Chapter 4

BEGOE(1+2)-$S1$: Embedded ensemble of finite interacting bosons with spin one degree of freedom

4.1 Introduction

In the previous chapter, BEGOE for two species boson systems with F-spin degree of freedom generated by Hamiltonians that conserve the total F-spin of the m-boson systems [called BEGOE(1+2)-F] is introduced and its spectral properties are analyzed in detail [also see [41]]; F-spin for bosons is similar to the F-spin in the proton-neutron interacting boson model (pnIBM) of atomic nuclei [42, 62]. In order to apply BEGOE to spinor BEC discussed in [63, 64] and to analyze generic structures generated by the IBM-3 model of atomic nuclei (here spin S is equivalent to isospin T of the bosons in IBM-3) [65, 66], an important extension that needs to be investigated is BEGOE for a system of interacting bosons carrying spin-one degree of freedom. For a system with m number of bosons in Ω number of single particle(sp) levels, each

triply degenerate, we define EGOE of random matrices generated by random two-body interactions that conserve the total spin S. This random matrix ensemble is hereafter denoted by BEGOE(2)-$S1$. In the presence of a mean-field interactions, we have BEGOE(1+2)-$S1$. It is useful to add that for condensates in an optical trap, the single-mode approximation is often adopted for the studies of spin dynamics and many-body physics, where all spin components are assumed to share the same spatial dependence and only the spin components vary. Thus, in the mean-field approximation, one need not consider orbital angular momentum degree of freedom for spinor condensates [67, 68, 69]. However, it becomes important to incorporate the effects of orbital angular momentum and its coupling with spin degree of freedom via spin-orbit term for condensates in a magnetic or magneto-optical traps [69, 70]. In this situation BEGOE(1+2)-$S1$ has to be extended to include additional degrees of freedom like orbital angular momentum and this is for future. Besides possible applications to spinor BEC and atomic nuclei, BEGOE(1+2)-$S1$ ensemble is also of general interest as embedded ensembles for isolated finite many-body quantum systems are being used as generic models for many-body chaos [71, 72, 73].

This chapter reports the first results of the BEGOE(1 + 2)-$S1$ ensemble. In the next section, we introduce the embedded ensemble BEGOE(1 + 2)-$S1$ [also BEGOE(2)-$S1$] for m bosons in Ω number of sp levels that are triply degenerate with total spin S being a good symmetry and give a method for the numerical construction of this ensemble in fixed-(m,S) spaces. Thereafter discussed is the embedding algebra for the ensemble which is $U(\Omega)\bigotimes[SU(3) \supset SO(3)]$. Further sections contain the numerical results for the ensemble averaged eigenvalue density, nearest neighbor spacing distribution (NNSD), width of the fluctuations in energy centroids, and spectral variances. And lastly, two types of pairing in BEGOE(1 + 2)-$S1$ space are introduced and some numerical results for ground state structure and pairing structure are presented.

4.2 Definition and construction of BEGOE(1+2)-$S1$ Hamiltonian

We consider a system of m ($m > 2$) bosons with spin-one ($S = 1$) degree of freedom occupying Ω number of single particle states. For convenience, we use the notation "$\mathbf{s}$" for the spin quantum number of a single boson, "s" for the spin carried by a two boson system and for $m > 2$ boson systems, the notation "S" is used for the spin. It can be easily seen that $\mathbf{s} = 1$; $s = 0$, 1 and 2; $S = m$, $m - 1$, ..., 0. We use the same notations for the eigenvalues of $\hat{S}_z$ ('hat' denoting operator) and they are given by $m_{\mathbf{s}}$, m_s and M_S respectively. Now on, the space generated by the sp levels $i = 1, 2$, ..., Ω is referred as orbital space. Then the sp states of one boson are denoted by $|i; \mathbf{s} = 1, m_{\mathbf{s}}\rangle$ with $i = 1, 2, \ldots, \Omega$ and $m_{\mathbf{s}} = +1$, 0 and -1. We have Ω number of orbital degrees of freedom and three spin ($m_{\mathbf{s}}$) degrees of freedom, hence each state is a triply degenerate and thus the total number of sp states is $N = 3\Omega$. Going further, two boson (normalized) states that are symmetric in the total orbital $\times$ spin space are denoted by $|(ij); s, m_s\rangle$ with $s = 1 \times 1 = 0$, 1 and 2; however, for $i = j$ only $s = 0, 2$ are allowed. Also, $-s \leq m_s \leq s$. In terms of boson creation ($b^\dagger$) and annihilation (b) operators, the single particle (sp) states are $|i; 1, m_{\mathbf{s}}\rangle = b^\dagger_{i;s,m_{\mathbf{s}}}|0\rangle$. Similarly, the two-boson states are $|(ij); s, m_s\rangle = \frac{1}{\sqrt{(1+\delta_{ij})}}(b^\dagger_{i;1} b^\dagger_{j;1})^s_{m_s}|0\rangle$. It is to be noted that here we are using spin (angular momentum) coupled representation.

For the present case, we consider one- plus two-body Hamiltonians (H) preserving spin S for m-boson systems. Then the Hamiltonian operator is defined by $\widehat{H} = \widehat{h}(1) + \widehat{V}(2)$ with the mean-field one-body Hamiltonian $\widehat{h}(1)$ to be as follows:

$$\widehat{h}(1) = \sum_{i=1}^{\Omega} \epsilon_i \hat{n}_i \tag{4.2.1}$$

Here the single particle energies ϵ_i in Eq. (4.2.1) are independent of the $m_{\mathbf{s}}$ quantum number and therefore each sp level i is triply degenerate.

The single particle energies defining the one-body Hamiltonian $\hat{h}(1)$ in Eq. (4.2.4) are chosen to be $\epsilon_i = i + 1/i$, where $i = 1, 2, ..., \Omega$ just as in many of the earlier

papers on embedded ensembles [29, 38, 39, 41, 74, 75]. The second term $1/i$ is added in ϵ_i so that degeneracy can be avoided fairly as discussed in [74]. Also, one can choose the values of ϵ_i from the eigenvalues of a random matrix ensemble or also from the centre of a Gaussian Orthogonal Ensemble (GOE). While applying to the Bose Einstein Condensation (BEC) the use of a harmonic trap suggests equidistant levels with different degeneracies. Similarly, anisotropic traps (elongated or toroidal traps that are uniform along only one direction while still harmonic along the other two directions) [76, 77] result in non-equidistant energy levels with different degeneracies. In a way, it is possible to carry out BEGOE(1 + 2)-$S1$ calculations with various choices of ϵ_i. But one thing is to be noted that no matter we make a distinct choice of ϵ_i, all the statistical properties discussed in the present paper will be more or less the same provided the strength of interaction is sufficiently large (i.e., for $\lambda > \lambda_C$ as discussed ahead). Secondly, the $\widehat{n}_i$ are number operators and it can be represented by $\widehat{n}_i = \sum_{m_s} b^\dagger_{i;1,m_s} b_{i;1,m_s}$; the action of $\widehat{n}_i$ on the level i gives the number of bosons occupying i^{th} level. From Fig. (4.1 (a)) we can observe a sp spectrum for $\Omega = 4$ which are triply degenerate. Let us add that, in principle, it is possible to consider $\widehat{h}(1)$ with off-diagonal energies ϵ_{ij} . One can define a two-body Hamiltonian operator $\widehat{V}(2)$ preserving spin S is defined in Eq. (4.2.2). The contribution from the $V(2)$ part will in turn explain about the contribution from different spins (for the present case $S = 0, 1, 2$). Moreover, the presence of $V(2)$ (its strength) in the ensemble accounts for the onset of chaos in the system.

$$\widehat{V}(2) = \sum_{i,j,k,l;s,m_s}{}' \frac{V^S_{ijkl}}{\sqrt{(1+\delta_{ij})(1+\delta_{kl})}} (b^\dagger_{i:1} b^\dagger_{j:1})^S_{m_s} [(b^\dagger_{k:1} b^\dagger_{l:1})^S_{m_s}]^\dagger \tag{4.2.2}$$

Eq. (4.2.2) clearly defines the two body part which acts as a complexity generating term. Here the "prime" over the summation symbol in indicates that the summation over i, j , k, and l is restricted to $i \geq j$ and $k \geq l$ for $s = 0, 2$ and $i > j$ and $k > l$ for $s = 1$. The symmetrized (with respect to the total orbital $\times$ spin space) two-body matrix elements $V^s_{ijkl} = \langle (ij)s, m_s | \widehat{V}(2) | (kl)s, m_s \rangle$ are independent of the m_s quantum number and this ensures that $\widehat{V}(2)$ preserves spin S. It is clearly seen from Eq. (4.2.2)

that $\widehat{V}(2) = \widehat{V}^{s=0}(2) + \widehat{V}^{s=1}(2) + \widehat{V}^{s=2}(2)$. Then the matrix of $\widehat{V}(2)$ in two-boson spaces will be a direct sum of $s = 0, 1$, and 2 matrices. Hence we obtain a 3×3 block matrix where the three diagonal blocks correspond to $s = 0, 1$,and 2, respectively, whereas the off-diagonal blocks are zero. Dimensions of the $s = 0, 1$, and 2 matrices are $\Omega(\Omega + 1)/2$, $\Omega(\Omega 1)/2$, and $\Omega(\Omega + 1)/2$ (here m_s is not counted as the states with same s but different m_s will be degenerate). Fig. (4.1 (b)) shows an example of the $\widehat{V}(2)$ matrix in two-boson spaces. Matrix representation for $\widehat{H} = \widehat{h}(1) + \widehat{V}(2)$ in m-boson spaces with good S can be obtained, for example, using the basis,

$$|(v_1)^{n_1,S_1}, (v_2)^{n_2,S_2}, \ldots, (v_r)^{n_r,S_r}; \alpha, S, M_S\rangle \tag{4.2.3}$$

