S1$ model with spin one degree of freedom, all three for bosons. In the present chapter, we introduce an Embedded Gaussian Orthogonal Ensemble for fermions which focuses on the three body interaction. The idea of three body interaction basically comes from generic ensembles studied previously where interaction was upto two body. Ample amount of work is done for the EGOE(1+2) and the EGOE(1+2)-$s$ models, the former model of finite interacting fermions without spin and the latter model includes fermions with spin $s$ degree of freedom. See references [29, 46] for more details. For a many particle system, it is observed that the inter-particle interactions are mainly

one-body or two-body in character. But one may also consider the contribution from three body interaction, provided the system contains at least three particles. Although weaker than the two body, the three body forces cannot be totally neglected in any many particle system.

The interest in three-body interactions has been revived recently. The three-body forces plays an important role in the nuclear dynamics. It was suggested that in nuclear physics many-body forces that the three-body forces are important for saturation of nuclear matter [91]. It is impossible to obtain reasonable results for saturation density and equation of state [92, 93, 94] without effective repulsion due to three-body forces active in a dense medium. It is shown [95] that the super fluidity properties of neutron matter and neutron stars are sensitive to three-body forces. Even, the three-body forces also influence the quark dynamics [96]. A. Volya [97] also theorized the manifestation of three-body forces in $f_{7/2}$-shell nuclei and its contribution in several sectors like binding energies, seniority mixing, particle-hole symmetry, electromagnetic and particle transition rates. Moreover, two-body and three-body contributions were taken into consideration while probing the unitarity of the SRG (Similarity Renormalization Group) transformations [98]. All this research in turn provides a motivation to study the higher body interactions in many particle systems.

For the present investigation we probe an Embedded Gaussian Orthogonal fermionic Ensemble with three body interaction in the presence of mean field, thus named EGOE(1+3)-$s$ model. Here, $m$ is the number of fermions with spin $s = 1/2$ and $\Omega$ are the number of single particle states which are doubly degenerate. The next section of the chapter discusses the construction of the system Hamiltonian and the method here used is different from that of used in Chapter 3 and Chapter 4 where concepts of group theory were involved. We now only focus on the contribution from the three body interactions in addition to the mean field (one body) part of the system. After the construction of Hamiltonian we obtain the eigenvalue spectrum to study the various spectral properties which is discussed in the next section. For

the spectral properties, first of all we study the eigenvalue density and perceive information about the nature of the density for different spins. Secondly, we observe the average-fluctuation separation for state densities and thus get information about the type of fluctuations. The Nearest Neighbour Spacing Distribution (NNSD) is also studied here. Similarly, the long range $\Delta_3$ statistics is also observed for three different spins. It is important to note that for the entire investigation, the strength of three body forces is kept constant ($\lambda = 0.1$). Now we shall began with the construction of Hamiltonian.

## 5.2 Construction of Hamiltonian

Single particle states are denoted by $|i, m_s = \pm 1/2\rangle$ with $i = 1, 2, 3...\Omega$ and similarly there are anti-symmetric states which are denoted by $|(ijk); S, M_S\rangle_a$ with $S = 3/2$ or $S = 1/2$. For one plus two body Hamiltonian preserving $m$-particle spin $S$, the one body Hamiltonian is given as $h(1) = \sum_{i=1}^{\Omega} \epsilon_i \hat{n}_i$ where $i$ are doubly degenerate, $n_i$ are number operator and $\epsilon_i$ are single particle energies. Similarly the three body Hamiltonian $\widehat{V}(3)$ is defined by three body matrix elements,

$$\widehat{V}_{ijkpqr} = {}_a\langle (pqr)S, M_S|\widehat{V}(3)|(ijk)S, M_S\rangle_a \tag{5.2.1}$$

with three particle spins $S = 3/2$ or $S = 1/2$ and they are independent of the $M_S$ quantum number. Thus,

$$\widehat{V}(3) = \lambda_{3/2}\widehat{V}(3)^{3/2} + \lambda_{1/2}\widehat{V}(3)^{1/2} \tag{5.2.2}$$

