Chapter 3

BEGOE(1+2)-F: Embedded bosonic ensemble with fictitious spin

3.1 Introduction

The random matrix ensembles, generally referred as Gaussian ensembles for m-particle system, involve interactions up to m-body and are dominated by the m-body interactions. In real many-particle systems such as nuclei and atoms, the Hamiltonian consists of a mean field one-body part defined by finite set of single particle states plus complexity generating two-body part. The concept of embedded ensemble (EE) introduced by French and co-workers takes care of this fact. In two-body EE, the basic two-body Hamiltonian is represented by one of the classical Gaussian ensemble and then many body matrix elements are generated using geometry of the m-particle space [21]. Initially the EE (in particular EGOE(1+2)- one plus two-body random Embedded Gaussian Orthogonal Ensemble) were used for interacting finite fermion systems and calculations done are compared with the results for the dilute limit ($m \to \infty$, $N \to \infty$, and $m/N \to 0$, where N is the number of single particle

states and m is the number of particles) of the embedded ensemble theory. For more detail study on fermionic EGOE(1+2) see [29].

On the other hand, for the finite interacting Boson system, there are only a few model and EE investigations reported in the literature. Alhassid and others [9, 10, 11, 12] employing the interacting boson model (IBM) of atomic nuclei and Borgonovi et al. [7, 8] using a simple symmetrized two coupled rotors model investigated the nature of level fluctuations, occupation numbers etc. in Boson systems. Similarly Bijker and Frank [13] employing an ensemble of one- and two- body interactions preserving angular momentum in the *sd*-boson space of IBM, investigated the probability for the occurrence of ordered structures vibrations and rotations in nuclei. Interest in the Bosonic systems is further enhanced due to the experimental study of atomic Bose-Einstein condensation [18].

In dilute limit both the Fermion as well as Boson systems are expected to behave similarly. However in addition to the dilute limit, dense limit ($m \to \infty$, $N \to \infty$, and $m/N \to \infty$) is feasible only for Bosons and therefore it is more interesting. EGOE(1+2) for spinless dense Bose systems are denoted by BEGOE(1+2) [18].

Some of the generic results obtained for BEGOE(1+2) in the dense limit are surmise as follows. The study of analytic expressions for lower order moments of eigenvalue distribution [15] suggests that for dense boson systems, 2-body part of the interaction dominates over the 1-body part of the interaction. In addition to this, the different statistical properties based on eigenvalues as well as eigenfunctions have also been studied as a function of λ, the two-body interaction strength with a non-degenerate but fixed single particle energies ε_i. It has been shown that for dense boson systems, as λ, the strength of the 2- body interaction in BEGOE(1+2) is increased, the level fluctuations exhibit transition from Poisson to GOE. The value of the critical strength parameter, λ_C, is obtained using the width of the nearest neighbour spacing distribution (NNSD) [30] and the critical strength parameter variation seems to agree with the JS criterion [31]. There exists another transition point, beyond λ_C, for the strength of the 2-body part, which can be obtained using the information entropy,

occupancy entropy and NPC variation; the transition corresponds to shift of the strength function from BreitWigner form to Gaussian form at $\lambda = \lambda_F >> \lambda_C$. The strength function can be represented by the one parameter interpolation of BW form to Gaussian form [32]. The interpolated strength function is used to calculate the occupancy variation and it is seen to agree with exact results for all values of $\lambda \geq \lambda_C$. As in fermionic EGOE(1 + 2),the finite dense interacting boson systems also generate a third chaos marker $\lambda = \lambda_t$, a point or a region where different definitions of entropy, temperature, specific heat and other thermodynamic variables give the same results, i.e. where thermalization occurs [33]. Embedded ensembles are usually studied using only 2-body interactions, i.e., by allowing the single particle energies to be degenerate. Some of the generic results obtained for BEGOE(2) in the dense limit are: (i) The state density for dense boson systems is a Gaussian [15, 16, 17]; (ii) the fluctuations are of GOE type; (iii) the state density exhibit average-fluctuation separation, with smoothed density being a corrected Gaussian, The nature of this separation is understood analytically using the binary correlation approximation [17]; (iv) the Bosonic ensembles tend to be ergodic in the dense limit. With the finite number of single-particle states, the fluctuations do not vanish for $m \to \infty$; however the magnitude of fluctuations goes on decreasing with increasing N [34]. This is derived by calculating the first four moments (the centroid, the variance, the skewness and the excess), using 5000 Bosons and varying single particle states $N = 2 - 12$, of the eigenvalue-distribution for BEGOE(2) using the trace propagation equations given in the appendix of [15].

Embedded ensembles both for spin-less Fermion, EGOE(1+2), and for spin-less Bosons ensembles, BEGOE(1+2), do not have extra parameters except m, N and λ. In addition to mean field and two-body interactions, realistic quantum systems have additional symmetries. Recently, the GOE version, the embedded Gaussian orthogonal ensemble of one plus two-body interactions with spin degree of freedom, denoted by EGOE(1+2)-**s**, has received considerable attention [35, 36, 37, 38, 39]. Going beyond the embedded ensembles for spinless boson systems, a model for Boson

systems with additional symmetry was defined, a BEGOE for two species boson systems, by assigning a spin $f = \frac{1}{2}$ degree of freedom to each Boson, i.e. embedded Gaussian orthogonal two-body random matrix ensemble for Bosons with good spin $f = \frac{1}{2}$ [called BEGOE(1+2)-F] was first introduced by Chavda et al [40] and spectral properties of BEGOE(1+2)-F were studied in detail using numerical and analytical methods in [41]. Here the spin may be real or fictitious. This ensemble is useful in obtaining several physical conclusions. The two-species boson systems are used as a model in nuclear structure physics [42]. The two species here are proton and neutron bosons, and the corresponding model is called the protonneutron interacting boson model (pnIBM). Here the bosons, in addition to F-spin, also carry angular momenta $0\hbar$ and $2\hbar$. There is good evidence that heavy nuclei preserve F-spin [43, 44, 45]. In the present chapter spectral properties of BEGOE(1+2)-F are studied. In the present study, the one body part is defined by random single particle energies unlike in [ ]. Here, the construction of m-particle BEGOE(1+2)-F hamiltonian in m basis reduces to the problem of BEGOE(1+2) for spinless boson systems. For completeness, the Construction of Hamiltonian for spinless BEGOE(1+2) is describe in the next section.

3.2 Construction of Hamiltonian for spinless BEGOE(1+2)

Consider spinless m-interacting Bosons in N single particle states $|i\rangle$, $i = 1, \ldots, N$ with single particle energies ε_i and interacting via one- plus two-body interaction. The number of states in m particle system corresponding to the space (N, m) is $d(N, m) = \binom{N+m-1}{m}$. For example $d(4, 11) = 364$, $d(5, 10) = 1001$, $d(6, 12) = 6188$ and $d(10, 20) = 10015005$. A basis state in m-particle space, due to product nature of the states in occupation number representation, is denoted by $\left|\prod_{i=1}^{N} m_i\right\rangle = |m_1, m_2, \ldots, m_N\rangle$, where m_i is the number of Bosons in i'th single particle state. Here, we have $\sum m_i = m$. The m-particle Hamiltonian, $H(m)$, is a matrix with dimensionality $d(N, m)$. The one-body part of the interaction $h(1)$, described by the single particle energies

ε_i, is given by, $h(1) = \sum_{i=1}^{N} \varepsilon_i b_i^\dagger b_i = \sum_{i=1}^{N} \varepsilon_i n_i$, where $b_i^\dagger$ and b_i are the Boson-creation operator and the Boson-destruction operator respectively for single particle state $|i\rangle$. The operator $n_i = b_i^\dagger b_i$ is the number operator of the i'th single-particle state. The operators $b_i^\dagger$ and b_i, obey the commutation rules given by $\left[b_i, b_j^\dagger\right] = \hat{\delta}_{ij}$ and $\left[b_i, b_j\right] = \left[b_i^\dagger, b_j^\dagger\right] = \hat{0}$. The two-body hamiltonian $V(2)$ is defined in terms of the matrix elements in the two particle basis states $|i,j\rangle \equiv |m_i = 1, m_j = 1\rangle$ if $i \neq j$ or $|m_i = 2\rangle$ if $i = j$,

$$V(2) = \sum_{\substack{i \leq j \\ k \leq l}} \frac{\langle ij|\, V(2)|\, kl\rangle_s}{\sqrt{1+\delta_{ij}}\sqrt{1+\delta_{kl}}} b_i^\dagger b_j^\dagger b_k b_l = \frac{1}{4} \sum V_{ijkl}\, b_i^\dagger b_j^\dagger b_k b_l \;, \tag{3.2.1}$$

where the symmetrized two-body matrix elements (TBME) $\langle ij|\, V(2)|\, kl\rangle_s$ have following symmetries;

$$\langle ij|\, V(2)|\, kl\rangle_f = \langle kl|\, V(2)|\, ij\rangle_f = \langle ji|\, V(2)|\, lk\rangle_f = \langle ij|\, V(2)|\, lk\rangle_f. \tag{3.2.2}$$