Here v_i are the occupied sp levels, n_i are number of bosons in the v_i level, $m = \sum_i n_i, S_i = n_i, n_i - 2, \ldots, 0$, or 1, and $S = S_1 \times S_2 \times \ldots S_r$. Also, in Eq. (4.2.3) "α" are additional labels needed for complete specification of the basis states. As $\widehat{H}$ preserves spin S, matrix elements of $\widehat{H}$ in the basis given by Eq. (4.2.3) will be independent of M_S quantum number. A method to calculate H matrix dimension $d(\Omega, m, S)$ is discussed ahead. Note that $d(\Omega, m, S)$ gives the number of basis states for a given (Ω, m, S) without counting M_S values, i.e.,H matrix dimension for a given (Ω, m, S). Thus, by now using the above information , one can define the BEGOE(1 + 2)-$S1$ ensemble. Here { } denotes an ensemble and the operator of BEGOE(1+2)-$S1$ is given as:

$$\left\{\widehat{H}(1+2)\right\}_{BEGOE(1+2)-S1} = \widehat{h}(1) + \lambda_0 \left\{\widehat{V}^{s=0}(2)\right\} + \lambda_1 \left\{\widehat{V}^{s=1}(2)\right\} + \lambda_2 \left\{\widehat{V}^{s=2}(2)\right\}, \tag{4.2.4}$$

Here, we should keep one thing in mind that the matrices $\widehat{V}^{s=0}(2)$, $\widehat{V}^{s=1}(2)$, and $\widehat{V}^{s=2}(2)$ in two-boson spaces [i.e., the three diagonal block matrices in Fig. 4.1 (b)] are independent GOEs with unit variance. This means that all the two-body matrix elements V^s_{ijkl} defining the three parts of $\widehat{V}(2)$ are zero centered independent Gaussian variables with variance unity for off-diagonal matrix elements and variance two for the diagonal matrix elements. In Eq. (4.2.4), λ_0, λ_1, and λ_2 are the strengths of the $s = 0, 1$, and 2 parts of $\widehat{V}(2)$, respectively. The mean-field Hamiltonian $\widehat{h}(1)$ in

Eq. (4.2.4) is defined by the sp energies ϵ_i [see Eq. (4.2.1)] with average spacing Δ. Without loss of generality, we put $\Delta = 1$ so that λ_0, λ_1, and λ_2 are in the units of Δ. The action of each member of $\widehat{H}_{BEGOE(1+2)S1}$ on (m, S) basis states given by Eq. (4.2.3) will give a member of BEGOE(1 + 2)-$S1$ ensemble in a given (m, S) space and the ensemble of all these members then defines BEGOE(1 + 2)-$S1$ ensemble in a given (m, S) space. Here we choose total 100 members for the spectral analysis of given BEGOE(1+2)-$S1$ ensemble. One can see from Fig. (4.1 (c)) that the H matrix in m-boson spaces will be a direct sum matrix with the diagonal blocks representing the matrix with fixed S value. Note that the dimension of the diagonal blocks is $d(\Omega, m, S)$. Starting with H defined by Eq. (4.2.4) and using the basis given by Eq. (4.2.3), there will be an ensemble corresponding to each diagonal block in Fig. (4.1 (c)) and this is the BEGOE(1 + 2)-$S1$ ensemble. At this point if one simply wants BEGOE(2)-$S1$, then put the value of $\widehat{h}(1) = 0$ in Eq. (4.2.4). This ensemble can also be studied alternatively when one deals in very strong interaction limit.

Now we shall move ahead with the construction of the BEGOE(1+2)-$S1$ using the concepts of group theory. A simple method for constructing the BEGOE(1 + 2)-$S1$ ensemble in a given (m, S) space is to first construct the matrices in good M_S basis [see Eq. (4.2.7) ahead] and then employ for spin S projection the $\widehat{S}^2$ operator [its eigenvalues are $S(S + 1)$], as was done before for spin-1/2 fermion [39] and F-spin boson systems [41]. We need a many-particle basis here and for generating it first the 3Ω sp states are arranged such that the first Ω number of sp states have $m_{\mathbf{s}} = 1$, the next Ω number of sp states have $m_{\mathbf{s}} = 0$, and the remaining Ω sp states have $m_{\mathbf{s}} = -1$. Then, the sp states are $|r\rangle = |i = r, m_{\mathbf{s}} = 1\rangle$ for $r \leq \Omega$, $|r\rangle = |i = r - \Omega, m_{\mathbf{s}} = 0\rangle$ for $\Omega < r \leq 2\Omega$, and $|r\rangle = |i = r - 2\Omega, m_{\mathbf{s}} = -1\rangle$ for $2\Omega < r \leq 3\Omega$. With this, $r = 1, 2 \ldots, 3\Omega$. Now, the many-particle states for m bosons can be obtained by distributing m_1 bosons in the $m_{\mathbf{s}} = 1$ sp states, m_2 bosons in the $m_{\mathbf{s}} = 0$ sp states, and, similarly, m_3 bosons in the $m_{\mathbf{s}} = -1$ sp states with $m = m_1 + m_2 + m_3$. Let us denote each distribution of m_1 bosons in $m_{\mathbf{s}} = 1$ sp states by $\mathbf{m}_1 = (m_1^1, m_1^2, ..., m_1^\Omega)$, m_2 bosons in $m_{\mathbf{s}} = 0$ sp states by $\mathbf{m}_2 = (m_2^1, m_2^2, ..., m_2^\Omega)$,

and, similarly, $\mathbf{m}_3 = (m_3^1, m_3^2, ..., m_3^\Omega)$ for m_3 bosons in $m_\mathbf{s} = -1$ sp states. Note that $m_k = \sum_{i=1} m_k^i, k = 1, 2, 3$, and m_k^i is the number of bosons in the sp state $|r = i + (k-1)\Omega\rangle$. Each $[m_1, m_2, m_3]$ defines a m-particle configuration or basis state in occupation number representation with $M_S = (m_1 - m_3)$. To proceed further, the Hamiltonian operator defined by Eq. (4.2.4) is converted into M_S representation. Then $\widehat{h}(1) = \sum_{r=1}^{3\Omega} \epsilon_r^{'} \widehat{n}_r^{'}$ and the sp energies $\epsilon_q^{'} = \epsilon_{q+\Omega}^{'} = \epsilon_{q+2\Omega}^{'} = \epsilon_r$ for $1 \leq q \leq \Omega$. Similarly, for changing $\widehat{V}(2)$, first note that two-boson states in M_S representation can be written as $|i, m_\mathbf{s}; j, m_\mathbf{s}^{'}\rangle; m_s = m_\mathbf{s} + m_\mathbf{s}^{'}$. Then the two particle matrix elements are,

$$V^{'}_{i,m_{m_\mathbf{s}}^{f1};j,m_{m_\mathbf{s}}^{f2};k,m_{m_\mathbf{s}}^{i1};l,m_{m_\mathbf{s}}^{i2}}(2) = \langle i, m_{m_\mathbf{s}}^{f1}; j, m_{m_\mathbf{s}}^{f2} | \widehat{V}(2) | k, m_{m_\mathbf{s}}^{i1}; l, m_{m_\mathbf{s}}^{i2} \rangle \tag{4.2.5}$$

One can derive formulas for these in terms of $V_{ijkl}^s(2)$ by applying angular momentum algebra to expand $|i, m_\mathbf{s}; j, m_\mathbf{s}^{'}\rangle$ states in terms of the coupled two-particle states $|(ij)_s, m_s = m_\mathbf{s} + m_\mathbf{s}^{'}\rangle$ and normalizing them appropriately. The final formulas are

$$\begin{aligned}
V'_{i,1;j,1;k,1;\ell,1}(2) &= V_{ijkl}^{s=2}(2) \ , \\
V'_{i,1;j,0;k,1;\ell,0}(2) &= \frac{\sqrt{(1+\delta_{ij})(1+\delta_{k\ell})}}{2} \left[V_{ijkl}^{s=1}(2) + V_{ijkl}^{s=2}(2) \right] \ , \\
V'_{i,1;j,-1;k,1;\ell,-1}(2) &= \frac{\sqrt{(1+\delta_{ij})(1+\delta_{k\ell})}}{6} \left[2\, V_{ijkl}^{s=0}(2) + 3\, V_{ijkl}^{s=1}(2) + V_{ijkl}^{s=2}(2) \right] \ , \\
V'_{i,0;j,0;k,0;\ell,0}(2) &= \left[\frac{1}{3}\, V_{ijkl}^{s=0}(2) + \frac{2}{3}\, V_{ijkl}^{s=2}(2) \right] \ , \\
V'_{i,1;j,-1;k,0;\ell,0}(2) &= \frac{\sqrt{(1+\delta_{ij})}}{3} \left[V_{ijkl}^{s=2}(2) - V_{ijkl}^{s=0}(2) \right] \ .
\end{aligned} \tag{4.2.6}$$