Here the sum is a direct sum. Then the EGOE(1+3)-$s$ is defined by :

$$\widehat{H}_{EGOE(1+3)-s} = h(1) + \lambda_0\widehat{V}(3)^{3/2} + \lambda_1\widehat{V}(3)^{1/2} \tag{5.2.3}$$

Three body Hamiltonian in fermionic space can be constructed and is described ahead. First of all three particle states are generated in the $m_s$ basis, with $i = 1, 2, ...N$ for $m_s = 1/2$ and $i = N+1, N+2, ...2N$ for $m_s = -1/2$ part of the orbits. Similarly,

$m$-particle configurations are generated as described in [36, 46]. In the construction of three body in fermionic space, $S = 3/2$ corresponds to three fermions in different single particle orbits, say 1, 2 and 3. Here $S = 3/2$ , $M_S = 3/2$ corresponds to $|1_+, 2_+, 3_+\rangle \equiv |1, 2, 3\rangle$ in $m_s$ basis. And similarly, $S = 3/2$ , $M_S = -3/2$ corresponds to $|1_-, 2_-, 3_-\rangle \equiv |1+N, 2+N, 3+N\rangle$ in $m_s$ basis. To generate $S = 3/2$ , $M_S = 3/2$ state, apply $S_-$ operator on $|1, 2, 3\rangle$ which generates linear combination of $|1_-, 2_+, 3_+\rangle$, $|1_+, 2_-, 3_+\rangle$ and $|1_+, 2_+, 3_-\rangle$, equivalently linear combination of $|1+N, 2, 3\rangle$, $|1, 2+N, 3\rangle$ and $|1, 2, 3+N\rangle$ in $m_s$ basis.

Therefore, $|3/2, 1/2\rangle$ with fermions in orbits 1,2,3 is equal to

$$|3/2, 1/2\rangle = \frac{1}{\sqrt{3}}\Big\{|2, 3, 1+N\rangle - |1, 3, 2+N\rangle + |1, 2, 3+N\rangle\Big\}. \tag{5.2.4}$$

Similarly $S = 3/2$ , $M_S = -1/2$ state can be written in $m_s$ basis (after applying $S_-$ operator to $|3/2, 1/2\rangle$ state). So,

$$|3/2, -1/2\rangle = \frac{1}{\sqrt{3}}\Big\{-|3, 1+N, 2+N\rangle + |2, 1+N, 3+N\rangle - |1, 2+N, 3+N\rangle\Big\}. \tag{5.2.5}$$

Obviously, $S = 1/2$ , $M_S = 1/2$ states are $|1^2, 2\rangle$, $|1^2, 3\rangle$, $|1, 2^2\rangle$, $|2^2, 3\rangle$, $|1, 3^2\rangle$ and $|2, 3^2\rangle$. But there are also two $S = 1/2$ , $M_S = 1/2$ states with three fermions in different orbits. Using CG coefficients, one can see that,

$$-\sqrt{\tfrac{2}{3}}|1, 2, 3+N\rangle - \sqrt{\tfrac{1}{6}}|1, 3, 2+N\rangle + \sqrt{\tfrac{1}{6}}|2, 3, 1+N\rangle$$

is orthogonal to $S = 3/2$ , $M_S = 1/2$ state and hence it is $S = 1/2$ , $M_S = +1/2$ state. Second $S = 1/2$ , $M_S = 1/2$ state, orthogonal to $S = 3/2$ , $M_S = 1/2$ state and also orthogonal to just given $S = 1/2$ , $M_S = 1/2$ state is,

$$+\frac{1}{\sqrt{2}}|1, 3, 2+N\rangle - \frac{1}{\sqrt{2}}|2, 3, 1+N\rangle.$$

One can see that this is orthogonal to previous $M_S = 1/2$ states. So now we have description of three particle states in three orbits (6 single particle states) according to $|S, M_S\rangle$ basis.

