Chapter 6

Number of Principal Components, Information Entropy and Fidelity Decay

6.1 Introduction

In finite interacting many-particle systems, the chaos measures like number of principal components (NPC), information entropy, fidelity decay etc. can be determined by examining the general features of the strength functions [31]. In the past, NPC and information entropy have been studied analytically as well as numerically for fermion systems using EGOE(1+2) by Kota and Sahu [151]. Also for bosonic systems, NPC and information entropy have been studied both analytically and numerically using BEGOE(1+2) by Chavda, Potbhare and Kota [62]. Recently, the lower-order bivariate reduced moments of the transition strengths are examined for the action of a transition operator on the eigenstates generated by EGOE(k) and it is shown that the ensemble averaged distribution of transition strengths follows a bivariate q-normal distribution f_{biv-qN} and a formula for NPC in the transition strengths from a state is obtained [83]. Very recently, it is shown that the conditional q-normal density f_{CqN} can be used to represent strength functions in the strong coupling limit [84]. In chapter 5 we have studied these strength functions in detail and demonstrated that in strong coupling domain (in the thermalization region), the strength functions in EGOE(1+ k) and BEGOE(1+ k) indeed can be represented by the conditional q-normal distribution f_{CqN} in the dense interacting boson systems interacting via k-body interaction. Moving ahead in the present chapter we use this f_{CqN} form and parameters that enter in this form to describe various statistical quantities such as NPC, information entropy and fidelity decay that measure chaos in finite interacting particle systems.

Firstly, in section 6.2 we use these f_{CqN} forms to describe NPC and derive a two-parameter (ζ and q) analytical formula for NPC as a function of energy for k -body interaction utilizing the interpolating form f_{CqN} of the strength functions. This two parameter analytical formula is tested for both EGOE(1+ k) and BEGOE(1+ k) and the results are found to be in good agreement. Further in section 6.3, the localization length l_H related to the information entropy S^{info} is studied numerically using EGOE(1+ k) and BEGOE(1+ k) as a function of energy for k -body interaction utilizing the interpolating form f_{CqN} of the strength functions and results are compared with ensemble averaged EE results. Moving ahead in section 6.4 we utilize this interpolating form of the strength functions to describe the fidelity decay after k -body interaction quench for bosons using BEGOE(1+ k). The work on bosons presented in this chapter is based on [30] and that of fermions is under preparation to be published.

6.2 Number of Principal Components

The various layers of chaos in interacting particle systems [57, 154, 155] and for a system like atomic nuclei, are characterized using the number of principal components (NPC) in wavefunction. It is also known as inverse participation ratio (IPR). NPC for transition strengths is a measure of fluctuations in transition strength sums [83]. For an eigenstate $|E_i\rangle$ spread over the basis states $|\kappa\rangle$, with energies $\xi_\kappa = \langle\kappa|H|\kappa\rangle$, NPC is defined as,

$$\text{NPC}(E) = \left\{ \sum_{\kappa} |C_{\kappa}^{i}|^{4} \right\}^{-1} \tag{6.1}$$

It essentially gives the number of basis states $|\kappa\rangle$ that constitute an eigenstate with energy E . The GOE value for NPC is $d/3$. NPC is written as integrals involving strength functions over all ξ energies. It is a basis dependent quantity and it can be studied by examining the general features of the strength functions $F_\xi(E)$ discussed in chapter 5. The smooth form for NPC (E) can be written as [151],

$$\text{NPC}(E) = \frac{d}{3} \left\{ \int d\xi \; \frac{\rho^{H_\kappa}(\xi)[F_\xi(E)]^2}{[\rho^H(E)]^2} \right\}^{-1} \;, \tag{6.2}$$

In the above Eq.(6.2), $\rho^{H_\kappa}(\xi)$ and $\rho^{H}(E)$ are normalized spectral densities of diagonal Hamiltonian H_κ matrix and full Hamiltonian H matrix, respectively. Here E and ξ are zero centered and scaled by their corresponding widths. From chapter 5 we know that the spectral densities can be described by f_{qN} and the strength functions by f_{CqN} . Hence we replace $\rho^{H_\kappa}(\xi)$ and $\rho^{H}(E)$ by f_{qN} and $F_\xi(E)$ by f_{CqN} in the above equation Eq.(6.2).