All other $V^{'}$ matrix elements follow by symmetries. Now converting $|i, m_\mathbf{s}^i\rangle$ into $|r\rangle$ states, the $V^{'}$ matrix elements can be written as $\langle r, s | \widehat{V}(2) | t, u\rangle$ matrix elements and then $\widehat{V}(2)$ will be of the form given by Eq. (A.1) of [41]. Now it is easy to see that

the construction of a m-particle matrix for $\widehat{h}(1)+\widehat{V}(2)$ in (m_1, m_2, m_3) basis reduces to the problem of BEGOE(1 + 2) for spinless boson systems, and hence Eq. (A.3) of [41] will give the formulas for the matrix elements. Note that $\widehat{h}(1)$ is diagonal in the (m_1, m_2, m_3) basis and $\langle(m_1, m_2, m_3)|\widehat{h}(1)|(m_1, m_2, m_3)\rangle = \sum_{r=1}^{\Omega} \epsilon_r(m_1^r + m_2^r + m_3^r)$. To project out S, we consider the following basis as they will contain states with all S values.

$$|(m_1, m_2, m_3), M_S = 0\rangle \tag{4.2.7}$$

The dimension of this basis space is $\mathcal{D}(\Omega, m, M_S = 0) = \sum_S d(\Omega, m, S)$. In the basis defined by Eq. (4.2.7), both the H matrix and the $\hat{S}^2$ matrix are constructed using the procedure described above. Note that for the $\hat{S}^2$ operator, the diagonal two-particle matrix elements $V^s_{ijij}(2)$ are -4, -2 and 2 for $s = 0$, 1, and 2, respectively, and they are independent of i and j . Similarly, all the off-diagonal matrix elements are zero and the sp energies are $\epsilon_i = 2$ independent of i. Now diagonalizing the $\hat{S}^2$ matrix will give the unitary transformation required to convert the H matrix in the $M_S = 0$ basis [given by (4.2.7)] into good S basis [we use the fact that $\hat{S}^2$ eigenvalues are $S(S+1)$ with $S = 0, 1, \ldots, m$ for a given]. This then gives the m-boson matrix as a direct sum of matrices, one each for the allowed S values. Thus, the H matrix will be of the form shown in Fig. (4.1 (c)). This procedure has been implemented and computer programs are developed. One can say that the BEGOE(1 + 2)-$S1$ ensemble is defined by six parameters $(\Omega, m, S, \lambda_0, \lambda_1, \lambda_2)$ with λ_ss in units of Δ . Now we shall briefly discuss about the embedding algebra for the BEGOE(1 + 2)-$S1$ ensemble which will in fact help to calculate $d(\Omega, m, S)$. Once we have the dimensionality, the numerical results will be carried out in the further sections.

4.3 Embedding algebra

Embedding algebra for BEGOE(1+2)-$S1$ is not unique and following the earlier results for the IBM-3 model of atomic nuclei [65, 66], it is possible to identify two algebras. They are: (i) $U(3\Omega) \supset U(\Omega) \otimes [U(3) \supset SO(3)]$; (ii) $U(3\Omega) \supset SO(3\Omega) \supset$

$SO(\Omega) \otimes SO(3)$. Here we will consider case (i) and later we will consider (ii) in brief.

Initially, the spectrum generating algebra $U(3\Omega)$ is generated by the $(3\Omega)^2$ number of operators $u^k_q(i,j)$ where

$$u^k_q(i,j) = \left(b^\dagger_{i;\mathbf{s}=1}\tilde{b}_{j;\mathbf{s}=1}\right)^k_q \; ; k = 0,1,2 \;\text{ and }\; i,j = 1,2,\ldots,\Omega \; . \tag{4.3.1}$$

Note that u^k are given in angular momentum coupled representation with $k = \mathbf{s}\times\mathbf{s} = 0,1,2$. Also, $b^\dagger$s are one boson creation operators, bs are one boson annihilation operators and $\tilde{b}_{i;1,m_\mathbf{s}} = (-1)^{1+m_\mathbf{s}}\, b_{i;1,-m_\mathbf{s}}$. The quadratic Casimir invariant of $U(3\Omega)$ is $\hat{C}_2(U(3\Omega)) = \sum_{i,j,k}\; u^k(i,j)\cdot u^k(j,i)$. Note that $T^k\cdot U^k = (-1)^k\sqrt{(2k+1)}\,(T^kU^k)^0$. In terms of the number operator $\hat{n} = \sum_{i,m_\mathbf{s}}\; b^\dagger_{i;1,m_\mathbf{s}} b_{i;1,m_\mathbf{s}}$, we have $\hat{C}_2(U(3\Omega)) = \hat{n}(\hat{n}+3\Omega-1)$. All m-boson states transform as the symmetric irrep $\{m\}$ w.r.t. $U(3\Omega)$ algebra and therefore,

$$\left\langle \hat{C}_2(U(3\Omega))\right\rangle^{\{m\}} = m(m+3\Omega-1) \; . \tag{4.3.2}$$

Using the results given in [78] it is easy to write the generators of the algebras $U(\Omega)$ and $SU(3)$ in $U(3\Omega) \supset U(\Omega)\otimes SU(3)$. The $U(\Omega)$ generators are $g(i,j)$ where,

$$g(i,j) = \sqrt{3}\,\left(b^\dagger_{i;\mathbf{s}=1}\tilde{b}_{j;\mathbf{s}=1}\right)^0 \; ; i,j = 1,2,\ldots,\Omega \tag{4.3.3}$$

and they are Ω^2 in number. Similarly, $SU(3)$ algebra is generated by the eight operators $h^{k=1,2}_q$ where,

$$h^k_q = \sum_i \left(b^\dagger_{i;\mathbf{s}=1}\tilde{b}_{i;\mathbf{s}=1}\right)^k_q \; ; k = 1,2 \; . \tag{4.3.4}$$

It is useful to mention that $(h^0,\, h^1_q,\, h^2_{q'})$ generate $U(3)$ algebra and $U(3) \supset SU(3)$. The quadratic Casimir invariants of $U(\Omega)$ and $SU(3)$ algebras are, $\hat{C}_2(U(\Omega)) = \sum_{i,j}\; g(i,j)\cdot g(j,i)$ and $\hat{C}_2(SU(3)) = (3/2)\sum_{k=1,2}\; h^k\cdot h^k$ respectively. The irreps of $U(\Omega)$ can be represented by Young tableaux $\{f\} = \{f_1, f_2, \ldots, f_\Omega\}$, $\sum_i f_i = m$. However, as we are dealing with boson systems (i.e. the only allowed $U(3\Omega)$ irrep being $\{m\}$), the irreps of $U(\Omega)$ and $U(3)$ should be represented by the same $\{f\}$. Therefore, $\{f\}$ will be maximum of three rows. The $U(\Omega)$ and $SU(3)$ equivalence gives a relationship between their quadratic Casimir invariants, $\hat{C}_2(U(\Omega)) = \hat{C}_2(U(3)) + (\Omega-3)\,\hat{n}$

and $\hat{C}_2(U(3)) = \sum_{k=0,1,2} h^k \cdot h^k = (2/3)\hat{C}_2(SU(3)) + (1/3)\hat{n}^2$. Given the $U(\Omega)$ irrep $\{f_1\, f_2\, f_3\}$, the corresponding $SU(3)$ irrep in Elliott's notation [79] is given by $(\lambda\mu)$ where $\lambda = f_1 - f_2$ and $\mu = f_2 - f_3$. Thus,

$$\begin{aligned} &\{m\}_{U(3\Omega)} \to \Big[\{f_1\, f_2\, f_3\}_{U(\Omega)}\Big]\ \big[(\lambda\,\mu)_{SU(3)}\big]\ ; \\ &f_1 + f_2 + f_3 = m, \quad f_1 \geq f_2 \geq f_3 \geq 0\ , \\ &\lambda = f_1 - f_2, \ \ \mu = f_2 - f_3\ . \end{aligned} \tag{4.3.5}$$

Using Eq. (4.3.5) it is easy to write, for a given m, all the allowed $SU(3)$ and equivalently $U(\Omega)$ irreps. Eigenvalues of $\hat{C}_2(SU(3))$ are given by

$$\Big\langle \hat{C}_2(SU(3)) \Big\rangle^{(\lambda\,\mu)} = C_2(\lambda\,\mu) = \big[\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)\big]\ . \tag{4.3.6}$$

Let us add that the $SU(3)$ algebra also has a cubic invariant $C_2(SU(3))$ and its matrix elements are [80],

$$\Big\langle \hat{C}_3(SU(3)) \Big\rangle^{(\lambda\,\mu)} = C_3(\lambda\,\mu) = \frac{2}{9}\,(\lambda - \mu)(2\lambda + \mu + 3)(\lambda + 2\mu + 3)\ . \tag{4.3.7}$$

The $SO(3)$ subalgebra of $SU(3)$ generates spin S. The spin generators are

$$S^1_q = \sqrt{2}\, h^1_q\ , \quad \hat{S}^2 = C_2(SO(3)) = S^1 \cdot S^1, \quad \Big\langle \hat{S}^2 \Big\rangle^S = S(S+1)\ . \tag{4.3.8}$$

Given a $(\lambda\,\mu)$, the allowed S values follow from Elliott's rules [79] and this introduces a "K" quantum number,

$$\begin{aligned} K &= min(\lambda\,,\,\mu),\ min(\lambda\,,\,\mu) - 2,\ \ldots,\ 0 \ \text{ or } \ 1\ , \\ S &= max(\lambda\,,\,\mu),\ max(\lambda\,,\,\mu) - 2,\ \ldots,\ 0 \ \text{ or } \ 1 \ \text{ for } \ \ K = 0, \\ &= K, K+1, K+2, \ldots, K + max(\lambda\,,\,\mu) \ \text{ for } \ \ K \neq 0\ . \end{aligned} \tag{4.3.9}$$