$S = 3/2$ : one level with four states $M_S = 3/2, 1/2, -1/2, -3/2$.

$S = 1/2$ : two levels with all three particles in different orbits making up totally four states because of two values of $M_S = \pm 1/2$.

$S = 1/2$ : six levels of type $1^2, 2$ and so on, each having two states with $M_S = \pm 1/2$ thus making up twelve states.

This describes all 20 states in $|S, M_S\rangle$ basis of three particles in three orbits in a linear combination of 20 states in $|m_s\rangle$ basis. Thus one has generated a unitary transformation matrix from $|S, M_S\rangle$ basis to $m_s$ basis, for three particles in orbits 1, 2, 3. This then can be easily generalized to 3-particles in any orbits. Thus the three particle state in $|S, M_S\rangle$ basis can be expressed as superposition in $|m_s\rangle$ basis,

$$|S, M_S\rangle_\alpha = \sum_i U_{i\alpha} |m_s\rangle_i \tag{5.2.6}$$

where $U_{i\alpha}$ is the unitary transformation matrix.

The three-body Hamiltonian preserving $S$ and $M_S$ is generated in the $|S, M_S\rangle$ basis where $H_{\alpha\beta}$ is as usual a Gaussian variate with zero mean and variance $(1+\delta_{\alpha\beta})$.

$$H = \sum_{\alpha,\beta} H_{\alpha\beta} |\alpha\rangle\langle\beta| \tag{5.2.7}$$

where $|\alpha\rangle$, $|\beta\rangle$ are three particle states in $|S, M_S\rangle$ basis. This Hamiltonian is then transferred to $|m_s\rangle$ basis using Unitary transformation matrix defined above.

$$\begin{aligned} H &= \textstyle\sum_{\alpha,\beta} \left( \sum_i |i\rangle\langle i| \right) |\alpha\rangle H_{\alpha\beta} \langle\beta| \left( \sum_j |j\rangle\langle j| \right) \\ &= \textstyle\sum_{ij} |i\rangle \left( \sum_{\alpha,\beta} \langle i|\alpha\rangle H_{\alpha\beta} \langle\beta|j\rangle \right) \langle j| \\ &= \textstyle\sum_{ij} |i\rangle H_{ij} \langle j| \end{aligned} \tag{5.2.8}$$

where $H_{ij}$ is the matrix element in $m_s$ basis with $|i\rangle$ and $|j\rangle$ being the three particle states in the $m_s$ basis.

$$H_{ij} = \sum_{\alpha,\beta} U_{i\alpha} H_{\alpha\beta} U^{-1}_{\beta j}. \tag{5.2.9}$$

This three-body Hamiltonian in $m_s$ basis is then used to obtain $H$ in $m$-particle space, which then is diagonalized to obtain eigenvalues and eigenvectors.

## 5.3 Spectral Analysis

The method of constructing the Hamiltonian for the present EGOE(1+3)-$s$ ensemble is now known to us. The Hamiltonian matrix is further diagonalized to obtain the eigenvalues which form the eigenvalue spectrum. Based on the eigenvalue spectrum, one can procure about the spacing statistics and density of the energy levels. This information about the ensemble explicitly briefs us about the chaotic nature of the system. We have chosen a finite system of $(m, \Omega) = (8, 8)$ where $\Omega$ are doubly degenerate because of the $spin = 1/2$ degree of freedom. Hence we can express the total number of states as $N = 2\Omega$. We now begin with the analysis of Eigenvalue density for the present system of EGOE(1+3)-$s$.