The form of the Hamiltonian with $\langle ij|\, V(2)|\, kl\rangle_f$ is used for generating the m-particle Hamiltonian, while the V_{ijkl} type matrix elements, which are related to the two-body matrix elements $\langle ij|V(2)|kl\rangle$ by $V_{ijkl} = \langle ij|\, V(2)|\, kl\rangle_f \sqrt{1+\delta_{ij}}\sqrt{1+\delta_{kl}}$, are generally used for calculation of moments of eigenvalue distribution and for evaluating traces of various operators products. The number of independent TBME are $ime(N) = d_2(d_2+1)/2$ with two-particle space dimension $d_2 = d(N,2) = N(N+1)/2$; for example with $N = 4$, $ime(4) = 55$ and with $N = 5$, $ime(5) = 120$. The m-particle Hamiltonian matrix $H(m)$ is then defined by m particle matrix elements $\langle m'_1, m'_2, \ldots, m'_N|H|m_1, m_2, \ldots, m_N\rangle$. Many of the matrix elements of $H(m)$ for $m > 2$ are zero due to two-body nature of the interaction; particularly the matrix elements with $|m_i - m'_i| > 2$ and $\sum_{i=1}^{N} |m_i - m'_i| > 4$ are identically zero. The non-zero matrix elements of $H(m)$ linear combinations of the single particle energies and the TBMEs,

$$\begin{aligned} &\langle m'_1, m'_2, \ldots, m'_N|H|m_1, m_2, \ldots, m_N\rangle \\ &= \sum_i \varepsilon_i \left\langle \Pi m'_p|b_i^\dagger b_i|\Pi m_p \right\rangle + \lambda \sum_{i\leq j,\ k\leq l} \frac{\langle ij|V(2)|kl\rangle}{\sqrt{1+\delta_{ij}}\sqrt{1+\delta_{kl}}} \left\langle \Pi m'_p|b_i^\dagger b_j^\dagger b_k b_l|\Pi m_p \right\rangle. \end{aligned} \tag{3.2.3}$$

where $b_i^\dagger|m_i\rangle = \sqrt{m_i+1}|m_i+1\rangle$ and $b_i|m_i\rangle = \sqrt{m_i}|m_i-1\rangle$ and λ is the strength of two-body interaction. The ensemble of $H(m)$ is obtained by taking the defining TBME's as Gaussian random variables with

$$\overline{\langle kl|V(2)|ij\rangle} = 0 \quad \text{and } \overline{|\langle kl|V(2)|ij\rangle|^2} = \upsilon^2\left(1+\delta_{(ij),(kl)}\right) \ , \tag{3.2.4}$$

where the bar indicates an ensemble average. We have used $\upsilon^2 = 1$ throughout our calculations. Thus $V(2)$ is a Gaussian orthogonal ensemble, GOE(2), and $H(m)$ is an embedded Gaussian orthogonal ensemble of two-body interaction with strength λ plus the single particle hamiltonian $h(1)$.

3.3 Definition and Construction of BEGOE(1+2)-F

Consider a system of $m(m>2)$ interacting bosons distributed in Ω number of single particle states. As stated earlier we have considered a fictitious spin quantum number of $f = 1/2$ associated with each boson hence our single particle orbits are actually doubly degenerate i.e. total number of orbitals are $N = 2\Omega$. They are denoted as $|i, m_f = \pm 1/2\rangle = b^\dagger_{i,m_f}|0\rangle$ with $i = 1, 2, 3, \Omega$, as followed by the EGOE(1+2)-s model [36]. And the two particle symmetric states are denoted as $|(ij)f, m_f\rangle_{symm} = (b^\dagger_{i,}b^\dagger_j)^f_{m_f}|0\rangle$ with $f = 0$ and $f = 1$ [46]. The total dimensionality of the two particle space with $m_f = 0$ is Ω^2 which is same in the case of EGOE(2)-s but the matrix dimension of $f = 0$ and $f = 1$ is $\Omega(\Omega-1)/2$ and $\Omega(\Omega+1)/2$ respectively.

Now the Hamiltonian of the system and its construction is explored as follows. The Hamiltonian here is a combination of the mean field part and the two body random matrix ensemble and hence called as the one plus two body Hamiltonian where the spin is preserved. The one body Hamiltonian is defined as:

$$h(1) = \sum_{i=1}^{\Omega} \epsilon_i \hat{n}_i \tag{3.3.1}$$

where i are the orbits which are doubly degenerate, n_i are the number operators and ϵ_i are the single particle energies.
Similarly, a two-body Hamiltonian operator $\widehat{V}(2)$ preserving F-spin is given by,

$$\widehat{V}(2) = \sum_{i,j,k,l;f,m_f}{}' \frac{V^f_{ijkl}}{\sqrt{(1+\delta_{ij})(1+\delta_{kl})}} (b^\dagger_i b^\dagger_j)^f_{m_f} [(b^\dagger_k b^\dagger_l)^f_{m_f}]^\dagger \tag{3.3.2}$$

The 'prime' over the summation symbol in the above Eq. (3.3.2) indicates that the summation over i, j, k and l is restricted to $i \geq j$ and $k \geq l$ for $f = 1$ and $i > j$ and $k > l$ for $f = 0$.

The two body symmetrized matrix elements (with respect to the total orbital spin space) $\widehat{V}(2)$ can be defined as under:

$$V(2)^f_{ijkl} = {}_{symm}\langle (kl)f, m_f | (ij)f, m_f \rangle_{symm} \tag{3.3.3}$$

They are independent of the m_f quantum number which confirms the spin preserving nature of $\widehat{V}(2)$. It is important to note that the two body Hamiltonian $\widehat{V}(2)$ includes contribution from both the spins (here) $f = 0$ and $f = 1$ which can be seen from the Eq.(3.3.2). Thus we can write like $\{V(2)\} = \{V(2)^{f=0}\} + \{V(2)^{f=1}\}$. The $\{\}$ bracket denotes an ensemble whereas the 'sum' here is actually the 'direct sum'. From this, we can construct a schematic diagram of the two body Hamiltonian $\widehat{V}(2)$ Fig. (3.1) which consists of a 2×2 block matrix or rather two diagonal blocks that correspond to $f = 0$ and $f = 1$, whereas he off-diagonal blocks represent the zero values. Hence one can say that the matrix is a direct sum of $f = 0$ and $f = 1$ matrices.

Now, by defining these two parts of the two-body Hamiltonian to be independent GOEs in the 2-particle space, BEGOE(2)-F for a given (m, F) system is generated. On the basis of this, the two-body Hamiltonian $\{V(2)\}$ is then further propagated to the m-particle spaces with a given spin F by using the geometry (direct product structure) of the m-particle space. This propagation up to $m = 10$ is also shown in a block form in Fig. (3.1).

Thus from (3.3.1) and (3.3.3) the final Hamiltonian for BEGOE(1+2)-F,

$$H_{BEGOE(1+2)-F} = h(1) + \lambda_0\{V(2)^{f=0}\} + \lambda_1\{V(2)^{f=1}\} \tag{3.3.4}$$

Here, $\{V(2)^{f=0}\}$ and $\{V(2)^{f=1}\}$ are GOEs with unit variance. The two body interaction strengths λ_0 and λ_1 are associated with the spin 0 and 1 part respectively. While doing the numerical calculations, it is this interaction strength which is varied and a transition is observed from Poisson to Wigner (GOE). Moreover, the mean field part $(h)1$ in the above equation is defined by single particle energies ε_i with average spacing Δ. Without loss of generality, we put the value of $\Delta = 1$ so that the values of interaction strengths (λ_f) are in the units of Δ. In principle, many other choices for the sp energies are possible. Thus our present ensemble BEGOE(1+2)-F solely depends upon these parameters (i) doubly degenerate orbitals (Ω) (ii) Number of particles (bosons here) (m) (iii) Spin (F) (iv) The values of interaction strength $(\lambda_0$ and $\lambda_1)$. Now we are in a position to define the dimensionality $d(m, F)$ of our system which varies accordingly for different configuration of (m, F). It is the number of states in (m, F) space which have $(2F + 1)$-fold degeneracy

The dimensionality can be expressed as below:

$$d(m,F) = \frac{(2F+1)}{(\Omega-1)} \begin{pmatrix} \Omega + (m/2+F+1) - 2 \\ m/2+F+1 \end{pmatrix} \begin{pmatrix} \Omega + (m/2-F) - 2 \\ m/2-F \end{pmatrix} \tag{3.3.5}$$

It can be verified that the above expression for dimensionality satisfies the sum rule $\Sigma_F(2F + 1)\ d(m, F) = \binom{N+m-1}{m}$. For example, for a system of $m = 10$ bosons possessing spin $f = 1/2$ each, and there are $\Omega = 5$ single particle states then the dimensions are 1176, 3150, 4125, 3850, 2574 and 1001 for spins $F = 0$, 1, 2, 3, 4 and 5 respectively. Taking another configuration say e.g. $m = 6$ distributed in $\Omega = 3$ we get the dimensionality as 10, 27, 35 and 28 for $F = 0$, 1, 2 and 3 respectively.

Now, its time to discuss about the construction of the Hamiltonian matrix whose energy eigenvalue spectrum is to be further analyzed. Here, for a given (m, F) the single particle states are arranged in such a manner that, those states for which $m_f = 1/2$ buildup the first half of the total Ω states whereas the remaining states have $m_f = -1/2$. Thus a state say r can be in general defined as for $r \leq \Omega$, $|r\rangle = |i = r, m_f = 1/2\rangle$ and for for $r > \Omega$, $|r\rangle = |i = r - \Omega, m_f = -1/2\rangle$. Also, with respect to the the product nature of the states, the m-particle configurations can be defined in terms of the occupation number as

$$\left| \prod_{r=1}^{N=2\Omega} m_r \right\rangle = |m_1, m_2, \ldots, m_\Omega, m_{\Omega+1}, \ldots, m_{2\Omega}\rangle \,, \tag{3.3.6}$$

where m_r can take values 0 or 1 or ...or $m/2$ for even m (and $m/2+1$ for odd m) with $\sum_r^N m_r = m$. The corresponding M_F values are, $M_F = \frac{1}{2}\left[\sum_{r=1}^{\Omega} m_r - \sum_{r'=\Omega+1}^{2\Omega} m_{r'}\right]$. One thing should be marked here that, the configurations with $M_F = 0$ will contain states with all F values for even m whereas configurations with $M_F = \frac{1}{2}$ are for odd m. For the present model, we have with us $m = 10$ (even) number of bosons thus we shall deal with configurations for which $M_F = 0$ and thus we construct the m particle Hamiltonian matrix using the basis defined by configurations with M_F=0. For this matrix, the dimension of its basis space is obtained by adding the dimensionalities of all the respective spins (F) for a given set of (Ω, m). Hence from Eq.(3.3.5) discussed above, one can calculate the value of $\sum_F d(m, F)$. For example, for $(m = 10, \Omega = 4)$ we have $\sum_F d(m, F)$=3136, for $(m = 12, \Omega = 4)$ we have $\sum_F d(m, F)$=7056, for $(m = 10, \Omega = 5)$ we have $\sum_F d(m, F)$=15876.