Eq. (4.3.9) gives $d_{(\lambda\,\mu)}(S)$, the number of times a given S appears in a $(\lambda\,\mu)$ irrep. Similarly the number of sub-states that belong to a $U(\Omega)$ irrep $\{f_1\, f_2\, f_3\}$ are given by $d_\Omega(f_1\, f_2\, f_3)$ where [81],

$$d_\Omega(f_1\, f_2\, f_3) = \begin{vmatrix} d_\Omega(f_1) & d_\Omega(f_1+1) & d_\Omega(f_1+2) \\ d_\Omega(f_2-1) & d_\Omega(f_2) & d_\Omega(f_2+1) \\ d_\Omega(f_3-2) & d_\Omega(f_3-1) & d_\Omega(f_3) \end{vmatrix}\ . \tag{4.3.10}$$

Here, $d_\Omega(\{g\}) = \binom{\Omega+g-1}{m}$ and $d_\Omega(\{g\}) = 0$ for $g < 0$. Note that the determinant in Eq. (4.3.10) involves only symmetric $U(\Omega)$ irreps. Using the $U(3\Omega) \supset U(\Omega) \otimes [U(3) \supset SO(3)]$ algebra, m bosons states can be written as $|m; \{f_1\, f_2\, f_3\}\, \alpha; (\lambda\, \mu)\, K\; S\, M_S\rangle$; The number of α values is $d_\Omega(f_1\, f_2\, f_3)$, K values follow from Eq. (4.3.9) and $-S \leq M_S \leq S$. Note that m and $(\lambda\, \mu)$ give a unique $\{f_1\, f_2\, f_3\}$. Therefore H-matrix dimension in fixed-(m, S) space is given by $d(m, S) = \sum_{\{f_1\, f_2\, f_3\} \in m} d_\Omega(f_1\, f_2\, f_3)\; d_{(\lambda\, \mu)}(S)$ and they will satisfy the sum rule $\sum_S (2S+1) d(m, S) = \binom{3\Omega+m-1}{m}$. Also, the dimension $D(m, M_S = 0)$ of the H-matrix in the basis discussed earlier is $D(m, M_S = 0) = \sum_{S \in m} d(m, S)$. For example, for $(\Omega = 4,\ m = 8)$, the dimensions $d(m, S)$ for $S = 0-8$ are 714, 1260, 2100, 1855, 1841, 1144, 840, 315 and 165 respectively. Similarly, for $(\Omega = 6,\ m = 10)$, the dimensions for $S = 0 - 10$ are 51309, 123585, 183771, 189630, 178290, 133497, 94347, 51645, 27027, 9009 and 3003 respectively. These give $\mathcal{D}(\Omega = 4, m = 8, M_S = 0) = 10234$ and $\mathcal{D}(\Omega = 6, m = 10, M_S = 0) = 1045113$. Because of these very large dimensions, numerical analysis of BEGOE(1+2)-S1 ensemble is not that simple. Due to such computational constraints, we limit ourselves only up to a finite system of $(m, \Omega) = (8, 4)$ which is fairly large to establish appropriate conclusions.

4.4 Spectral Analysis

4.4.1 Eigenvalue density

In the previous chapters, we have already understood the procedure for calculating the Eigenvalue density and its importance. For present system of spin one bosons we have $(m, \Omega) = (8, 4)$. The value of interaction strength λ_s where $s = 0, 1$ and 2 is set to be constant i.e. $\lambda_0 = \lambda_1 = \lambda_2 = 0.2$. For constructing the ensemble averaged eigenvalue densities, the spectra of each member of the ensemble are first zero centered and then scaled to unit width. The normalized eigenvalues $\hat{E}$ where $\hat{E} = \frac{E - E_c}{\sigma}$, $E_c(m, S)$ denotes the centroid and σ denotes the square root of variance or the spectral width(m, S). A histogram of these eigenvalues is then compared with

the respective curves of Gaussian and Edgeworth corrected Gaussian. The bin size for the histogram is set to be 0.2. The expressions for Gaussian and Edgeworth corrected Gaussian densities are given as follows:

$$\rho_G(\hat{E}) = \frac{1}{\sqrt{2\pi}\sigma}\ exp\left(\frac{(E-E_c)^2}{2\sigma^2}\right) \tag{4.4.1}$$

$$\rho_{EDG}(\hat{E}) = \rho_G(\hat{E})\left\{1 + \left[\frac{\gamma_1}{6}He_3(E)\right] + \left[\frac{\gamma_2}{24}He_4(E) + \frac{\gamma_1^2}{72}He_6(E)\right]\right\} \tag{4.4.2}$$

The expression of corrected Gaussian in Eq. (4.4.2) contains contribution from the three Hermite polynomials He_3, He_4 and He_6. The calculations here are done for a 100 member ensemble of $m = 8$ spin one bosons distributed in $\Omega = 4$. For all the spins $S = 0, 1, 2, ...8$, the eigenvalue density is obtained and plotted in the form of histogram as mentioned earlier. With this, we also obtain the values of γ_1 (skewness) and γ_2 (excess) for different spin cases. Fig. (4.2) displays the nature of the eigenvalue density with respect to the normalized eigenvalues for three spins, one lowest spin ($S = 0$), one largest spin ($S = 8$) as well as one intermediate spin value ($S = 4$). The respective values of γ_1, γ_2 and dimensionality (d) are stated for each spin. One can observe that in each figure, the nature of state density is similar i.e. close to Gaussian, irrespective of the value of spin. The histograms in black denote the calculated energy eigenvalues, the curve in red (color) shows the Gaussian distribution whereas the curve in green (color) is the Edgeworth Gaussian distribution defined above. It is to be remembered that here the value of two body strength is kept fixed, hence $\lambda_0 = \lambda_1 = \lambda_2 = 0.2$. Hence one can conclude that the energy eigenvalues are arranged in such a manner in the spectrum that they form a shape which is close to the Gaussian distribution, no matter whatever be the value of spin. This is an important generic feature also observed in the case of BEGOE(2) and BEGOE(1+2) systems studied in the past.

4.4.2 Nearest Neighbour Spacing Distribution

The method of Nearest Neighbour Spacing Distribution (NNSD) is a test to check whether the system is integrable (or regular) or is in a chaotic regime and this method is very well discussed in the earlier chapters. Also from the previous done work it is seen that the NNSD is of GOE type for spinless BEGOE(1 + 2) [17] and BEGOE(1 + 2)-F [41] for a sufficiently strong two-body interaction strength. We shall now evaluate the spacing distribution of the energy eigenvalues for the present system of spin one bosons and compare with the theoretical curves of Poisson and GOE.

We have with us an ensemble of 100 members for a system of $(\Omega, m) = (4, 8)$ and the spacing distribution $P(x)$ has been calculated, where x is in the units of local mean spacing. As we have excess use of the notation "s" in this chapter, the symbol for spacing in NNSDs is chosen to be "x" here. For different values of spin ranging from $S = 0, 1, 2, ...8$ the nature of the spacing has been calculated. Again the two body interaction strengths λ_0, λ_1 and λ_2 corresponding to the spin 0, 1 and 2 parts of $\widehat{V}(2)$ are kept fixed equal to $\lambda_0 = \lambda_1 = \lambda_2 = \lambda = 0.2$ which is sufficiently strong. The NNSDs are obtained by unfolding each spectrum in the ensemble, using the method described in [17], with the smoothed density as a corrected Gaussian with corrections involving up to sixth order moments of the density function. For present calculations the major 80% of the eigenvalues are employed, i.e. we drop the 10% portion from both ends of the spectrum. From the Fig. (4.3) we can observe the nature of spacing distribution is consistent for all the spins; here we are mainly showing three prominent spins $S = 0$, $S = 4$ and $S = 8$. The histograms (in black) represent the calculated results which when matched with the theoretical curves show that they are close with the GOE (Wigner) distribution (red color curve), whereas the results are far different from the Poisson distribution (green color curve).

It is thus plausible to conclude that for finite interacting boson systems the eigenvalue density will be generically of Gaussian form and fluctuations, with sufficiently strong interaction, follow GOE. But if mean field is taken into consideration as discussed in [34, 41], the interaction strength has to be larger than a critical value for the

fluctuations to change from Poisson-like to GOE. For this, in the present system calculations are carried out for various values of λ. As discussed in [30], we have another parameter other than λ which is denoted by Λ. The analysis is done for selected spins that are at the extremities, spin 0 and 8. The Fig. 4.4 clearly displays that for small values of λ the nature of NNSD is Poisson-like. Slowly as the value of λ increases, one starts observing the onset of chaos and as we further approach, we see complete chaos (GOE nature of NNSD) for $\lambda = 0.05$ and $\Lambda = 0.88$ (co-incidently values are coming out to be same for spin 0 and 8). Here it should be marked that the transition point ($\lambda_C = 0.031$ for spin 0 and 0.029 for spin 8) is determined empirically, keeping the value of NNSD variance $\sigma^2(0)$ equal to 0.37. For Poisson type, it has the value of 1, and for GOE, it is known to be equal to 0.27. The choice of the critical variance $\sigma^2(0) = 0.37$ of NNSD comes from the (2×2) random matrix model constructed in [30] for the Poisson to GOE transition in NNSD. As stated in this model, from the formula for $\sigma^2(0 : \Lambda)$ it is easily seen that $\sigma^2(0)$= 0.37 for $\Lambda = 0.3$ which is clearly verified from the Fig. (4.4). Thus for a considerably high value of interaction strength λ (as in the present case where $\lambda = 0.2 \gg \lambda_C$) level fluctuations follow GOE, and this proves to be a generic behavior in all the above mentioned examples of bosonic ensembles.