### 5.3.1 Eigenvalue density

The study of Eigenvalue density serves a platform that helps us in realization of various spectral properties of the system. For present system of fermions we have with us $m = 8$ fermions which are to be distributed in $\Omega = 8$ number of orbitals. The method and formulae to calculate the eigenvalue density is not discussed as it is already elaborated in previous chapters. The calculations here are done for a 20 member ensemble due to computational constraints. Fig. (5.2) shows the nature of the eigenvalue density for three different spins $S = 0$, $S = 1$ and $S = 2$ where the three body interaction strength is kept fixed as $\lambda = 0.1$. The mere reason of not showing these results for spins $S = 3$ and $S = 4$ is the small dimensionality that they possess but they have been verified. Similarly, choice of the three body interaction strength $\lambda$ is such that the ensemble falls in the chaotic regime i.e. for the strong interaction limit. Histograms in Fig. (5.2) show the eigenvalue densities and they are compared with the Gaussian (red colour) and Edgeworth (ED) corrected Gaussian (blue colour) forms. One can observe that the eigenvalues density are close to the Gaussian form whereas the agreements with the ED Gaussian is also remarkable. Moreover, the spins here does not have much contribution to the form of density and

well it is independent of the value of spin. This again is a generic feature as observed in the case of EGOE(2), EGOE (1+2) and EGOE(1+2)-$s$ systems, studied in the past [46]. Also, the respective values of $\gamma_1$, $\gamma_2$ and dimensionality ($d$) are stated for each spin.

### 5.3.2 Average-fluctuation separation

As far as now, we are clear that the generic density for the EGOE(1+3)-$s$ system of fermions is close to Gaussian. Here we only deal with the dilute limit which does not distinguish the bosonic systems from the fermionic systems and they behave in a similar pattern. Following the method used by Mon and French the results for separation between average and fluctuations are shown here. With the generic form of density being close to Gaussian, the normalized state density for the EGOEs can be in general represented by the Gram-Charlier (G-C) expansion [99]

$$\rho(E) = \rho_G(E)\left\{1 + \sum_{\zeta\geq 3}(\zeta!)^{-1}S_\zeta He_\zeta(\hat{E})\right\} \tag{5.3.1}$$

For the present investigation we have with us a fermionic system with $m = 8$ fermions in $\Omega = 8$ number of single particles that are doubly degenerate. We already discussed the nature of eigenvalue density and it comes out to be close to Gaussian. Thus now we study the average-fluctuation separation by keeping fix the three body interaction strength $\lambda = 0.1$. A plot of $\Delta(E) = [F(E) - \overline{F(E)}]$ versus $\hat{E}$ is generated for two spins $S = 0$ and $S = 1$. While doing calculations, the spectra are first zero centered and scaled to unit width. Results are shown in Fig. (5.3) for zeroth order (Gaussian), order 3, 4 and 6 corrections to the asymptotic Gaussian density of states. In the ensemble calculations, level motion for each order is obtained for each member separately and the figures show the results from all 20 members. The values of root-meansquare deviations $\Delta_{RMS}$ are also shown in the figures. It is clearly seen that with increasing order the $\Delta_{RMS}$ decreases rapidly at first and then changes slowly indicating a sharp separation between the smooth and fluctuating parts of

the distribution function [49]. For spin 0, the value of $\Delta_{RMS} = 0.95$ confirms the fluctuations to be of GOE type. This onset of GOE is actually observed well in the $4^{th}$ order. Similarly for spin 1, again approaching $6^{th}$ order we get the GOE nature of fluctuations for which the value of $\Delta_{RMS}$ comes out to be 0.97. In a way one can conclude that for EGOE(1+3)-$s$ system, the average-fluctuation separation is observed and the fluctuations after removing the smoothed part are of GOE type.

### 5.3.3 Nature of Nearest Neighbour Spacing Distribution

The method of Nearest Neighbor Spacing Distribution (NNSD) is a test to check whether system belongs to the chaotic domain or to the regular domain. For the present fermionic ensemble we have $(m, \Omega) = (8, 8)$. The value of the three body interaction strength ($\lambda$) is kept fixed to 0.1 which serves to be a strong interaction limit. Fig. (5.4) shows the trend of NNSD for three major values of spin i.e. $s = 0, 1$ and 2. It is seen that the histograms (black color) are in close agreement with the Wigner curve (red color). This behavior is consistent and is independent of the value of spin. Thus, the short-range spacing statistics show GOE nature for $\lambda = 0.1$ irrespective of the rank of spin. The bin size kept during these calculations is 0.2.