This gives the following formula of NPC in terms of f_{qN} and f_{CqN} [83, 84],

$$\mathrm{NPC}(E) = \frac{d}{3}\left\{\int_{S(q)} d\xi \; \frac{f_{qN}(\xi|q)[f_{CqN}(E|\xi;\zeta,q)]^2}{f_{qN}(E|q)}\right\}^{-1} , \tag{6.3}$$

In general, q's in the above equation need not be same [83, 84]. However, in the thermalization region, with $\zeta^2 \leq 1/2$, one can approximate the fourth moment γ_2 (discussed in chapter 5) as $\gamma_2 \approx (q-1)$. Then, the formula for q given in chapter 4 is valid for f_{qN} as well as for f_{CqN}. Also, the results of γ_2 presented in chapter 5 corroborate this claim. With this, it is possible to simplify Eq.(6.3) and a simple two parameter formula, valid in chaotic domain, for NPC can be written as,

$$\mathrm{NPC}(E) = \frac{d}{3}\left\{\sum_{n=0}^{\infty} \frac{\zeta^{2n}}{[n]_q!} H_n^2(E|q)\right\}^{-1} , \tag{6.4}$$

We can observe from the above Eq.(6.4) that as $\zeta \to 0$ NPC(E) approaches GOE value i.e. $d/3$. Also, f_{qN} and f_{CqN} in Eq.(6.3) reduce to Gaussian when $q \to 1$ and in this case Eq.(6.4) gives similar results obtained for $k = 2$ in [151].

6.2.1 Results for Fermion Systems

We now test this formula with numerical ensemble averaged EGOE(1+k) results for fermion systems. Ensemble averaged NPC results are presented in Figs. 6.1 - 6.5, by considering a 100 member EGOE(1+k) ensemble with $m = 6$ fermions in $N = 12$ sp states example for different values of λ for all values of body rank i.e. $k = 2$ to $k = m = 6$. The ensemble averaged NPC values are shown with blue solid circles. The continuous lines are obtained using the theoretical expression given by Eq. (6.4) and the ensemble averaged ζ and q values used for this analysis are given in the corresponding figures. One can see from the results that for fixed k (i) when the value of λ is very small, i.e. when the one-body part of the interaction dominates the k-body interaction, numerical NPC values are zero and the theoretical curve is far away from the numerical results indicating that the wavefunctions are completely localized; (ii) with further increase in λ, the theoretical estimate for NPC in the chaotic domain is much above the ensemble averaged curve indicating that the chaos has not yet set in; (iii) However, with sufficiently large λ, we see that the ensemble averaged curve is matching with the theoretical estimate given by Eq. (6.4), indicating that system is in chaotic domain corresponding to the thermalization region given by $\zeta^2 \sim 1/2$ [27] and the strength functions $F_\xi(E)$ are well represented by f_{CqN}. Again with further increase in λ, the match between the theoretical chaotic domain estimate and the ensemble averaged values is very well in the bulk part of the spectrum ($|E| < 2$) for all values of k with deviations near the spectrum tails. This indicates that the wavefunctions are delocalized in this region. Hence, in the chaotic domain, the energy variation of NPC(E) using Eq. (6.4) is essentially given by two parameters, ζ and q .