At this stage, the given (1+2)-body Hamiltonian defined by $(\varepsilon_i, V(2)_{ijkl}^{f=0,1})$ is converted to the $\left|i, m_f = \pm\frac{1}{2}\right\rangle$ basis. This can be done by changing ε_i to ε_r with the index r defined as above and changing $V(2)_{ijkl}^{f=0,1}$ to $V_{im_i,jm_j,km_k,lm_l} = \langle im_i, jm_j|\, V(2)|\, km_k, lm_l\rangle$. This procedure is done by using the following equations (as stated in Eq.(4) of [36]):

$$\begin{aligned} V_{i\frac{1}{2},j\frac{1}{2},k\frac{1}{2},l\frac{1}{2}} &= V_{ijkl}^{f=1} \,; \\ V_{i-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2},l-\frac{1}{2}} &= V_{ijkl}^{f=1} \,, \\ V_{i\frac{1}{2},j-\frac{1}{2},k\frac{1}{2},l-\frac{1}{2}} &= \frac{\sqrt{(1+\delta_{ij})(1+\delta_{kl})}}{2}\{V_{ijkl}^{f=1} + V_{ijkl}^{f=0}\} \,. \end{aligned} \tag{3.3.7}$$

All other matrix elements will be zero except for the following symmetries,

$$\begin{aligned} V_{im_i,jm_j,km_k,lm_l} &= V_{km_k,lm_l,im_i,jm_j} \\ V_{jm_j,im_i,lm_l,km_k} &= V_{im_i,jm_j,lm_l,km_k}. \end{aligned} \tag{3.3.8}$$

Using $(\varepsilon_r, V_{im_i,jm_j,km_k,lm_l})$'s, construction of the m particle H matrix in the basis defined by Eq. (3.3.6) reduces to the problem of BEGOE$(1+2)$ for spinless Boson systems. One can obtain the (m, F) matrices from the (m, M_F) matrices just by

projecting spin F using the F^2 operator. For the F^2 operator, it is easy to recognize that $\varepsilon_i = 3/4$ independent of i, $V(2)^{f=0}_{ijij} = -3/2$ and $V(2)^{f=1}_{ijij} = 1/2$ independent of (ij) and all other matrix elements are zero. Previously, such a procedure was done for the fermionic systems with spin degree of freedom. Alternatively, it is possible to construct the H matrix directly in a good F basis using angular-momentum algebra. But it is noted that, for BEGOE(1 + 2)-F only the M_F representation is used for constructing the H matrices [41]. Hence for the present investigation, considering the above terms for the F^2 operator, the m particle matrix with $M_F = 0$ for even m ($M_F = \frac{1}{2}$ for odd m) is constructed and diagonalized. This gives a direct sum of unitary matrices and the unitary matrix that corresponds to a given F is identified by the eigenvalue $F(F + 1)$. Applying the unitary transformation defined by this unitary matrix, the m particle H matrix with $M_F = 0$ for even m ($M_F = \frac{1}{2}$ for odd m) is transformed to the basis with good F values.

Using the above methodology the numerical construction of the Hamiltonian matrix is done for a 50 member BEGOE(1+2)-F with $(m = 10, \Omega = 4)$. The single particle energies ε_i are taken as independent gaussian random variables with mean equal to i and variance equal to $1/2$. Further, the eigenvalues are obtained by proper diagonalization of the Hamiltonian matrix. Moreover, for the present study we choose the value of interaction strength for the two spin values $F = 0$ and $F = 1$ to be equal i.e. $\lambda = \lambda_0 = \lambda_1$. Now we shall study the spectral properties. The eigenvalue density is constructed by making the centroid of all the members of the ensemble to be zero and variance equal to unity. The smooth density here being a corrected Gaussian with corrections involving up to 6^{th} order moments of the density function [47], the spectra of each member is unfolded using procedure in [17] and then NNSD and Δ_3 are calculated.

3.4 Spectral Analysis

3.4.1 Eigenvalue density for BEGOE(2)-F

Using BEGOE(2)-F ensemble defined by $h(1) = 0$, we have constructed the fixed-F density of states, $\rho^{m,F}(E)$, for the system ($m = 10, \Omega = 4$) for all values of F ($F_{max} = m/2$) and they are found to be close to Gaussian as shown in Fig. (3.2). Here in each calculation 50 member ensemble is used. In construction of the eigenvalue density, the spectra of each member of the ensemble is first zero centered and scaled to unit width and then histograms are generated using a bin-size of 0.2. The strengths of the two-body interaction in the $f = 0$ and $f = 1$ channels are chosen to be equal i.e. $\lambda_0 = \lambda_1$. Results are shown for all F values. The ensemble-averaged values of skewness (γ_1) is found to be very close to zero for all F values. The ensemble-averaged values of the excess parameter (γ_2) are also shown in the figure. In the plots, the state densities, for a given spin F, are normalized to dimension $d(m, F)$. The BEGOE(2) results are compared with Gaussian and Edgeworth(ED)(using Eq.(C.2) of Ref.[29]) corrected Gaussian. The ensemble averaged values of skewness(γ_1) and excess(γ_2) parameters are as shown in figure. It is seen that the agreement between exact and ED is very well. In the past, it has been shown that the Gaussian form is generic for embedded ensembles of spinless Fermion and Boson systems [29, 34]. Recently, using Fermion EGOE(1+2)-**s**, it is shown that fixed-(m, S), ensemble averaged state density takes Gaussian form [36, 39]. Hence from the results obtained for BEGOE(2)-F, it can be concluded that the Gaussian form is also generic for the ensembles with good quantum numbers.

One of the most significant feature of embedded ensembles for interacting particle systems is the separation of information regarding energy levels (also for other observables) into averages and fluctuations. Using the binary correlation approximation, the nature of separation is understood analytically both for fermion systems in the dilute limit with EGOE(2) [19, 21] as well as for boson systems in the dense limit with BEGOE(2) [17]. Numerical examples are also given in [17, 48]. Most recently,

it is demonstrated, using Periodogram analysis for various embedded random matrix ensembles EGOE(1+2), EGOE(1+2)-**s**, BEGOE(1+2) and a Shell model example, that interacting particle systems exhibit a well defined averagefluctuation separation [49]. Similar generic behavior is also expected in case of BEGOE(2)-F. In the further sections, the numerical results for the spacing statistics for BEGOE(1+2)-F and average - fluctuation separation in energy levels for BEGOE(2)-F ensemble are presented.

3.4.2 Average-fluctuation separation in BEGOE(2)-F

The average-fluctuation separation in BEGOE(2)-F is obtained numerically using the method given in [19]. Given a normalized state density $\rho(E)$, it is possible to expand it in terms of its smoothed form being gaussian $\overline{\rho(E)} = \rho_G(E) = \sqrt{2\pi\sigma^2}\exp\left(-\left(E-\epsilon\right)^2/(2\sigma^2)\right)$, using the Gram-Charlier expansion (GC) we have,

$$\rho(E) = \rho_G(E)\left\{1 + \sum_{\zeta\geq 3}\left(\zeta!\right)^{-1} S_\zeta He_\zeta(\hat{E})\right\} . \tag{3.4.1}$$

In above Eq. $\hat{E} = (E-\epsilon)/\sigma$ is the standardized energy variable. The centroid $\epsilon = \langle H\rangle^m$ and the variance $\sigma^2 = \langle H^2\rangle^m - \epsilon^2$ of the Gaussian $\rho_G(E)$ are the same as that of $\rho(E)$. He_ζ are Hermite polynomials and S_ζ are, in principle, related to higher moments of the state density $\rho(E)$. Here Eq. 3.4.1 can be applied to BEGOE(2)-F by noting that $\overline{\rho(E)} = \rho_G(E)$ for all spin (F) values. Since S_ζ's change from member to member of ensemble, one can treat them as independent zero-centered random variables,

$$\overline{S_\zeta} = 0, \;\; \overline{S_\zeta S_{\zeta'}} = 0 \text{ for } \;\; \zeta \neq \zeta'. \tag{3.4.2}$$

This is consistent with the result $\overline{\rho(E)} = \rho_G(E)$ where the 'bar' denotes an ensemble average. Each ζ term in Eq. 3.4.1 represents an excitation 'mode' and the wavelength of the modes is proportional to ζ^{-1}. The distribution function $F(E)$, the integrated version of $\rho(E)$, is given by $F(E) = d\int_{-\infty}^{E}\rho(E')dE'$ where d is the dimensionality. Deviation of a given level with energy E from its smoothed (with respect to the

ensemble) counterpart $\overline{E}$ gives the level motion. In terms of $F(E)$ and the local mean spacing $\overline{D(E)}$, we have $\delta E = E - \overline{E} = [F(E) - \overline{F(E)}]\overline{D(E)}$. Then the variance of the level motion is given by the ensemble average of $(\delta E)^2/\overline{D(E)}^2$. Using Eq. 3.4.1 and adding centroid and variance fluctuations, we have

$$\overline{(\delta E)^2/\overline{D(E)}^2} = \overline{[F(E) - \overline{F(E)}]} = d^2\sigma^2[\rho_G(E)]^2 \left\{ \sum_{\zeta \geq 1} (\zeta!)^{-2}\, \overline{S_\zeta^2}[He_{\zeta-1}(\hat{E})]^2 \right\} . \tag{3.4.3}$$

Hence $\overline{S_\zeta^2}$'s are needed for embedded ensembles and they are related to covariances $\overline{\langle H\rangle^p \langle H\rangle^q} - \overline{\langle H\rangle^p}\ \overline{\langle H\rangle^q}$. For binary correlation approximation results, see refs.[19, 21] for fermion systems in dilute limit and ref.[17] for boson systems in dense limit.