### 5.3.4 Behaviour of Dyson Mehta ($\Delta_3$) statistics

For the present ensemble EGOE(1+3)-$s$ of fermions, we also study the spacing distribution of energy levels that are not nearest neighbors of each other, nor are they the next nearest neighbors. The $\Delta_3$ statistics relates the long range energy levels and this method is already discussed in Chapter 3. For a fixed $(m, \Omega)$=$(8, 8)$ a plot of $\Delta_3$ versus $L$ is constructed for different values of spins $S = 0, 1, 2$ for $L \leq 40$ , again keeping the strength of the three body strength constant to be $\lambda = 0.1$. In Fig. (5.5), the nature of statistic is near to GOE (red curve) as stated in Eq. (3.4.16) of Chapter 3. The blue curve in the figure denote the Poisson distribution described by Eq. (3.4.17), whereas the black dots reflect the calculated values. Again we can observe the same pattern as it was in the nature of NNSDs, the nature of spacing

statistics is independent of the value of spin for interaction strength $\lambda = 0.1$. The study here is limited to 20 members due to computational constraints.

## 5.4 Conclusion

This chapter brings a study on one plus three-body fermionic ensemble EGOE(1+3)-$s$, where fermions are assigned with spin degree of freedom. The basic method of constructing Hamiltonian for a given $(m, \Omega)$=$(8, 8)$ is employed in the good spin basis, not using group theory.

The eigenvalue density of the system is computed for fix value of three body interaction strength ($\lambda = 0.1$) and it is observed that the eigenvalue density (smooth part) is close to Gaussian irrespective of the value of spin. Also, apart from the smooth part of the density the average-fluctuations are evaluated for $S = 0$ and $S = 1$. The graphical results depict that as the order ($\zeta$) in the Gram-Charlier expansion is increased, the fluctuations exhibit GOE behavior. Regarding the spacing statistics, we analyze both - the short range (NNSD) and long range ($\Delta_3$) statistics under the same interaction limit ($\lambda = 0.1$). For NNSDs, we observe a consistent Wigner form for every value of spin. Similarly, the Dyson Mehta ($\Delta_3$) statistics also show the same pattern like NNSDs. Thus, a study of EGOE(1+3)-$s$ with some of its preliminary results is done in this chapter.