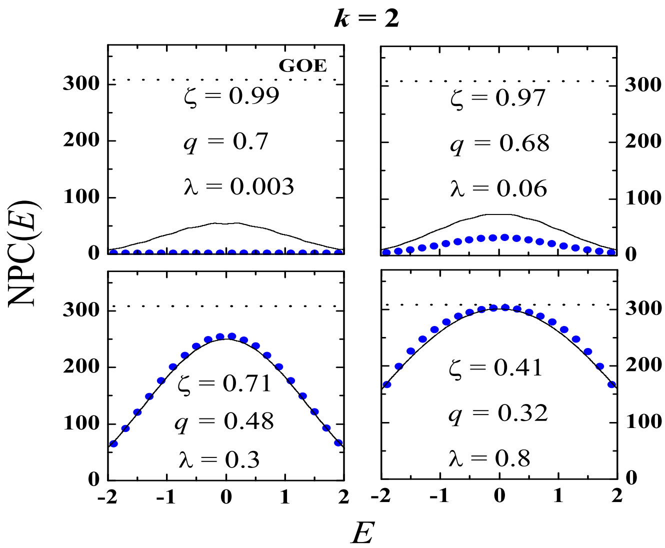

Figure 6.1: Ensemble averaged NPC as a function of normalized energy E for EGOE(1+ k) with $m = 6$ fermions distributed in $N = 12$ sp states for body rank $k = 2$. Here an ensemble of 100 members is considered. Results are shown for various values of λ . Here blue solid circles correspond to numerical ensemble averaged EGOE(1+ k) results and the continuous black curves correspond to theoretical curves obtained via Eq.(6.4). Dotted lines in each graph represent GOE estimate.

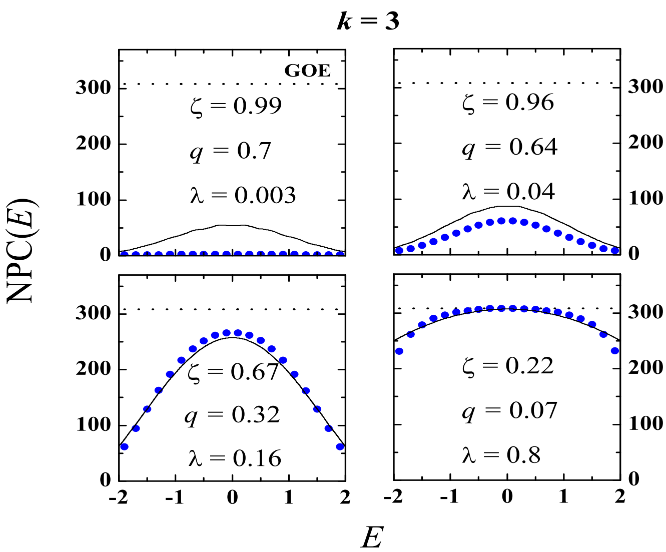

Figure 6.2: Same as Fig. 6.1 but for body rank $k = 3$.

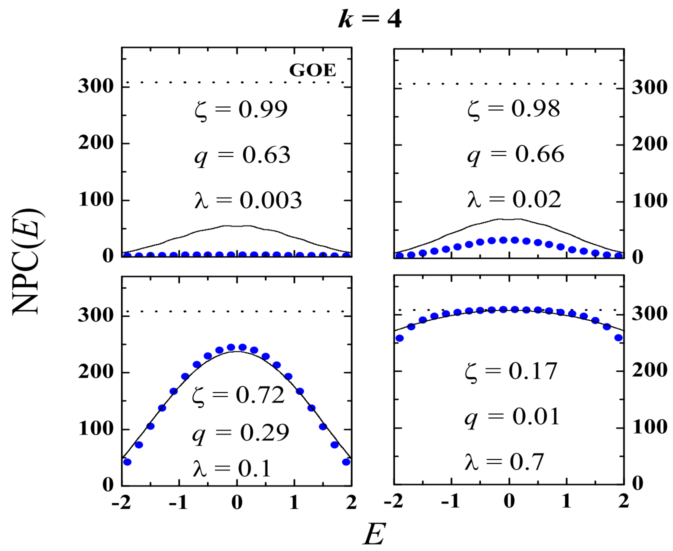

Figure 6.3: Same as Fig. 6.1 but for body rank $k = 4$.