In order to study average-fluctuation separation in energy levels, we consider BEGOE(2) for $m = 10$ bosons and $\Omega = 4$ single particle levels with various m particle spins F. Using the fixed-F eigenvalues, obtained from matrix diagonalization, one can calculate the exact distribution function $F(E)$. Also the smooth distribution function $\overline{F(E)}$ for a given order ζ_0 is obtained from the truncated Gram-Charlier expansion for the smooth density of states $\overline{\rho(E)}$,

$$\overline{\rho(E)} = \rho_G(E) \left\{ 1 + \sum_{\zeta \geq 3}^{\zeta_0} (\zeta!)^{-1}\, S_\zeta He_\zeta(\hat{E}) \right\} . \tag{3.4.4}$$

Now $\Delta(E) = F(E) - \overline{F(E)}$. In order to compare the exact staircase function $F(E)$ with a continuous one, we require that the continuous function $\overline{F(E)}$ pass through the mid-points of the jumps in the discrete function. Therefore $F(E_i) = (i - 1/2)$ for the i'th eigenvalue and this is employed in the analysis just as in [17, 48]. As the spectrum is discrete, the S_ζ's in Eq. (3.4.4) are determined, for a given ζ_0, by minimizing $\sum_i^d \{\Delta(E_i)^2\}$ with respect to the S_ζ's. These optimized S_ζ's, for a given ζ_0 order, give $\Delta(E)$ vs E, i.e. level motion [see Eq. (3)]. In the calculations for given F value, the spectra of each member of the ensemble is first zero centered and scaled to unit width and then ensemble average is carried out. In Figs. 3.3, results are shown for F= 0 and 2, for Gaussian as well as for orders : 3, 4, and 6 corrections to the asymptotic Gaussian density of states. In the ensemble calculations, level motion for

Table 3.1: Root Mean Square deviations, Δ_{RMS}, in $\Delta(\hat{E})$ (averaged over the eigenvalues of all the members of the ensemble) for each F value are as given below. The results for each order are obtained for a 100 member BEGOE(2)-F generated by the system $\Omega = 4, m = 10$.

F	Gaussian	Order 3	Order 4	Order 5	Order 6
0	2.0880	1.2724	0.6960	0.6344	0.6055
1	4.7162	2.5976	1.0145	0.8158	0.7522
2	6.2876	3.1934	1.1619	0.8919	0.7962
3	7.4281	3.6307	1.2990	0.9344	0.8197
4	6.6693	3.3582	1.3158	0.9762	0.8218
5	3.5519	1.8955	0.9272	0.7729	0.6858

each order is obtained for each member separately and the figures show the results from all the members. The root-means quare deviations Δ_{RMS} in $\Delta(\hat{E})$ are also shown in the figures. It is clearly seen that with increasing order the Δ_{RMS} decreases rapidly at first and then changes slowly indicating a sharp separation between the smooth and fluctuating parts of the distribution function. The averaged Δ_{RMS} for all F values upto the 6'th order correction to the asymptotic Gaussian density of states are also shown in Table 3.1. It is clearly seen that the example considered here, $\Delta_{RMS} \sim 1$, for all values of F, by the time 4-6th order corrections are added to ρ_G. As $\Delta_{RMS} \sim 0.88$ for GOE, this implies the onset of GOE fluctuations by the time the 4-6th order correction has been added to the asymptotic Gaussian density.

3.4.3 Fixed-F eigenvalue centroids and spectral variances

For constructing fixed-F Gaussian densities, one needs energy centroid $E_c(m, F) = \langle H \rangle^{m,F}$ and variance $\sigma^2(m, F) = \langle H^2 \rangle^{m,F} - [\langle H \rangle^{m,F}]^2$. These quantities can be calculated for any (Ω, m, F) with $m > 4$ without constructing the H matrix in space (m, F). The construction of H matrix is tested by calculating fixes-F energy centroid and spectral variance. This follows from the fact that simple propagation is possible for $E_c(m, F)$ in terms of corresponding E_c for $m \leq 2$ and for $\sigma^2(m, F)$ in terms of

Table 3.2: Inputs to calculate fixed-F centroids and variances for given Ω

m	F	$\langle H\rangle^{(m,F)}$
1	1/2	$\langle h(1)\rangle^{(1,1/2)}$
2	0,1	$\langle H\rangle^{(2,1)}$, $\langle H\rangle^{(2,0)}$
3	1/2,3/2	$\langle H\rangle^{(3,1/2)}$, $\langle H\rangle^{(3,3/2)}$
4	0,1,2	$\langle H\rangle^{(4,0)}$, $\langle H\rangle^{(4,1)}$, $\langle H\rangle^{(4,2)}$

corresponding σ^2 for $m \leq 4$. Given a one plus two body Hamiltonian H, the fixed-F centroid $E_c(m,F)$ will be a second order polynomial (in particle rank) in m and $F(F+1)$. Then

$$E_c(m,F) = a_0 + a_1 m + a_2 m^2 + a_3 F(F+1). \tag{3.4.5}$$

Solving for the 'a's in terms of the centroids in one- and two-particle spaces, the propagation formula for the eigenvalue centroids is [41],

$$\begin{aligned}
E_c(m,F) &= a_0 + \langle h(1)\rangle^{1,\frac{1}{2}}\ m + \langle V(2)\rangle^{2,0}\ P_1^0(m,F) + \langle V(2)\rangle^{2,1}\ P_1^1(m,F)\ ; \\
P_1^0(m,F) &= [m(m+2)/8 - F(F+1)/2]\ , \quad P_1^1(m,F) = [3m(m-2)/8 + F(F+1)/2]\ , \\
\langle h(1)\rangle^{1,\frac{1}{2}} &= \overline{\epsilon} = (1/\Omega)\ \sum_i \epsilon_i\ , \\
\langle V(2)\rangle^{2,0} &= [\Omega(\Omega-1)/2]^{-1}\ \sum_{i<j} V_{ijij}^{f=0}\ , \quad \langle V(2)\rangle^{2,1} = [\Omega(\Omega+1)/2]^{-1}\ \sum_{i\leq j} V_{ijij}^{f=1}
\end{aligned} \tag{3.4.6}$$

Similarly the spectral variance $\sigma^2(m,F)$ will be a fourth order polynomial in m and $F(F+1)$. This gives

$$\sigma^2(m,F) = \sum_{p=0}^{4} a_p m^p + \sum_{q=0}^{2} b_q m^q F(F+1) + c_0[F\,(F+1)]^2 \tag{3.4.7}$$

. The parameters (a_i, b_i, c_0) follow from $\sigma^2(m,F)$ for $m \leq 4$ and to determine these inputs one has to construct H matrices for m up to 4. Expression for fixed-F variances

is given by [41],

$$\sigma^2_{H=h(1)+V(2)}(m,F) =$$

$$\begin{aligned}
&\frac{(\Omega-2)m(\Omega+m/2)-2\Omega\,F(F+1)}{(\Omega-1)\Omega(\Omega+1)}\ \sum_i\ \tilde{\epsilon}_i^2\\
&+\frac{(\Omega+m/2)\,[m(m+2)-4F(F+1)]}{2(\Omega-1)\Omega(\Omega+1)}\ \sum_i\ \tilde{\epsilon}_i\lambda_{i,i}(0)\\
&+\frac{(\Omega+2)(\Omega+m/2)\,[3m(m-2)+4F(F+1)]-8\Omega(m-1)F(F+1)}{2(\Omega-1)\Omega(\Omega+1)(\Omega+2)}\ \sum_i\ \tilde{\epsilon}_i\lambda_{i,i}(1)\\
&+\frac{[(m+2)(\Omega+m/2)/2-F(F+1)]\,[m(m+2)-4F(F+1)]}{8(\Omega-2)(\Omega-1)\Omega(\Omega+1)}\ \sum_{i,j}\ \lambda^2_{i,j}(0)\\
&+\{8\Omega(m-1)(\Omega+2m-4)F(F+1)+\\
&\frac{(\Omega-2)\,[3(m-2)(\Omega+m/2)/2+F(F+1)]\,[3m(m-2)+4F(F+1)]\}}{8(\Omega-1)\Omega(\Omega+1)(\Omega+2)^2}\ \sum_{i,j}\ \lambda^2_{i,j}(1)\\
&+\frac{[3(m-2)(\Omega+m/2)/2+F(F+1)]\,[m(m+2)-4F(F+1)]}{4(\Omega-1)\Omega(\Omega+1)(\Omega+2)}\ \sum_{i,j}\ \lambda_{i,j}(0)\lambda_{i,j}(1)\\
&+P_2^0(m,\Omega,F)\ \left\langle\left(V^{\nu=2,f=0}\right)^2\right\rangle^{2,0}+P_2^1(m,\Omega,F)\ \left\langle\left(V^{\nu=2,f=1}\right)^2\right\rangle^{2,1}\ ;
\end{aligned}$$

$$\begin{aligned}
&P_2^0(m,\Omega,F)=[(\Omega+m/2)(\Omega+m/2-1)-F(F+1)]\,[m(m+2)-4F(F+1)]\,/[8\Omega(\Omega+1)]\ ,\\
&P_2^1(m,\Omega,F)=\{F^2(F+1)^2(3\Omega^2+7\Omega+6)/2+3m(m-2)(\Omega+m/2)(\Omega+m/2+1)\times\\
&(\Omega-1)(\Omega-2)/8+[F(F+1)/2]\,[(5\Omega+3)(\Omega-2)(\Omega+m/2)m+\Omega(\Omega-1)(\Omega+1)(\Omega-6)]\}/\\
&[(\Omega-1)\Omega(\Omega+2)(\Omega+3)]\ .
\end{aligned}\tag{3.4.8}$$