4.5 Propagation of Energy centroids and Spectral variance

As seen in the previous section, the eigenvalue density is close to Gaussian, it is useful to derive formulas for energy centroids and spectral variances in terms of single particle energies ϵ_i and the two-particle $V(2)$ matrix elements V^s_{ijkl}. They will also allow us to study, numerically (also analytically if the ensemble averages can be carried out analytically), fluctuations in energy centroids and spectral variances. For a given one plus two-body Hamiltonian H, propagation equation for the fixed-(m, S) energy centroids $\langle H \rangle^{m,S}$ in terms of the scalars $\hat{n}$ and S^2 operators [with their eigenvalues m

and $S(S+1)]$ is not possible. This is easily seen from the fact that upto 2 bosons, we have 5 states ($m=0, S=0$; $m=1, S=1$; $m=2, S=0,1,2$) but only 4 scalar operators (1, $\hat{n}$, $\hat{n}^2$, $\hat{S}^2$). For the missing operator we can use $\hat{C}_2(SU(3))$ but then only fixed-$(m,(\lambda\,\mu)S)$ averages will propagate [82].
The propagation equation is,

$$\begin{aligned}
\left\langle \widehat{H}(1+2)\right\rangle^{m,(\lambda\,\mu),S} &= \left\langle \widehat{h}(1)+\widehat{V}(2)\right\rangle^{m,(\lambda,\mu),S} = m\,\left\langle \widehat{h}(1)\right\rangle^{1,(10),1} \\
&+\left[-\frac{m}{6}+\frac{m^2}{18}+\frac{C_2(\lambda\,\mu)}{9}-\frac{S(S+1)}{6}\right]\,\left\langle \widehat{V}(2)\right\rangle^{2,(20),0} \\
&+\left[-\frac{5m}{6}+\frac{5m^2}{18}+\frac{C_2(\lambda\,\mu)}{18}+\frac{S(S+1)}{6}\right]\,\left\langle \widehat{V}(2)\right\rangle^{2,(20),2} \\
&+\left[\frac{m}{2}+\frac{m^2}{6}-\frac{C_2(\lambda\,\mu)}{6}\right]\,\left\langle \widehat{V}(2)\right\rangle^{2,(01),1}\,.
\end{aligned} \tag{4.5.1}$$

Now summing over all $(\lambda\,\mu)$ irreps that contain a given S will give $\left\langle \widehat{H}(1+2)\right\rangle^{m,S}$. This is used to verify the codes we have developed for constructing BEGOE(1+2)-$S1$ members. Propagation equation for spectral variances $\left\langle [\widehat{H}(1+2)]^2\right\rangle^{m,S}$ is more complicated. Just as with energy centroids, it is possible to propagate the variances $\left\langle [\widehat{H}(1+2)]^2\right\rangle^{m,(\lambda\,\mu),S}$. However, this requires the so called $SU(3)\supset SO(3)$ integrity basis operators $\hat{X}_3$ and $\hat{X}_4$ that are $three-$ and $four-$body in nature respectively. This makes the method complicated [83]. One can also use the (m_1,m_2,m_3) configurations in such a way that the m particle space can be decomposed using configurations (m_1,m_2,m_3) with $m_1+m_2+m_3=m$ where m_1 particles are in the unitary orbit with Ω number of sp states all carrying $m_{\mathbf{s}}=+1$ quantum number, m_2 particles in the unitary orbit with $m_{\mathbf{s}}=0$ and m_3 particles in the unitary orbit with $m_{\mathbf{s}}=-1$. The total M_S for such a configuration is (m_1-m_3). Propagation of the traces $\langle\langle H^p\rangle\rangle^{m_1,m_2,m_3}$, $p=1,2$ in terms of ϵ_i and $V'_{i,m_{\mathbf{s}}^{f1};j,m_{\mathbf{s}}^{f2};k,m_{\mathbf{s}}^{i1};\ell,m_{\mathbf{s}}^{i2}}(2)$ follow from the results given in [84, 85]. The Appendix B at the end gives the formula. These in turn will give the

traces $\langle\langle H^p\rangle\rangle^{m,M_S}$, $(p=1,2)$ as,

$$\langle\langle H^p\rangle\rangle^{m,M_S} = \sum_{fixed\ \ M_S} \langle H^p\rangle^{(m_1,m_2,m_3)} d(\Omega : m_1,m_2,m_3) = \sum_{S=M_S}^{S_{max}} d(\Omega,m,S)\langle H^p\rangle^{(m,S)}. \tag{4.5.2}$$

Where $\sum_{fixed\ \ M_S} d(\Omega : m_1,m_2,m_3) = D(\Omega,m,M_S) = \sum_{S=M_S}^{S_{max}} D(\Omega,m,M_S)$ and $d(\Omega,m,S)$ is the number of levels with fixed S value for m-particles.

Hence $d(m,S) = D(\Omega,m,M_S=S) - D(\Omega,m,M_S=S+1)$. Using these, fixed (m,S) energy centroids and spectral variances are given by

$$\langle H^p\rangle^{m,S} = \frac{\langle\langle H^p\rangle\rangle^{m,M_{S=S}} - \langle\langle H^p\rangle\rangle^{m,M_{S=S+1}}}{d(\Omega,m,M_S)} \tag{4.5.3}$$

It is easy to see that this procedure also gives $d(\Omega,m,S)$ as the dimension of a (m_1,m_2,m_3) configuration is $\binom{\Omega+m_1-1}{m_1}\binom{\Omega+m_2-1}{m_2}\binom{\Omega+m_3-1}{m_3}$ and it has been implemented for numerical calculations of $d(m,S)$, $E_c(m,S)$ and $\sigma^2(m,S)$.

For different configurations such as $(\Omega=3,m=6)$, $(\Omega=4,m=6)$, $(\Omega=4,m=8)$, and $(\Omega=5,m=5)$, choosing $\epsilon_i = i+1/i$ and V^s_{ijkl} as random numbers, the H matrices are explicitly constructed as described earlier and $E_c(m,S)$ and $\sigma^2(m,S)$ are calculated for various S values. These are compared with the numbers obtained using the propagation formulas given in the Appendix B and the agreement is found to be exact, as expected. Going beyond this testing, using the propagation formulas we have calculated the variation of the ensemble averaged spectral variances with spin S for $m=8$, 12, 16 and $\Omega=4$. It should be mentioned that direct matrix construction for $m=12$ and 16 is not possible with the available computational facility. These calculations are possible because of the propagation equations. Results are shown in Fig. (4.5). It is clearly seen from the figure that the ensemble averaged variances are almost constant for lower spins ($S < S_{max}/2$) and increase with S close to the $S=S_{max}$; a similar result is known for fermion systems [86]. It is useful to note that the near constancy of widths is a feature of many-body chaos [74, 87, 88].

4.6 Study of fluctuations in energy centroids and spectral variances by analyzing self correlations

Calculation of energy centroids and spectral variances for each member of the ensemble will allow us to examine the covariances in these quantities. For example, normalized covariances are defined by

$$\Sigma_{pp}(m,S:m',S') = \frac{\overline{\langle H^p\rangle^{m,S}\,\langle H^p\rangle^{m',S'}} - \left\{\overline{\langle H^p\rangle^{m,S}}\right\}\,\left\{\overline{\langle H^p\rangle^{m',S'}}\right\}}{\left\{\overline{\langle H^2\rangle^{m,S}}\;\overline{\langle H^2\rangle^{m',S'}}\right\}^{p/2}} \tag{4.6.1}$$

Here, $p = 1$ gives normalized covariances in energy centroids and $p = 2$ gives normalized covariances in spectral variances. For $(m, S) = (m', S')$ they will give information about fluctuations and in particular about level motion in the ensemble [17]. For $(m, S) \neq (m', S')$, the covariances (cross correlations) are non-zero for BEGOE while they will be zero for independent GOE representation for the m boson Hamiltonian matrices with different m or S (with fixed Ω for both (m, S) and (m', S') systems so that the Hamiltonian in two-particle spaces remains the same). We hence computed energy centroids and spectral variances for 200 members BEGOE(2)-$S1$ ensemble for $\Omega = 4 - 6$ and $m = 8 - 20$ (from the formulas in Appendix B). In the numerical calculations, we use $\lambda_0 = \lambda_1 = \lambda_2$. The results for self-correlations in energy centroids ($[\Sigma_{11}(m, S : m, S)]^{1/2}$) and in spectral variances ($[\Sigma_{22}(m, S : m, S)]^{1/2}$) as a function of spin S/S_{max} are shown in Fig. (4.6). It is seen that the centroid fluctuations are large for $S = 0$ with $m >> \Omega$ (blue curve in figure) and decreases with increase in S value. However for small m (black curve in figure), the variation of $[\Sigma_{11}]^{1/2}$ with spin S is weak. For fix m, there is also decrease in $[\Sigma_{11}]^{1/2}$ as increase in Ω. On the other hand, $[\Sigma_{22}]^{1/2}$ are always smaller than $[\Sigma_{11}]^{1/2}$ just as for BEGOE(2)-F [41]. It is seen from Fig. (4.6) that for $\Omega = 6$, the width of the fluctuations in the variances are $10 - 13\%$. Similarly for large m, with Ω being very small, the widths are quite large, but they decrease fast with increasing Ω. Thus, the width of the fluctuations

in spectral widths is found to be much smaller, unlike the width of the fluctuations in energy centroids.