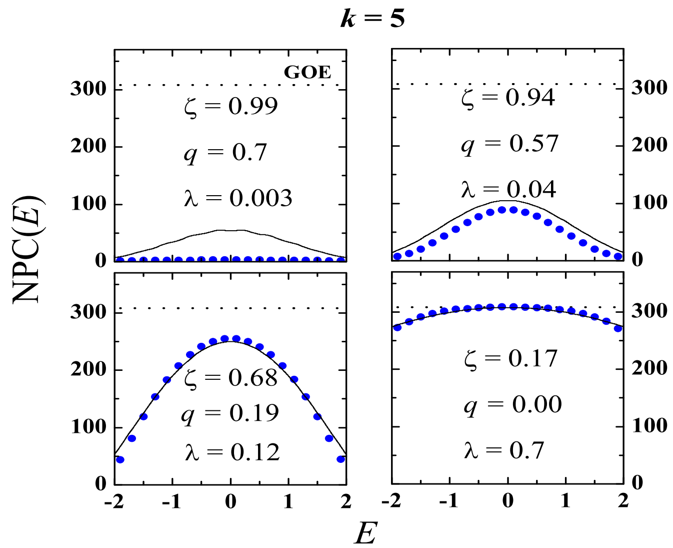

Figure 6.4: Same as Fig. 6.1 but for body rank $k = 5$.

6.2.2 Results for boson systems

We now proceed further and test this formula with numerical ensemble averaged BEGOE(1+k) results for boson systems. Ensemble averaged NPC results of BEGOE(1+k) ensemble are shown in Figs. 6.6 - 6.14. For this analysis, we have used a system of $m = 10$ bosons occupying $N = 5$ sp states. The analysis is done by considering 100 members of this ensemble and for different values of λ for all values of body rank of interaction i.e. $k = 2$ to $k = m = 10$. In these figures the ensemble averaged NPC values are presented with red solid circles and continuous black curves are obtained using the theoretical expression given by Eq. (6.4). Here in the case of BEGOE(1+k) also we

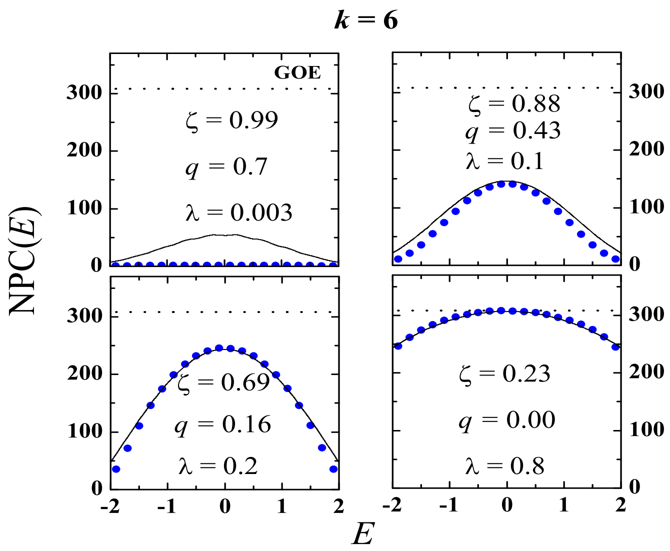

Figure 6.5: Same as Fig. 6.1 but for body rank $k = m = 6$.

can observe the localization to delocalization transition in wavefunctions as we gradually increase the interaction strength λ as observed in the case of EGOE(1+ k).