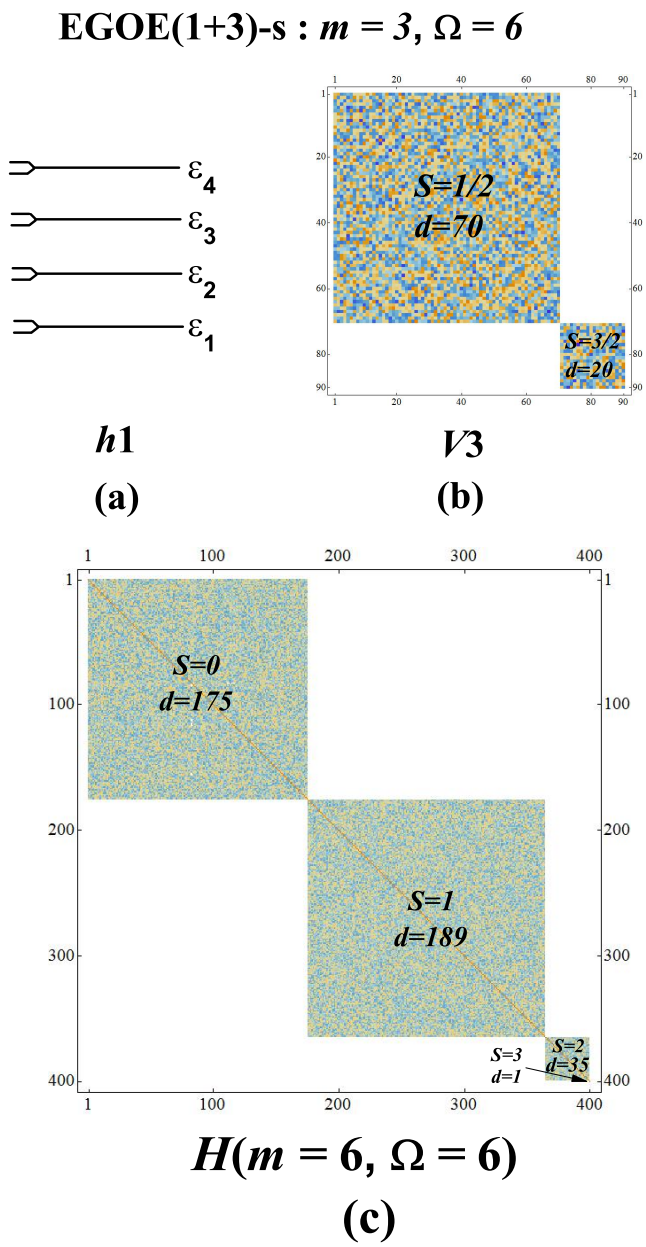


Figure 5.1: (a) Single-particle levels generated by $\hat{h}(1)$ for $\Omega = 6$. Each level is doubly degenerate; i.e., $N = 2\Omega$. (b) Matrix of $\hat{V}(3)$ in three-fermion space for $\Omega = 6$. (c) Decomposition of H matrix in $m$-particle space into a direct sum of matrices, each with a fixed $S$ value. There is a EGOE(1 + 3)-$s$ ensemble in each $(m, S)$ space corresponding to the diagonal blocks in (c). Note that the matrix elements in the off-diagonal blocks in (b) and (c) are all zero. The plots (b) and (c) are generated using MATHEMATICA and we have illustrated only one particular member of $V(3)$ and $H(m)$ ensembles that are employed in the numerical calculations.

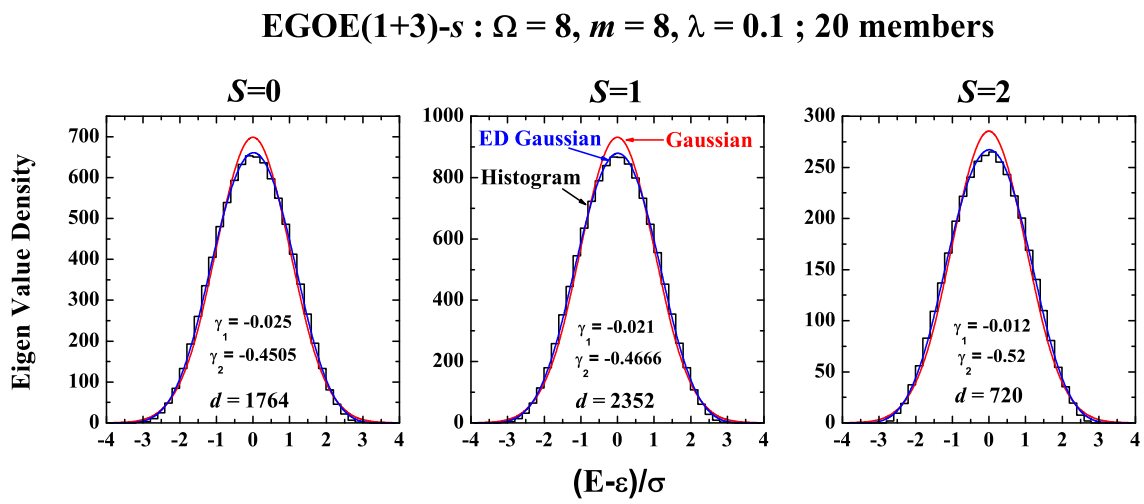


Figure 5.2: Figure shows an ensemble averaged eigenvalue density for a 20 member EGOE$(1+3)$-$s$ fermionic ensemble with $\Omega = 8$ and $m = 8$. The histograms show that calculated eigenvalue density which is close to Gaussian (blue color) or more precisely has a good match with the Edgeworth corrected Gaussian curve (red color). The form of density is independent of the value of spin which has been verified for three values of spin $S$=0,1 and 2