4.7 Pairing algebras for BEGOE(1+2)-$S1$

For the present ensemble of spin one bosons with one plus two body interaction, so far we studied the spectral properties like eigenvalue density and the nearest neighbor spacing distribution (NNSD). Further, in the BEGOE(1+2)-$S1$ space, we shall study about two different pairing symmetries that are supposed to be observed here. Each symmetry sector defines a particular type of pairing and they follow from the results in [66, 78, 89]. One of them corresponds to the $SO(\Omega)$algebra in $U(3\Omega) \supset [U(\Omega) \supset SO(\Omega)] \otimes [U(3) \supset SO(3)]$ and we refer to this as $SO(\Omega) - SU(3)$ pairing. The other corresponds to the $SO(3\Omega)$ in $U(3\Omega) \supset SO(3\Omega) \supset SO(\Omega) \otimes SO(3)$. Note that both the algebras have $SO(3)$ subalgebra that generates the spin S. The details of these pairing algebras are given in the Appendix B section which is at end of this thesis.

4.7.1 Expectation values of the pairing Hamiltonian in BEGOE(1+2)-$S1$

The importance of the expectation values can be understood from the fact that, for the pairing Hamiltonians in the eigenstates generated by random interactions, the expectation values will give information on regular structures generated by random forces and also they are a measure of chaos [38, 90]. Results for the expectation values of the two pairing Hamiltonians and also $\hat{C}_2(SU(3))$ in the eigenstates of the BEGOE(1+2) Hamiltonian defined by Eq. (4.2.4) are shown in Fig. (4.7). We have chosen the parameters in the region of chaos (see Fig. 4.3), i.e. $\lambda_0 = \lambda_1 = \lambda_2 = \lambda = 0.2$ so that fluctuations in the expectation values will be minimal [38]. It is seen that the expectation values are largest near the ground states and then decrease as we move towards the center of the spectrum. Due to finiteness of the model space, the curves are essentially symmetric around the center. The calculated results are in good

agreement with the prediction [41] that for boson systems (just as it was well verified for fermion systems [38]), expectation values will be ratio of Gaussians (see Eq. (43) in [41]). Results in the figure show that with repulsive pairing, ground states will be dominated by low seniority structure (small value for ω or $\omega_1 + \omega_2 + \omega_3$). Also, with random interactions, there is no clear distinction between the two different pairing structures. Variation with the λ parameter for larger systems (Ω and m large) may show the difference but these calculations are not attempted as the matrix dimensions will be very large. This exercise will be attempted in future. In addition, results in Fig. 4.7(c) show that random interactions give ground states with large value for the expectation value of $\hat{C}_2(SU(3))$ and this implies that ground states will be dominated by the $SU(3)$ irrep $(\lambda\mu) = (m0)$. Note that $\left\langle \hat{C}_2(SU(3)) \right\rangle^{(m0)} = m^2 + 3m$ and thus $\left\langle \hat{C}_2(SU(3)) \right\rangle^{(80)} = 88$ for $m = 8$. The expectation value near the ground state is close to this as seen from Fig. 4.7(c). This result is important for the IBM-3 model of atomic nuclei [65].

4.8 Conclusions

This chapter introduces an embedded Gaussian orthogonal ensemble of random matrices generated by random two-body interactions in presence of a mean-field for spin one boson systems [BEGOE(1+2)-$S1$]. For such an ensemble, some analytical formulation and numerical results are discussed here. For finite number of bosons (m) distributed in single particle states (Ω) that are triply degenerate some of the spectral properties are analyzed. Evaluating the form of eigenvalue density which is close to Gaussian, next studied is the Nearest Neighboring Spacing Distribution (NNSD) which follow GOE statistics for sufficiently strong interaction strength. Also calculated here are the lowest two moments of the two point function. Also, the preliminary aspects of one of the embedding algebras $SU(\Omega) \otimes SU(3)$ and also two pairing algebras in the space defining BEGOE(1+2)-$S1$ are discussed here. More detailed study of the effects of random interactions in presence of the two pairing

interactions can be useful for gaining new insights into IBM-3 model of atomic nuclei [65, 66].

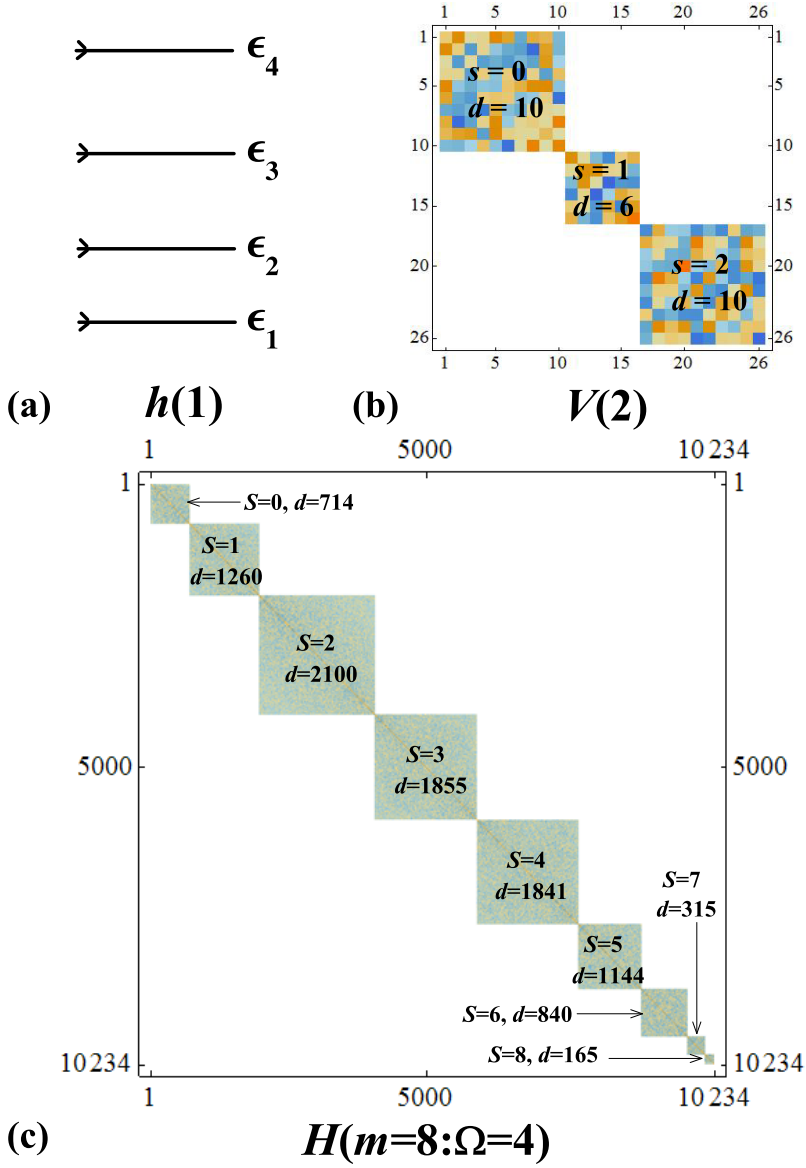

Figure 4.1: (a) Single-particle levels generated by $\hat{h}(1)$ for $\Omega = 4$. Here, each level is triply degenerate; i.e., $N = 3\Omega$. (b)Matrix of $\hat{V}(2)$ in two-boson space for $\Omega = 4$. (c) Decomposition of H matrix in m-particle space into a direct sum of matrices, each with a fixed S value. There is a BEGOE(1 + 2)-$S1$ ensemble in each (m, S) space corresponding to the diagonal blocks in (c). Note that the matrix elements in the off-diagonal blocks in (b) and (c) are all zero. In the matrix plots (b) and (c), generated using MATHEMATICA, we have used one particular member of $V(2)$ and $H(m)$ ensembles that are employed in the numerical calculations. As the figures are only for illustration, the scale defining the numerical value of the matrix elements is not shown

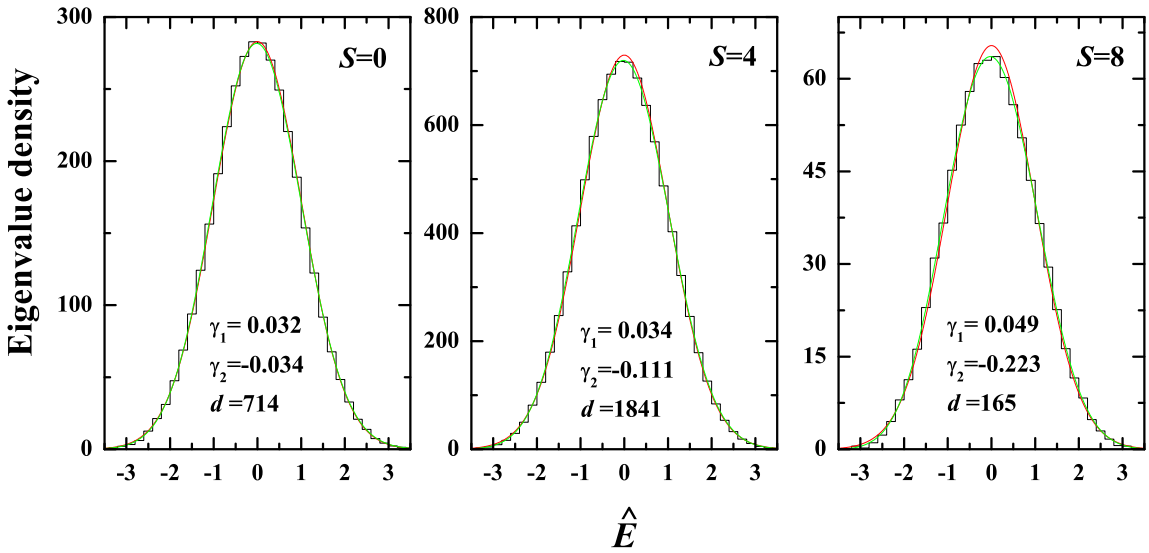

Figure 4.2: Ensemble averaged eigenvalue density $\rho^{m,S}(\hat{E})$ vs normalized energy $\hat{E} = \frac{E-Ec(m,S)}{\sigma(m,S)}$ for a 100 member BEGOE(1 + 2)-$S1$ ensemble with $\Omega = 4$,$m = 8$, and spins $S = 0, 4$, and 8. The strengths of the two-body interaction in the $s = 0$, $s = 1$, and $s = 2$ channels are chosen to be $\lambda_0 = \lambda_1 = \lambda_2 = 0.2$. Note that, in the eigenvalue density $\rho^{m,S}(\hat{E})$, the $(2S+1)$ degeneracy of the levels is not counted. The eigenvalue densities are compared to Gaussian (red) and Edgeworth (ED) corrected Gaussian (green) forms. The ensemble averaged values of skewness (γ_1) and excess (γ_2) parameters are shown in figure. In the plots, the eigenvalue densities, for a given spin S, are normalized to dimension d(m,S)