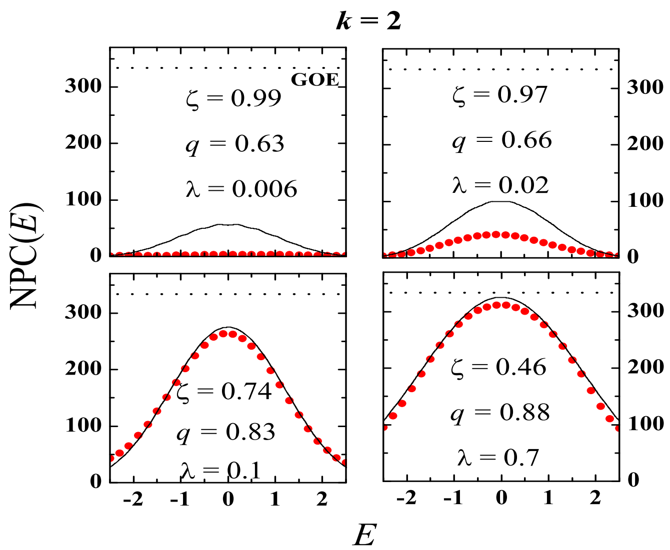

Figure 6.6: Ensemble averaged NPC vs. normalized energy E for a 100 member BEGOE(1+ k) with $m = 10$ interacting bosons distributed in $N = 5$ sp states for body rank $k = 2$. Here, results are shown for various values of λ. Ensemble averaged BEGOE(1+ k) results (red solid circles) are compared with the smooth forms obtained via Eq.(6.4). Dotted lines in each graph represent GOE estimate.

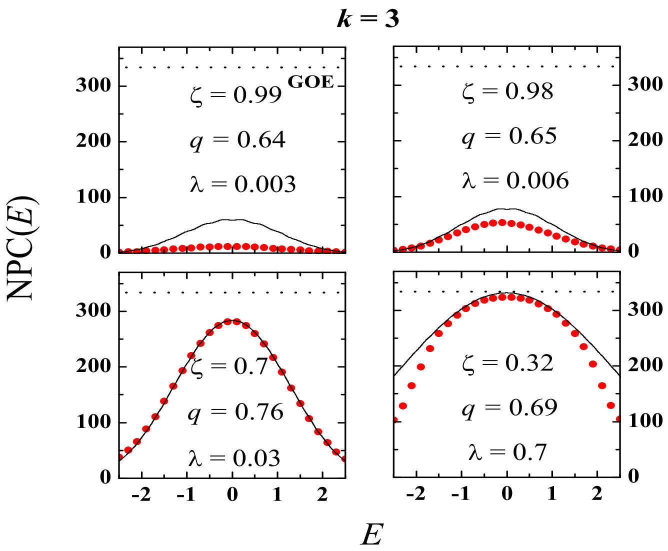

Figure 6.7: Same as Fig. 6.6 but for body rank $k = 3$.

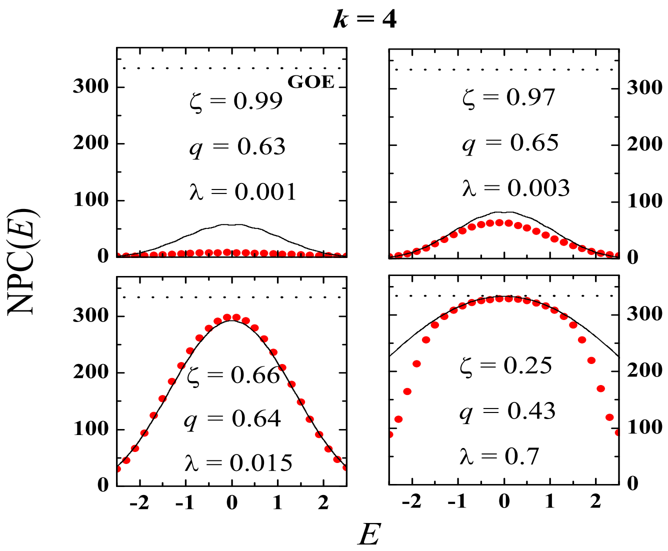

Figure 6.8: Same as Fig. 6.6 but for body rank $k = 4$.