And,

$$\begin{aligned}
&\tilde{\epsilon}_i=\epsilon_i-\overline{\epsilon}\\
&\lambda_{i,i}(f)=\sum_j\ V^f_{ijij}\,(1+\delta_{ij})\ -\ (\Omega)^{-1}\ \sum_{k,l}\ V^f_{klkl}\,(1+\delta_{kl})\\
&\lambda_{i,j}(f)=\sum_k\ \sqrt{(1+\delta_{ki})(1+\delta_{kj})}\,V^f_{kikj}\quad\text{for}\quad i\neq j\\
&V^{\nu=2,f}_{ijij}=V^f_{ijij}-\left[\langle V(2)\rangle^{2,f}+(\lambda_{i,i}(f)+\lambda_{j,j}(f))\left(\Omega-2(-1)^f\right)^{-1}\right]\\
&V^{\nu=2,f}_{kikj}=V^f_{kikj}-\left(\Omega-2(-1)^f\right)^{-1}\sqrt{(1+\delta_{ki})(1+\delta_{kj})}\,\lambda^f_{i,j}\quad\text{for}\quad i\neq j\\
&V^{\nu=2,f}_{ijkl}=V^f_{ijkl}\quad\text{for all other cases}\ .
\end{aligned}\tag{3.4.9}$$

3.4.4 Short range spacing statistics : NNSD and Brody distribution

After constructing the Hamiltonian matrix of the given system as explained above and calculating its eigenvalues, we now study the eigenvalue spectra to know about the spectral statistics of the system. It is interesting to check the transition of the present BEGOE(1+2)-F system from regular domain to chaotic domain which can be observed by changing the two body strength λ. For this we use the nearest neighbor spacing distribution (NNSD) which was explained in the previous chapter. Also in addition to this, some research groups also introduced hybrid methods to study such Poisson to Wigner(GOE) transitions like Brody [50] and Berry-Robnik [51]. For the present analysis we choose the Brody distribution, a quantitative measure for the repulsion between levels. The ansatz used here is given by

$$P(s,\omega) = A_\omega(\omega+1)s^\omega \; exp\; (-A_\omega \; s^{\omega+1}) \tag{3.4.10}$$

where ω is the Brody (repulsion) parameter and A_ω is known as the normalization constant. It is the value of ω which actually locates the regime of the system either chaotic or an integrable one. If the value of ω is set as 0 then we obtain a Poisson distribution (Eq. 3.4.11), whereas if the value of ω exceeds to 1 then there is an onset of chaos reflecting the Wigner surmise (Eq. 3.4.12).

$$P(s) = \; exp\; (-s) \tag{3.4.11}$$

$$P(s) = \frac{\pi}{2} \; s \; exp(-\frac{\pi}{4}s^2) \tag{3.4.12}$$

The probability distribution $P(s,\omega)$ can also be defined for negative values of $\omega(\omega > -1)$ and then can be said to have level attraction [52]. One thing should be clear that, Brody parameter (ω) even though un-physical indicates the level repulsion when positive ($0 < \omega < 1$) and exhibits inter-level attraction when negative ; the values of Brody parameter are calculated with respect to different λs here. For the present system of $(m,\Omega) = (10,4)$ the ensemble averaged NNSDs are calculated using 50 members and the histograms (in black) is compared with the theoretical curves of

Poisson (in blue) and Wigner(GOE) (in red) distributions as shown in Fig. (3.4). We plot for three prominent spins $F = 0$, $F = 2$ and $F = 5$ which are enough to give a fair conclusion. Say for spin $F = 0$,as the value of two body interaction strength λ increases from $\lambda = 0.01$, we start seeing that the histograms move away from the Poisson distribution and when the λ value reaches to $\lambda = 0.5$ the histograms are close to the Wigner surmise. Eventually the value of Brody parameter (ω) also follow the same trend by lifting itself from a integrable domain ($\omega = 0.11$) to a chaotic domain ($\omega = 0.96$). This transition is observed for all spins $F = 0, 1, 2...5$. The plot in Fig. (3.5) shows the nature of Brody parameter (ω) how it varies with respect to the two body interaction strength (λ). Here also we plot the graph only for $F = 0$ (in black), $F = 2$ (in red) and $F = 5$ (in blue). Moreover the green dashed line acts like a chaos marker at $\omega = 0.5$ and then compared are the corresponding values of λ. Thus we obtain chaos markers λ_C for $F = 0$, $F = 2$ and $F = 5$ which are 0.029, 0.027 and 0.023 respectively.

3.4.5 Long range spacing statistics : Δ_3 statistics

After studying the behavior of BEGOE(1+2)-F for the nearest eigenstates, we also probe the relation of those eigenstates which are far from each other. The Dyson-Mehta statistics popularly known as the $\Delta_3(L)$ statistics is a method which gives information about the long-range correlations among the eigenvalues. This statistical measure describing spectral rigidity proves to be a qualitative measure of the level of randomness in the system. The methodology is described here in brief.

If for a system, $N(E)$ is the number of levels having energy between zero and E and it is plotted against E. The resulting graph may give a staircase like pattern which reflects a good visual impression of the overall regularity of the level series. But for measuring the average level spacing, we draw a line having the same average slope as the staircase. Fluctuations in the level series then appear as deviations of the staircase from the straight line. From this, a statistic may be developed to check the deviations between the staircase and the straight line. This will in-fact serve to

test the overall irregularity of the observed level series and its agreement with the theoretical model or not.

One can choose the observed energy interval as $[-L, +L]$, with zero in the center [1]. For negative E, $N(E)$ is defined as minus the number of levels between E and zero. The main task in the staircase graph is to find out how much deviation is observed from the best fit line. Lets say it be,

$$y = N(E) \tag{3.4.13}$$

Whereas the straight line may be represented by,

$$y = AE + B \tag{3.4.14}$$

The above Eq. (3.4.14) should be fitted to Eq. (3.4.14) by the method of least square criterion for further analysis. The mean-square deviation of Eq. (3.4.13) from the best fit Eq. (3.4.14) is represented by Δ and is given as [1],

$$\Delta = Min_{A,B}\left(\frac{1}{2L}\int_{-L}^{+L}[N(E) - AE - B]^2 dE\right) \tag{3.4.15}$$

This statistic Δ measures the degree of irregularity that persists in the level series. Here, the value of parameters A and B used for the construction of straight line can be independently chosen to make a two parameter fit. The mean value of Δ is obtained from the procedure given in [1] and for GOE, it is given as,

$$\langle\Delta\rangle = \frac{1}{\pi^2} + \left(ln(2\pi s) + \gamma - \frac{5}{4} - \frac{\pi^2}{8}\right) + O(s^{-1}) \tag{3.4.16}$$

The above average value of Δ for different classical Gaussian Orthogonal Ensemble can be compared with the average Δ for a random sequence where the energy levels are not correlated. This describes a Poissonian distribution as under:

The equation for Poisson is given as:

$$\langle\Delta\rangle = \frac{s}{15} \tag{3.4.17}$$

For the present ensemble BEGOE(1+2)-F of bosons with a fictitious spin of $F = 1/2$ we study the spacing distribution of energy levels that are located far from each

other, unlike the NNSD distribution studied earlier. For a fixed (m, Ω)=(10, 4) a plot of Δ_3 versus L is constructed for different values of the two body interaction strength (λ). The various values of λ from low to high indicate the Poisson and GOE nature of the distribution respectively. In Fig. (3.6), lets say for spin $F = 0$, the value of λ for which the statistic is Poisson is near $\lambda = 0.01$ and as we gradually increase the value of λ to 0.5 we can observe a clear transition to chaos (GOE). The blue curve in the figure denote the Poisson distribution described by Eq. (3.4.17), the red curve represents the GOE as given in the first equation of Eq. (3.4.16) whereas the black dots reflect the calculated values. A similar study is also done for spin $F = 2$ and spin $F = 5$ as shown in the Fig. (3.6). It is to be noted that the study here is limited to 50 members due to computational constraints.