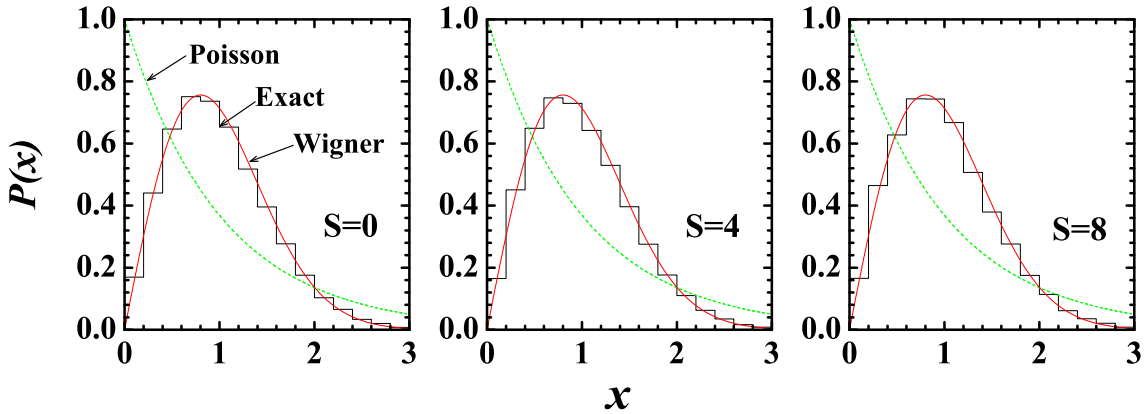

Figure 4.3: Ensemble averaged NNSD histogram for a 100 member BEGOE(1 + 2)-$S1$ with $m = 8$ and $\Omega = 4$ for two body interaction strength $\lambda_0 = \lambda_1 = \lambda_2 = 0.2$. Results are shown for the spin values $S = 0, 4$, and 8. Here, x is in the units of local mean spacing. Results are compared with Poisson and GOE (Wigner)forms. It is seen that in thr strong interaction limit $\lambda = 0.2$, the NNSDs are invariant of the spin value and has a consistent GOE nature.

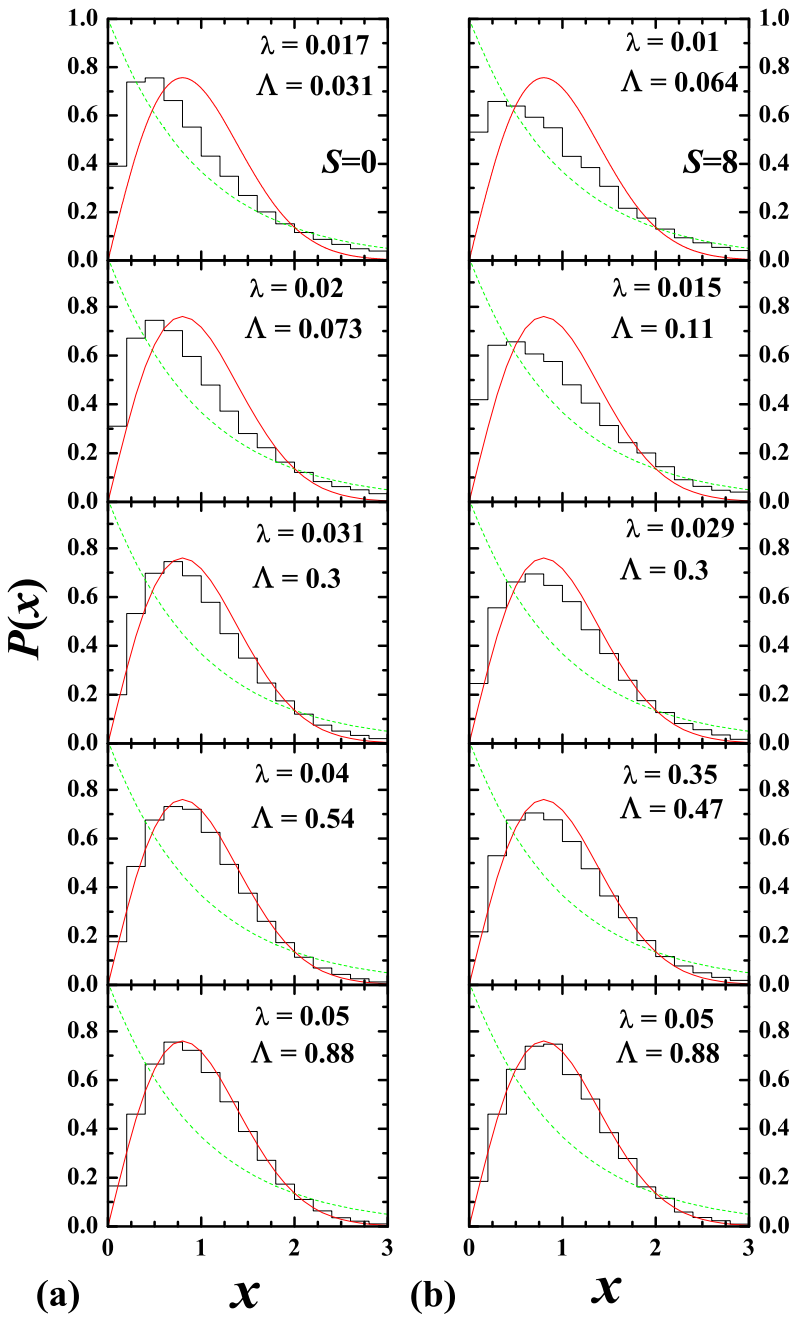

Figure 4.4: Ensemble averaged NNSD, for various values of λ, with $\Omega = 4$,$m = 8$ using 100 member BEGOE(1 + 2)-$S1$ ensemble. Results are shown for extreme spins: (a) $S = S_{min} = 0$ and (b) $S = S_{max} = 8$. Calculated NNSD are compared to the Poisson (green dashed lines) and Wigner (GOE) (red smooth lines) forms. Values of the interaction strength λ and the transition parameter Λ are given in the figure. The values of Λ are deduced as discussed in [39]. The chaos marker λ_C corresponds to $\Lambda = 0.3$ and its values, as shown in the figure, are 0.031 for $S = 0$ and 0.029 for $S = 8$. The bin size is 0.2 for the histograms

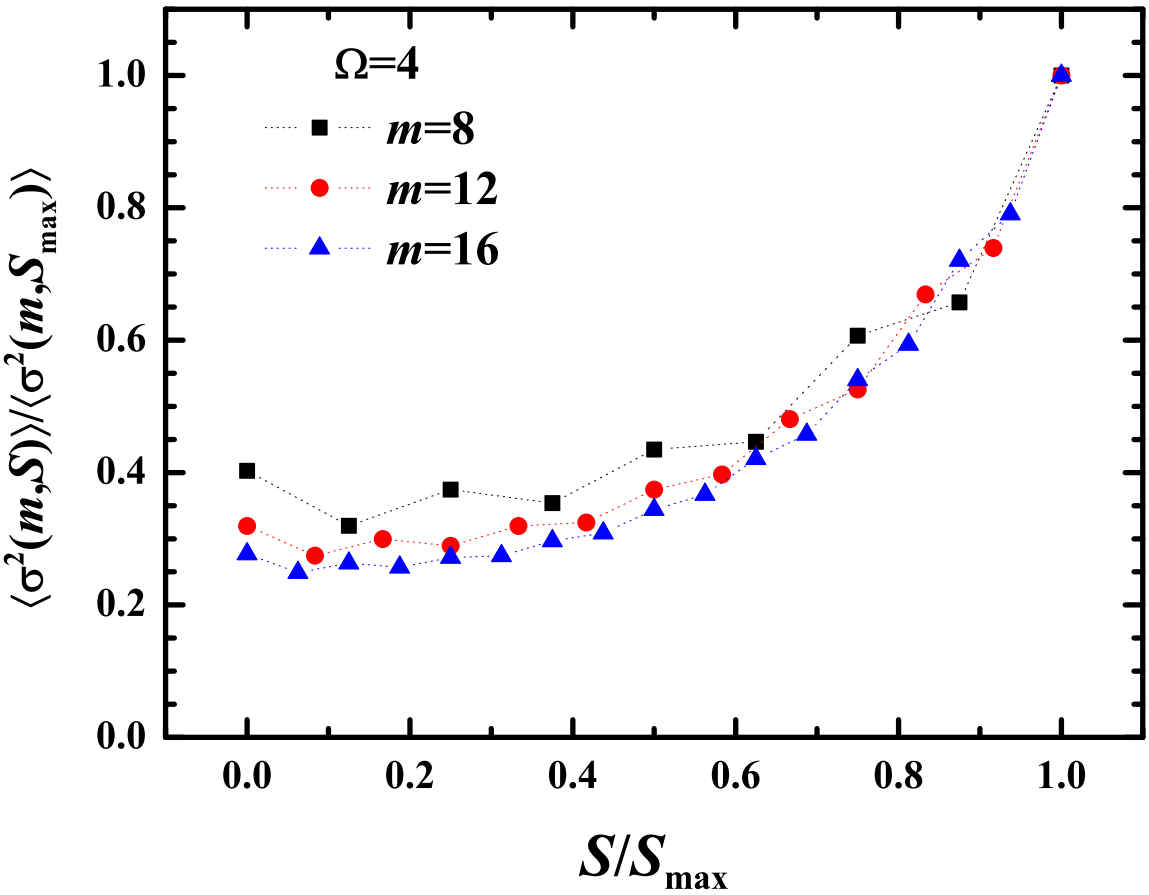

Figure 4.5: Ensemble averaged variances $\langle\sigma^2(m,S)\rangle$ / $\langle\sigma^2(m,S_{max})\rangle$ vs S/S_{max} for a 200 member BEGOE(2)-$S1$ ensemble with $\Omega = 4$ and $m = 8, 12$, and 16