6.3 Information Entropy

Another statistical quantity which characterizes the localization to delocalization transition in interacting particle systems is the information entropy (also known as Shannon entropy). This is also a basis dependent quantity just like NPC, discussed in the previous section. It is a measure of complexity (or disorder). The information entropy is defined by,

$$\begin{aligned} S^{\text{info}}(E) &= -\textstyle\sum_{\kappa} p_{\kappa}^{i} \ln p_{\kappa}^{i} \\ &= -\textstyle\sum_{\kappa} |C_{\kappa}^{i}|^2 \ln |C_{\kappa}^{i}|^2 \end{aligned} \tag{6.5}$$

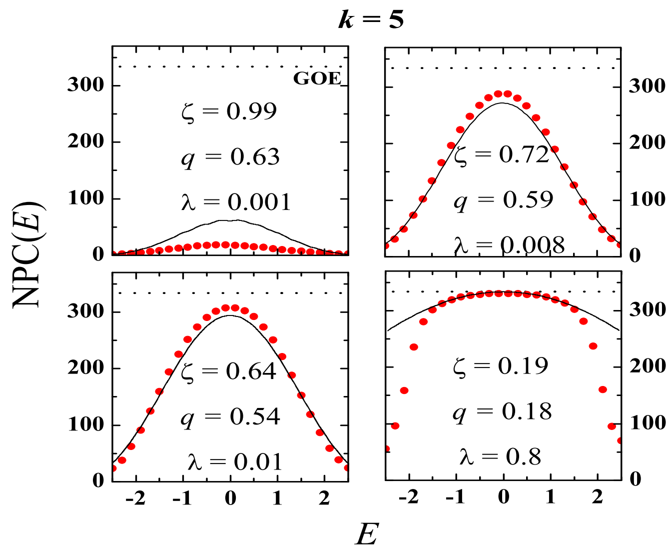

Figure 6.9: Same as Fig. 6.6 but for body rank $k=5$.

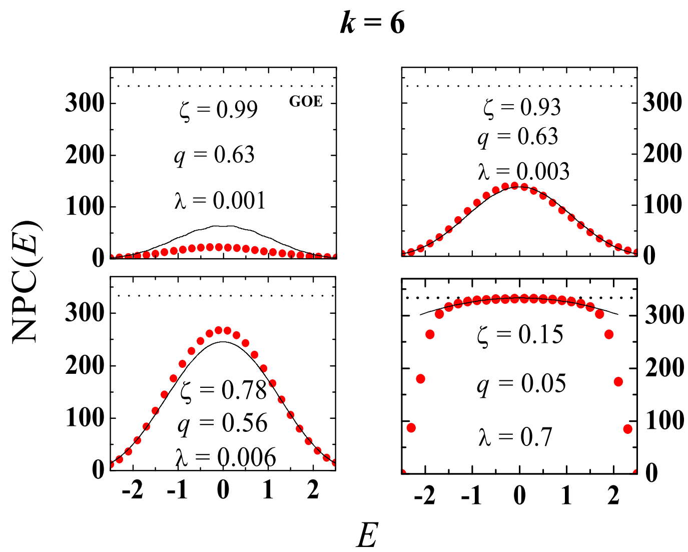

Figure 6.10: Same as Fig. 6.6 but for body rank $k=6$.