3.4.6 1/f noise and power spectrum analysis

Quantum chaos which is the study of quantum analogues of classically chaotic systems, is characterized by the fluctuation properties of the spectrum of its Hamiltonian operator [53]. Till now we have understood that when quantum systems tend to have a regular behavior, their spectral fluctuations are distributed according to the Poisson definition and we say that the energy levels are clustered together. Conversely, if these levels experience a repulsion and tend to repel each other then that is the onset of quantum chaos. A new method has been developed to characterize the Hamiltonian of the system where the eigenvalue spectra is compared with the time series [54, 55]. Periodogram analysis was first developed and applied in the analysis of astronomical data [60, 61]. Consider a set of data values y_i at energy values E_i respectively, where $i = 1, 2, ..., N$. The centroid $\bar{y}$ and variance σ^2 are defined as:

$$\bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i \quad ; \quad \sigma^2 = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{y})^2 \tag{3.4.18}$$

The Lomb and Scargle normalized periodogram is defined as function of ω as in [59, 60]

$$P(\omega) = \frac{1}{2\sigma^2}\left\{\frac{[\sum_{i=1}^{N}(y_i - \bar{y})cos\omega(E_i - \tau)]^2}{\sum_{i=1}^{N} cos^2\omega(E_i - \tau)} + \frac{[\sum_{i=1}^{N}(y_i - \bar{y})sin\omega(E_i - \tau)]^2}{\sum_{i=1}^{N} sin^2\omega(E_i - \tau)}\right\} \quad (3.4.19)$$

where the offset τ is determined for required angular frequency ($\omega = 2\pi f > 0$) by solving the following expression

$$tan(2\omega\tau) = \frac{\sum_{i=1}^{N} sin(2\omega E_i)}{\sum_{i=1}^{N} cos(2\omega E_i)} \quad (3.4.20)$$

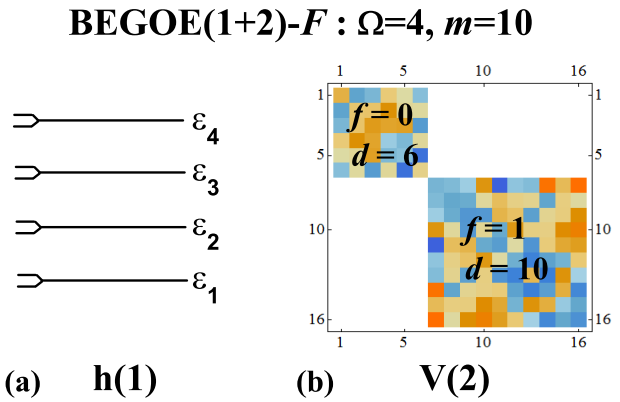
Here it is shown that the Fourier power spectrum $S(f)$ behaves as $1/f^{\alpha}$ in terms of the frequency f where the interesting part lies behind the presence of $1/f^{\alpha}$ noise in the fluctuating physical variables [56]. This kind of noise has been detected in condensed matter systems, traffic engineering, DNA sequence, quasar emissions, river discharge, human behavior, heartbeat and dynamic images, among many others. But even though possessing such an ubiquity, there is no universal explanation about this phenomenon except from being a generic manifestation of such complex quantum systems [57]. For characterizing the energy spectra of chaotic quantum systems by $1/f^{\alpha=1}$ noise, one may consider the sequence of discrete energy levels in a quantum system as a discrete time series, where the energy plays the role of time. Hence, the energy level fluctuations can be studied using traditional methods of time series analysis, like the study of the power spectrum. This type of analysis when made for typical chaotic quantum systems, the power spectrum showed a very accurate $1/f^{\alpha=1}$ behavior [54]. Also, from the analysis done in past [58] it is a established fact that the ensemble averaged power spectrum $\langle S(f)\rangle$ of the fluctuations in the cumulative level density goes as $1/f$ or $1/f^2$ depending on whether the system is classically chaotic or regular respectively. It is the value of slope α that helps us to procure the regime of the quantum system, where the regular part is represented by $\alpha = 2$ and it reduces to $\alpha = 1$ when the system is close to GOE. By drawing an analogy between the quantum spectrum and time series, the statistical self-similarity in the quantum spectrum has been explored using the detrended fluctuation analysis (DFA) and random matrix theory (RMT).

For the present system of BEGOE(1+2)-F which consists of $m = 10$ bosons in $\Omega = 4$ single particle states (that are doubly degenerate $N = 2\Omega$) we have total six values of spin viz. $F = 0, 1, 2, 3, 4$ and 5. Out of these we consider $F = 0$, $F = 2$ and $F = 5$ spins which give a significant contribution with their dimensionalities being 196, 750 and 286 respectively. For these three spins, we obtained a periodogram $P(f)$ using deviations between the smooth density function and the exact one, following the method described in [49] and detailed in Appendix A at the end.

We plot a log - log graph (Fig 3.7) of $logP(f)$ versus $logf$ to the straight line $logP(f) \propto a\ logf$ and hence determined the value of the slope a. The slopes a for $F = 0$, 2 and 5 are found to be -1.08, -1.11 and -1.12 respectively which makes it sure that the nature is GOE. And thus we conclude that the spectra of BEGOE(1 + 2)-F also exhibit $1/f$ behavior.

3.5 Conclusion

For one plus two-body bosonic ensemble BEGOE(1+2)-F, where bosons have fictitious spin, the basic construction of Hamiltonian is discussed here. For a fixed configuration of finite number of bosons (m) and sp states (Ω), the eigenvalue spectrum is obtained and spectral properties are studied. Initially, form of eigenvalue density is analyzed for BEGOE(2)-F (here one-body part is considered as zero), which is close to Gaussian. Also, the average-fluctuation is observed for BEGOE(2)-F and on same grounds the fixed-F eigenvalue centroids and spectral variances are analysed. Further, for BEGOE(1+2)-F, the spacing statistics for short range (i.e. NNSD) and long range (i.e. Δ_3 statistics) is studied. A clear transition is observed in both of them from Poisson to Gaussian as the value of λ increases. The spin dependence of transition point λ_C is obtained and it is seen that the value of the critical two-body strength parameter (λ_C) decreases as F-spin increases. Finally the $1/f$noise behaviour of the BEGOE(1 + 2)-F ensemble is verified using periodogram analysis and thus it is known that the spectra exhibits $1/f$ noise which is a sign of chaos.



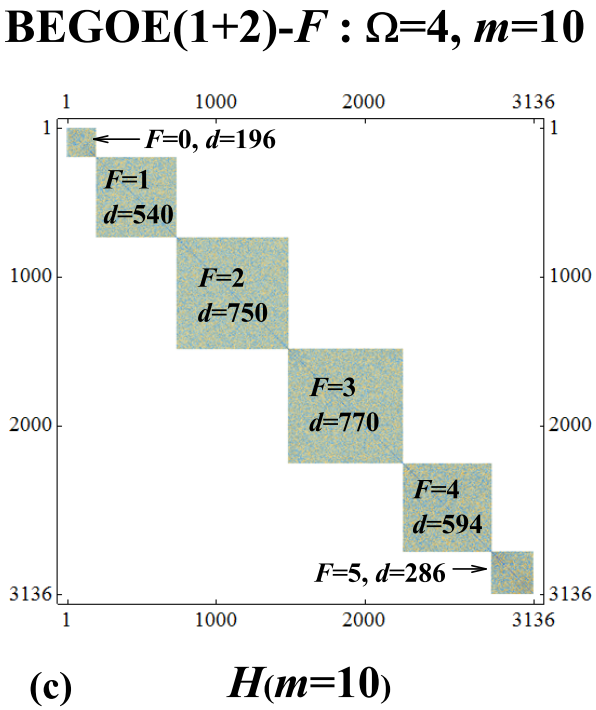


Figure 3.1: (a) Single-particle levels generated by $\hat{h}(1)$ for $\Omega = 4$. Each level is doubly degenerate; i.e., $N = 2\Omega$. (b) Matrix of $\hat{V}(2)$ in two-boson space for $\Omega = 4$. (c) Decomposition of H matrix in m-particle space into a direct sum of matrices, each with a fixed F value. There is a BEGOE(1 + 2)-F ensemble in each (m, F) space corresponding to the diagonal blocks in (c). Note that the matrix elements in the off-diagonal blocks in (b) and (c) are all zero. The plots (b) and (c) are generated using MATHEMATICA and we have illustrated only one particular member of $V(2)$ and $H(m)$ ensembles that are employed in the numerical calculations.

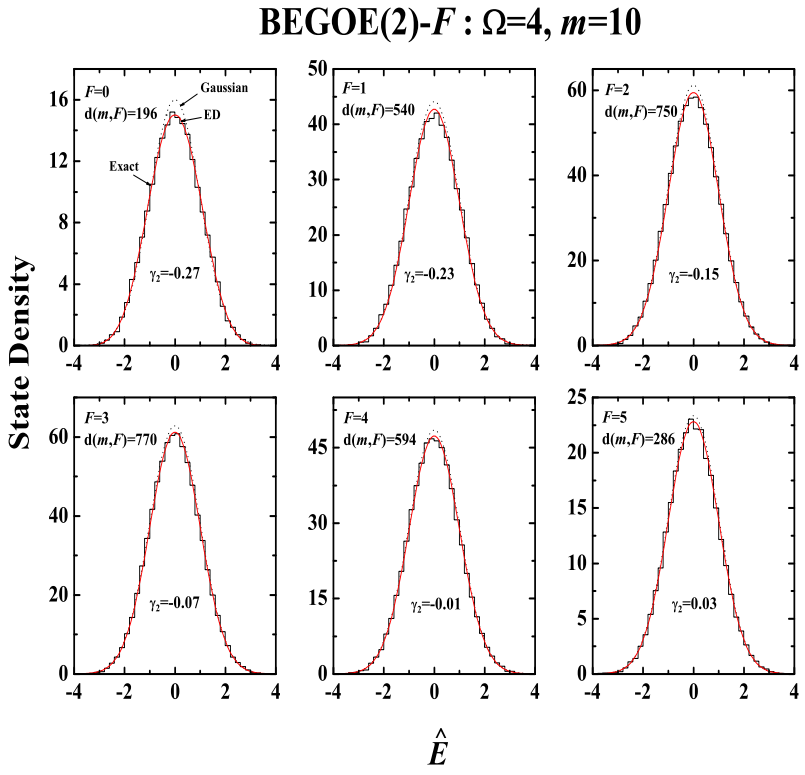


Figure 3.2: Ensemble averaged state density, $\rho^{m,F}(\hat{E})$ vs normalized energy, $\hat{E}$. Histograms represent ensemble averaged state density for 100 members BEGOE(2)-F generated by $m = 10$ Bosons in $\Omega = 4$ doubly degenerate single particle orbits. The dotted curves give Gaussian representation while The continuous curves are Edgeworth corrected Gaussians (ED).