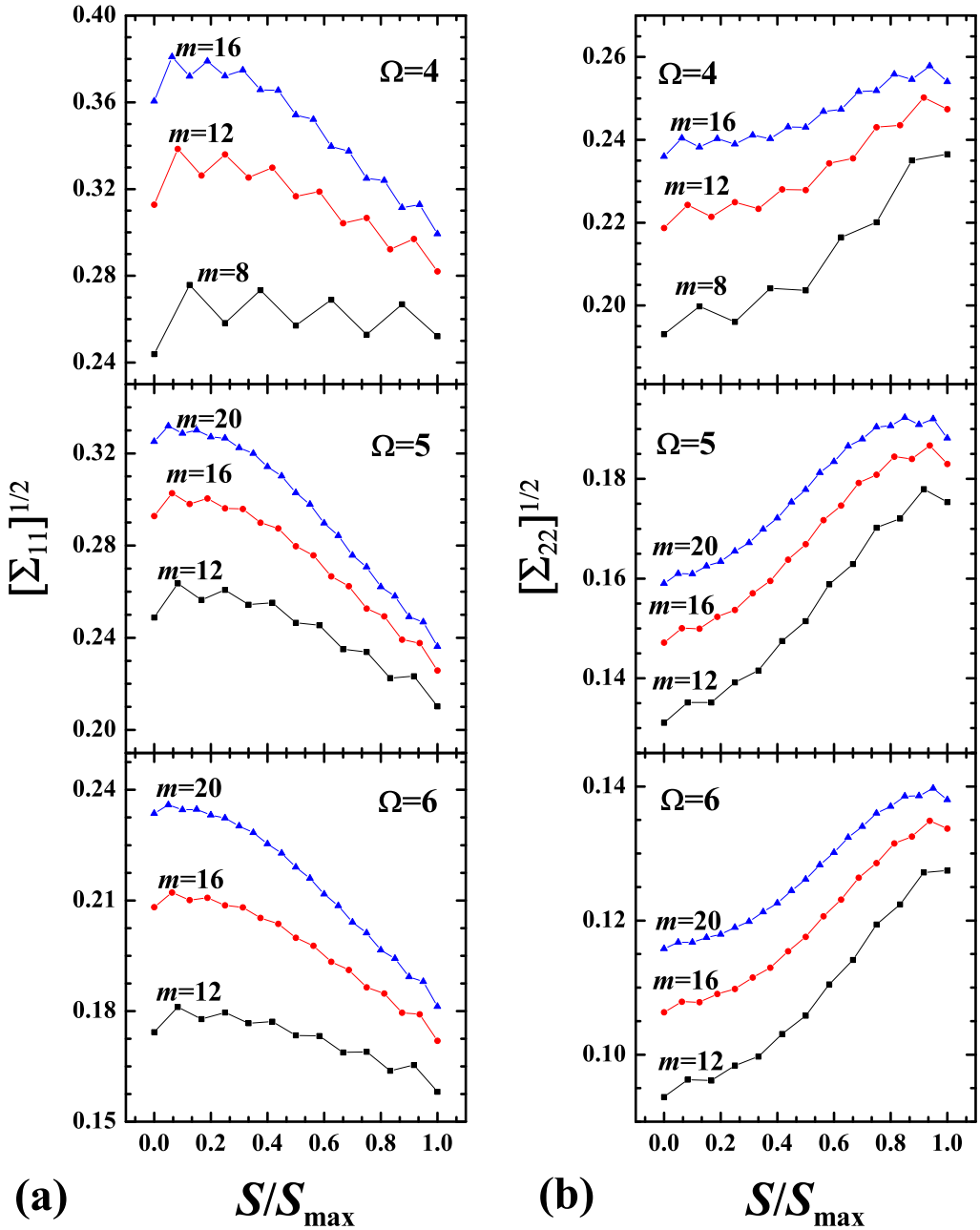

Figure 4.6: Normalized self-correlations in energy centroids - $[\Sigma_{11}]^{1/2}$ and in spectral variances - $[\Sigma_{22}]^{1/2}$ as a function of spin S for 200 member BEGOE(2)-$S1$ ensemble for various values of m and Ω.

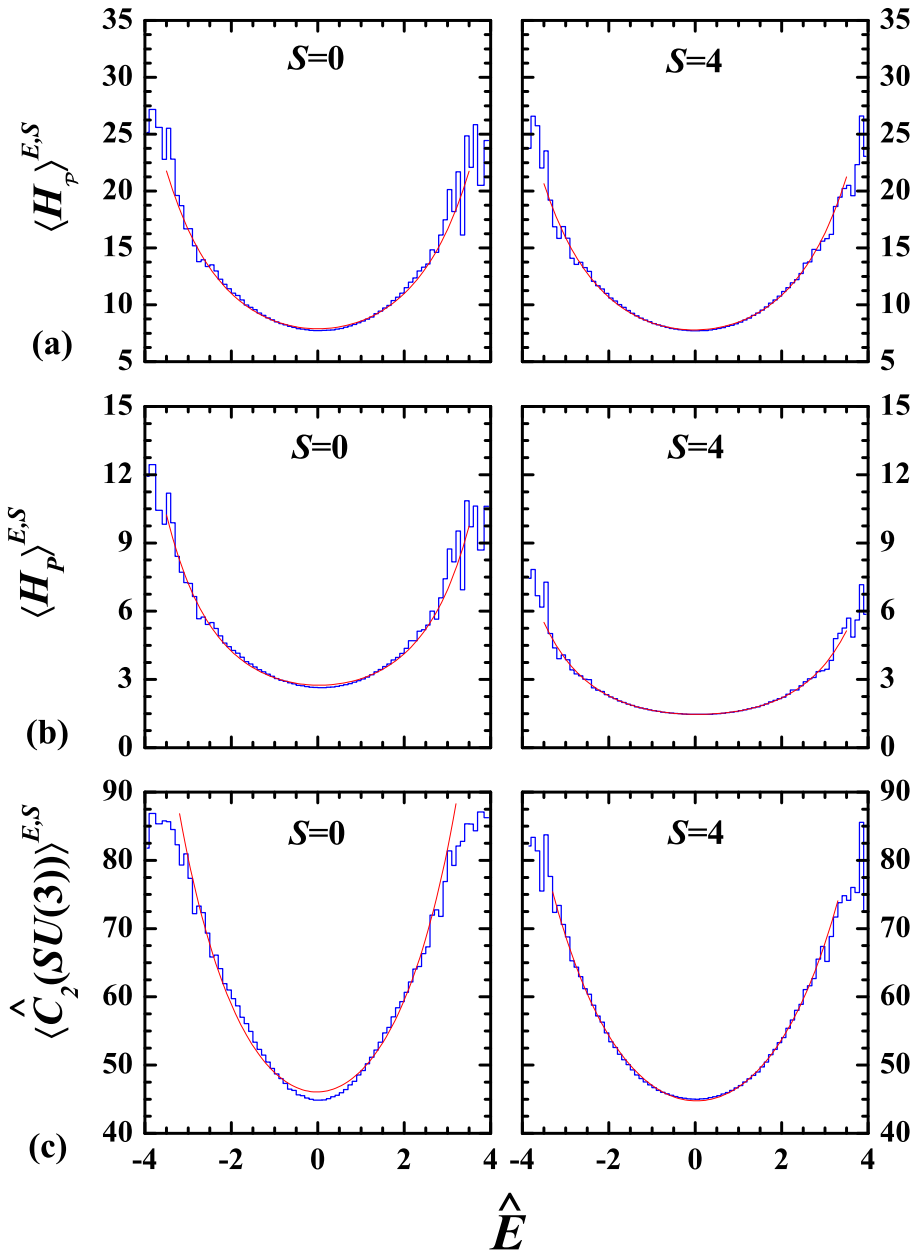

Figure 4.7: Expectation values of the two pairing Hamiltonians and $\hat{C}_2(SU(3))$ vs $\hat{E}$ for a 100 member BEGOE(1+2) systems with H defined by Eq. 4.2.4 and ($\Omega = 4, m = 8$). Results are shown for spins $S = 0$ and $S = 4$. (a) expectation values of $H_{\mathcal{P}}$, (b) expectation values of H_P and (c) expectation value of $\hat{C}_2(SU(3))$. Ensemble averaged results are shown by histograms while continuous curves are due to ratio of Gaussians given by EGOE theory [41]