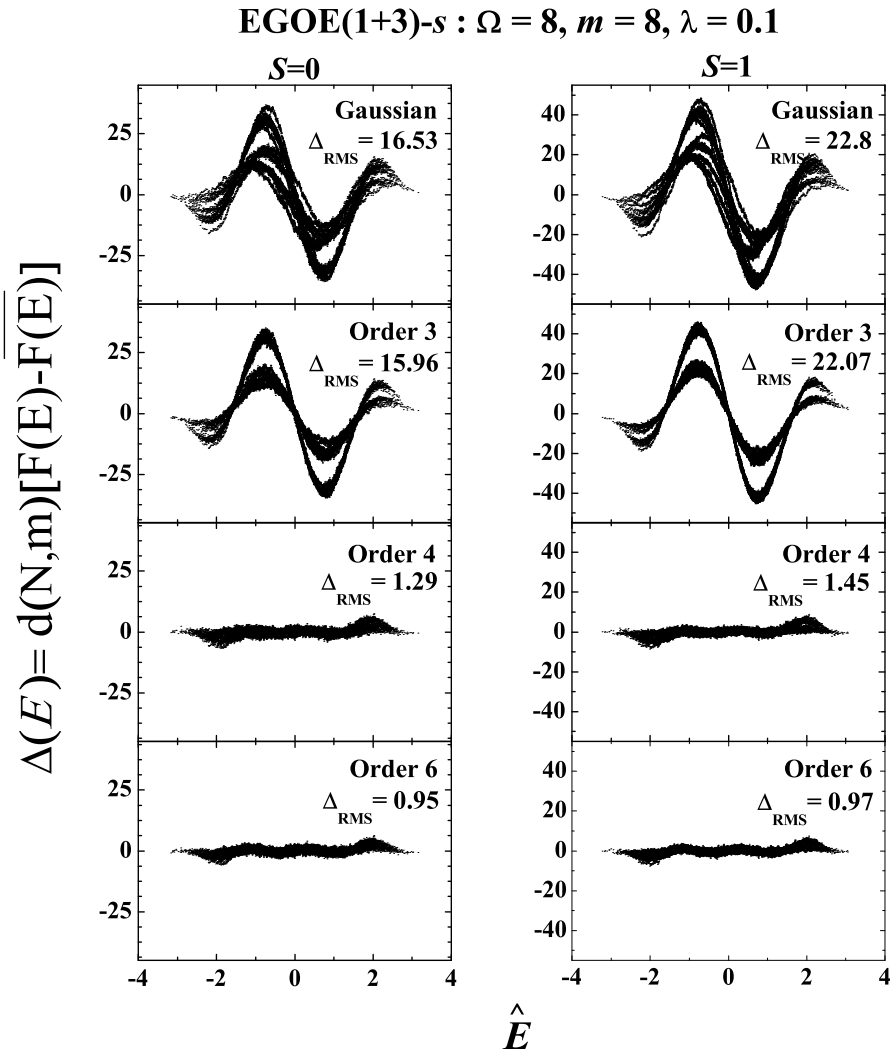


Figure 5.3: Figure shows the average-fluctuation separation for a 20 member EGOE(1 + 3)-$s$ fermionic ensemble with $\Omega = 8$ and $m = 8$. Starting with the zeroth order (Gaussian) the order of correction is gradually increased to third, fourth and sixth order. A fine transition is observed from Gaussian to GOE as we reach the sixth order. The value of $\Delta_{RMS}$ for each order is mentioned. This analysis is done for two significant spins $S = 0$ and $S = 1$