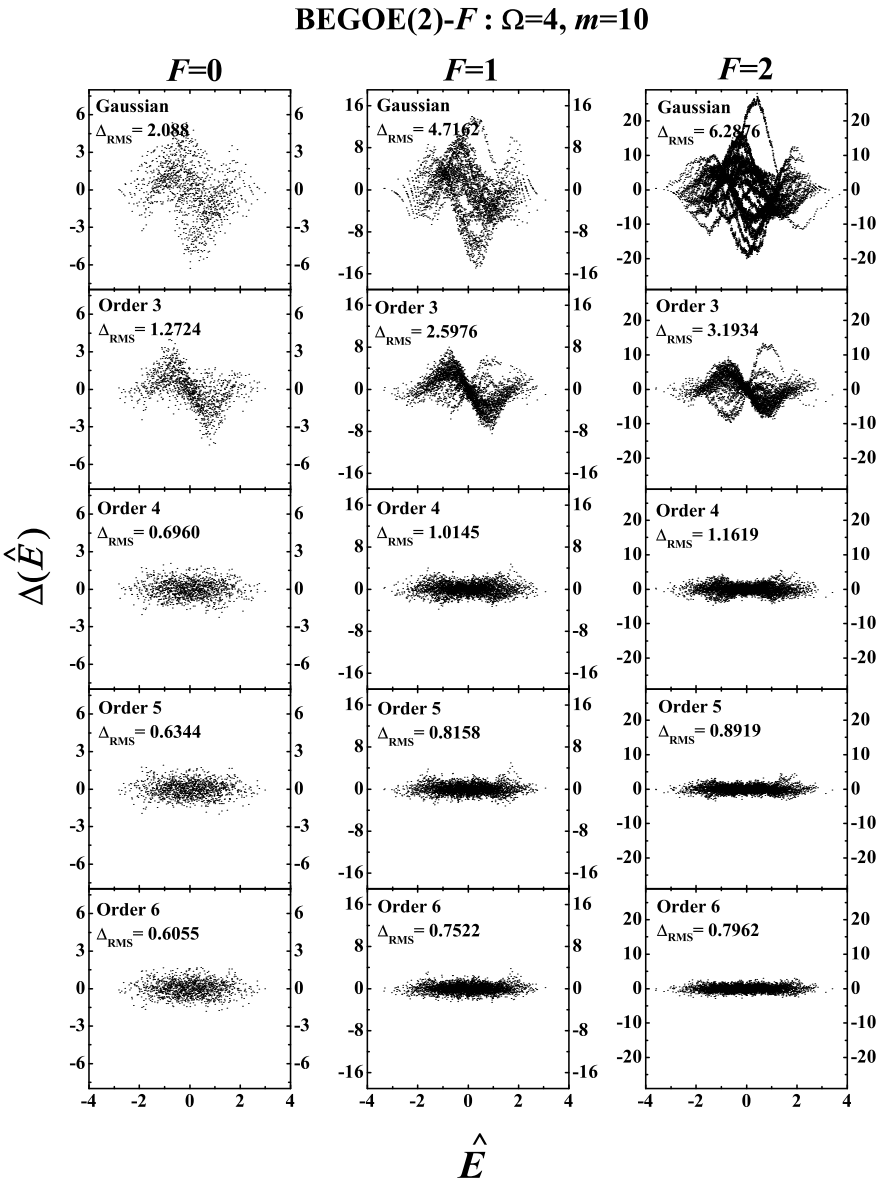


Figure 3.3: Level motion $\Delta(\hat{E})$ vs $\hat{E} = (E - \epsilon)/\sigma$ of the spectra of a 100 member BEGOE(2)-F generated using the system $(m = 10, \Omega = 4)$. Results are shown for m-particle spins F=0 and 2. The State-to-state deviation between the true distribution function and its Gaussian version as well as with $\overline{\rho^{m,F}(\hat{E})}$ including optimized correction upto third order, fourth order and sixth order are shown for all members of the ensemble. Values of the root mean square deviation Δ_{RMS} are as shown in figure.

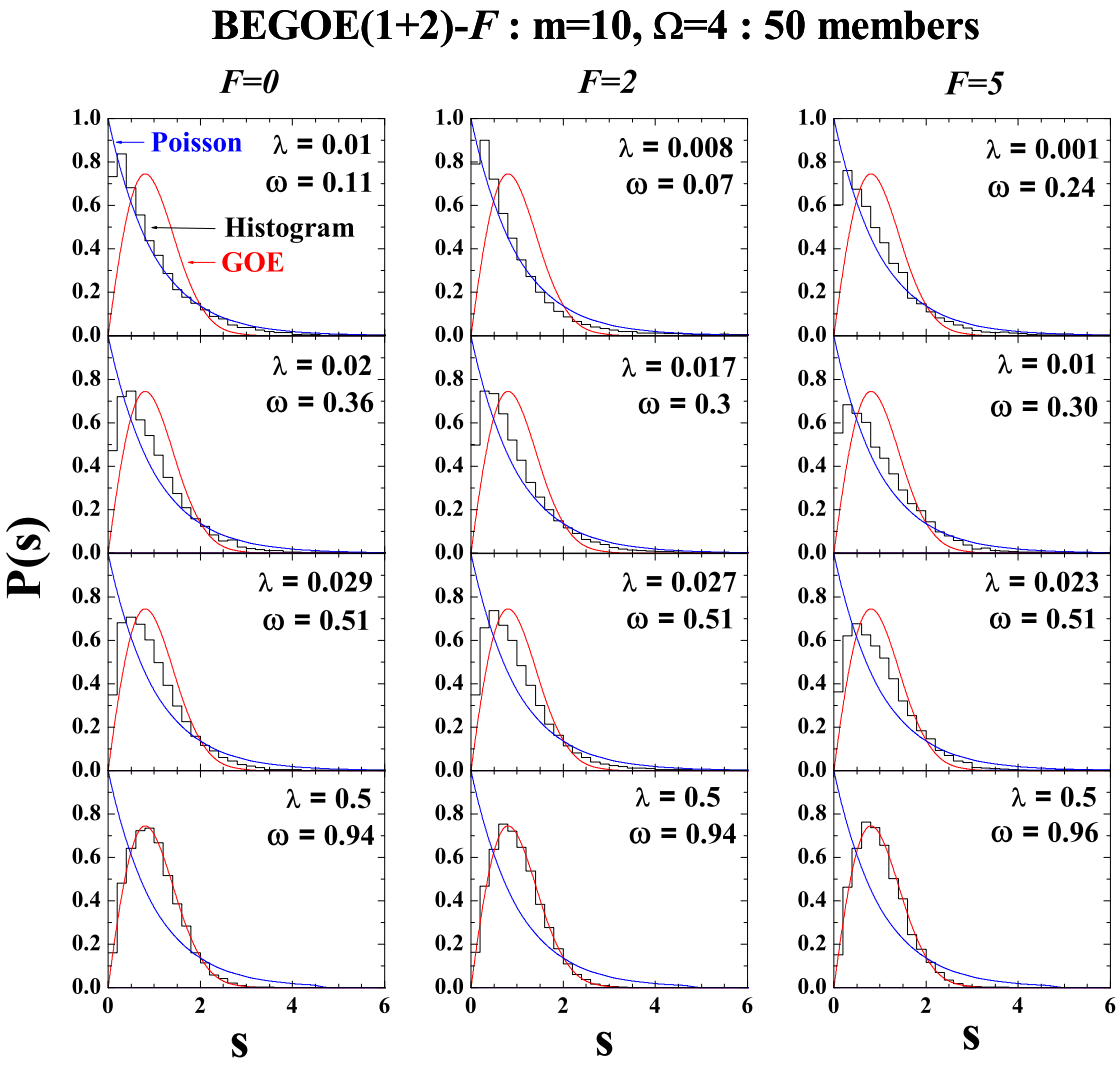


Figure 3.4: Nearest Neighbor Spacing Distribution for a 50 member BEGOE(1+2)-F ensemble with $\Omega = 4$ and $m = 10$. The plots are for $F = 0, 2$ and 5 respectively and they are compared with Poisson (blue) and GOE(red). The corresponding values of λ and ω are given. Bin size is taken as 0.2