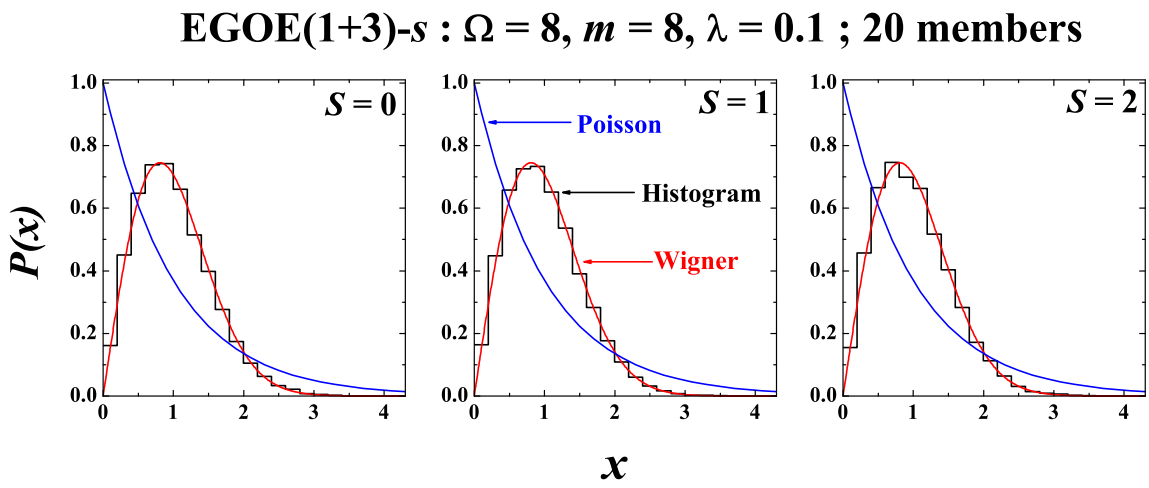


Figure 5.4: Figure shows the nearest neighbor spacing distribution (NNSD) for a 20 member EGOE(1 + 3)-$s$ fermionic ensemble with $\Omega = 8$ and $m = 8$. NNSD is calculated for three spins $S$=0,1 and 2. It is seen that the nature of spacing distribution $P(x)$ represented by the histogram fairly agrees with the Wigner curve (red color) and the system is far away from the Poisson part (blue curve)

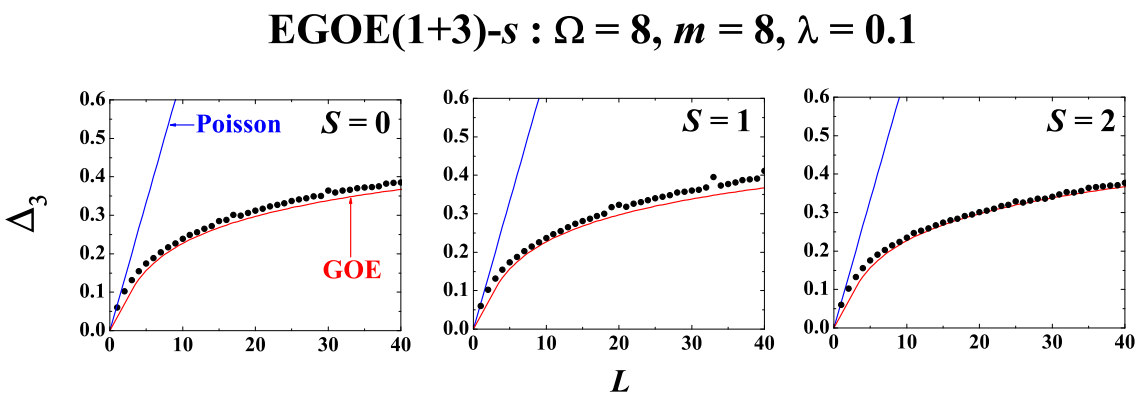


Figure 5.5: Figure shows the nature of $\Delta_3$ distribution with respect to the long range parameter $L$ for a 20 member EGOE$(1+3)$-$s$ fermionic ensemble with $\Omega = 8$ and $m = 8$ for three spins $S$=0,1 and 2. The dotted line in the figure represents the $\Delta_3$ distribution which is close to the GOE curve (red color) denoting chaos in the system, and the system is far away from the Poisson part (blue curve)