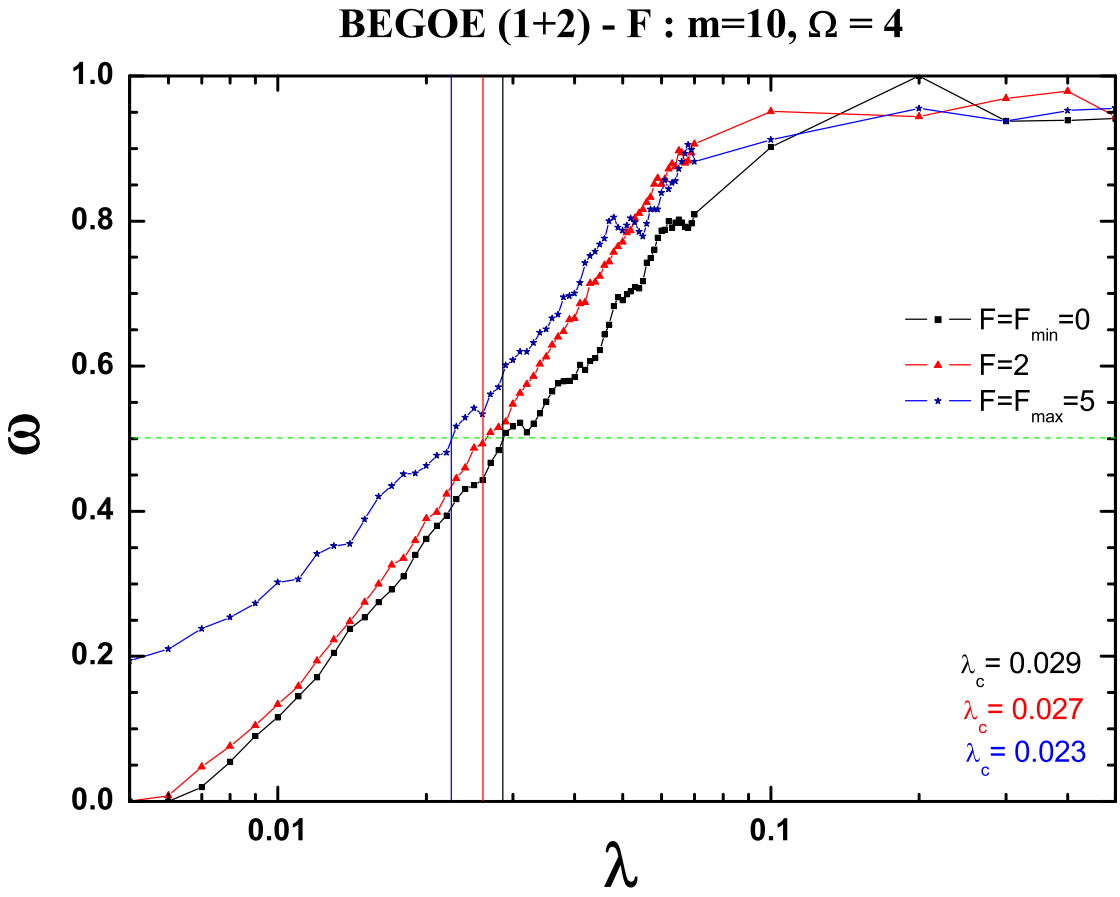


Figure 3.5: The average values of ω as a function of two body interaction strength λ are plotted for BEGOE(1+2)-F and $(\Omega, m) = (4, 10)$ using 50 members. The curves in black, red and blue are for $F = 0, 2$ and 5 respectively and their corresponding straight lines represent the transition point. The green dashed line is the chaos marker λ_C for $\omega = 0.5$

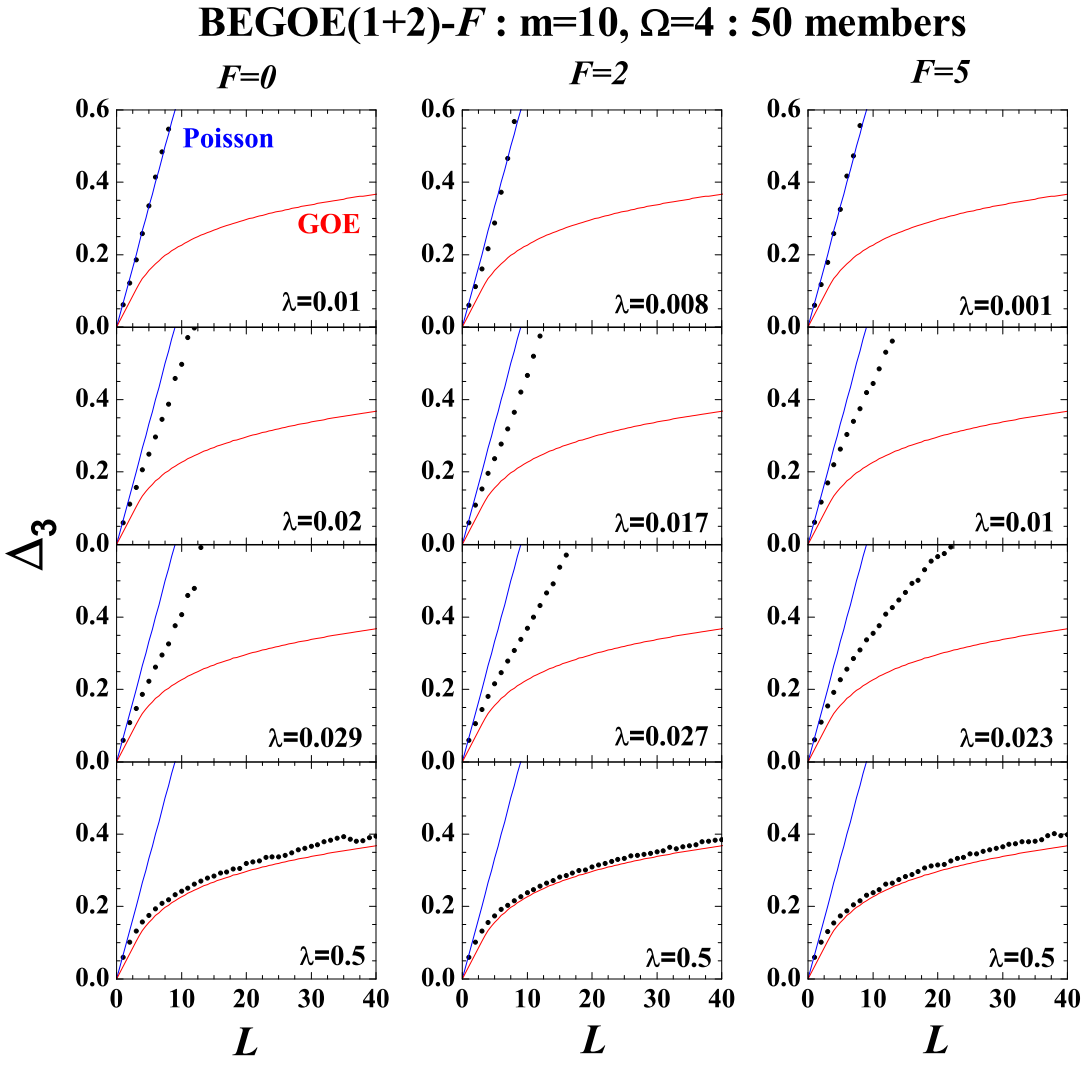


Figure 3.6: The $\overline{\Delta_3}$ statistics is plotted against L for a 50 member BEGOE(1+2)-F ensemble with $\Omega = 4$ and $m = 10$. The plots are for $F = 0, 2 and 5$ respectively and they are compared with Poisson (blue) and GOE(red).

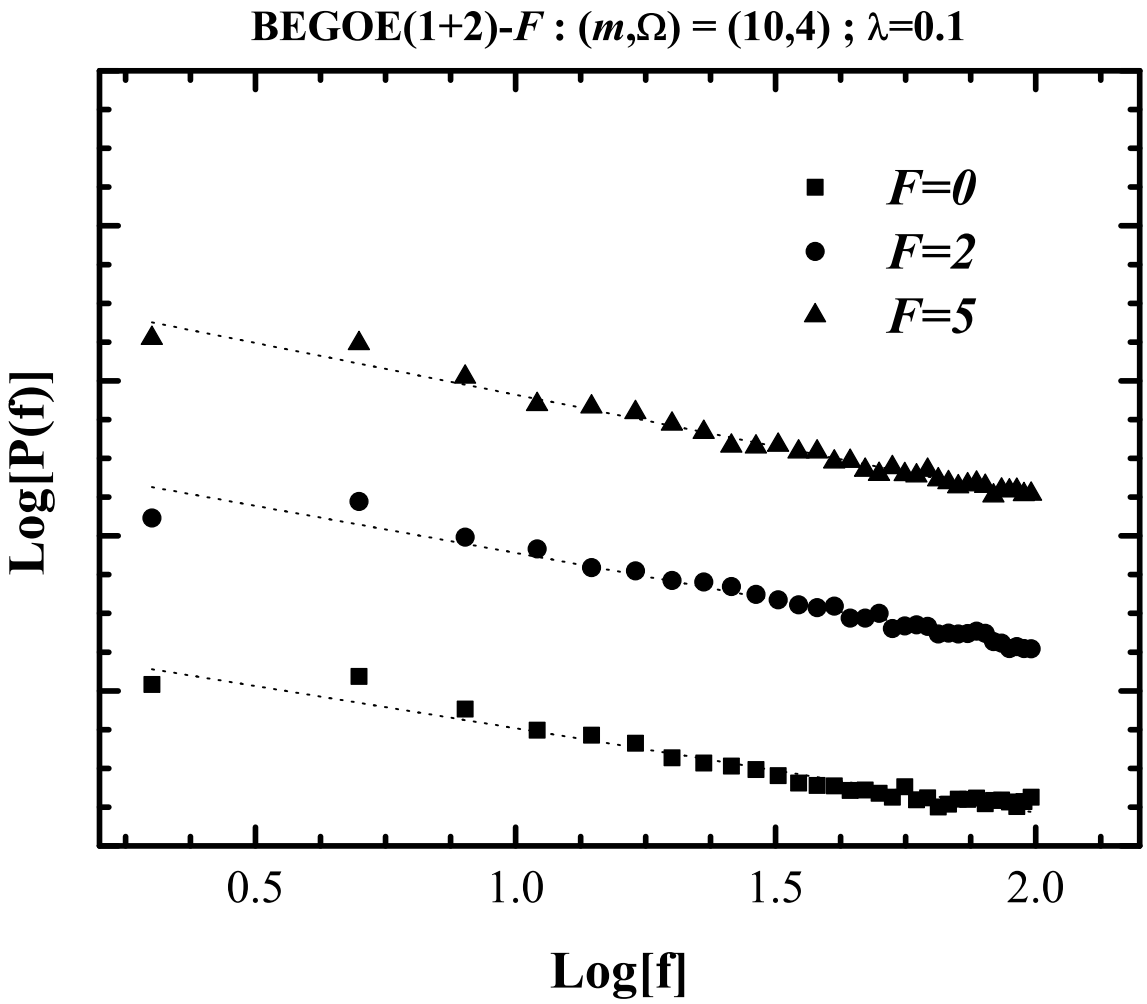


Figure 3.7: The $logP(f)$ versus $logf$ shows $1/f$ noise in spectra. The curves for $F = 2$ and $F = 5$ are shifted so that there is no overlap between the results of different F-spins. Results are obtained by binning the frequency (f) axis and averaging the power spectrum components in each bin