In the above Eq.(6.5), p^i_κ is the probability of basis state κ in the eigenstate at energy E_i. The localization length, l_H is related to $S^{\text{info}}(E)$ as follows,

$$l_H(E) = \exp\{S^{info}(E)\}/(0.48d). \tag{6.6}$$

The GOE value for l_H is 1. As in the case of NPC, l_H is also written as integrals involving strength functions over all ξ energies. Then the corresponding expression for l_H involving $F_\xi(E)$, can be written as [151],

$$l_H(E) = -\int d\xi \, \frac{F_\xi(E)\; \rho^{H_\kappa}(\xi)}{\rho^H(E)} \ln\left\{\frac{F_\xi(E)}{\rho^H(E)}\right\} . \tag{6.7}$$

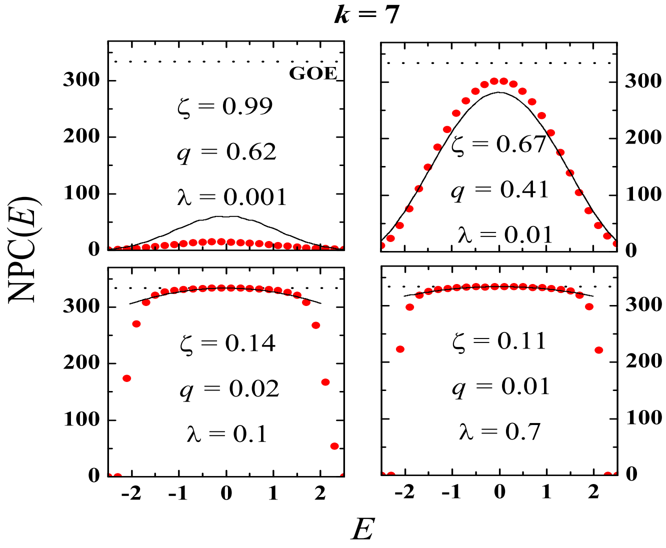

Figure 6.11: Same as Fig. 6.6 but for body rank $k = 7$.

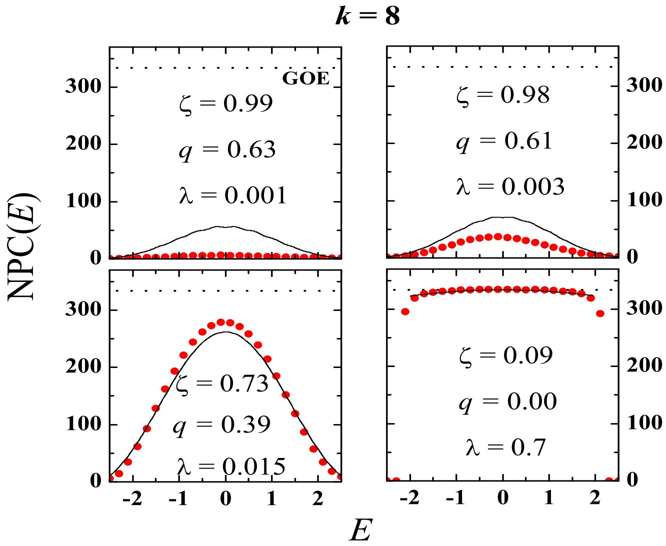

Figure 6.12: Same as Fig. 6.6 but for body rank $k = 8$.

In the above Eq.(6.7), $\rho^{H_\kappa}(\xi)$ and $\rho^{H}(E)$ are the normalized spectral densities of ξ-energies and by considering the full Hamiltonian respectively. Now we have seen in chapter 5 that the spectral densities can be described by f_{qN} and the strength functions by f_{CqN}. Therefore we replace $\rho^{H_\kappa}(\xi)$ and $\rho^{H}(E)$ by f_{qN} and $F_\xi(E)$ by f_{CqN} in Eq.(6.7) and we get formula for l_H valid in chaotic domain given by,

$$l_H(E) = -\int d\xi\, \frac{f_{CqN}(E|\xi;\zeta,q) f_{qN}(\xi|q)}{f_{qN}(E|q)}\, \ln\left\{\frac{f_{CqN}(E|\xi;\zeta,q)}{f_{qN}(E|q)}\right\}. \tag{6.8}$$

In principal it is possible to obatin a simplified formula of Eq.(6.8) however till now it is an open problem and therefore, it is evaluated numerically and results are compared with

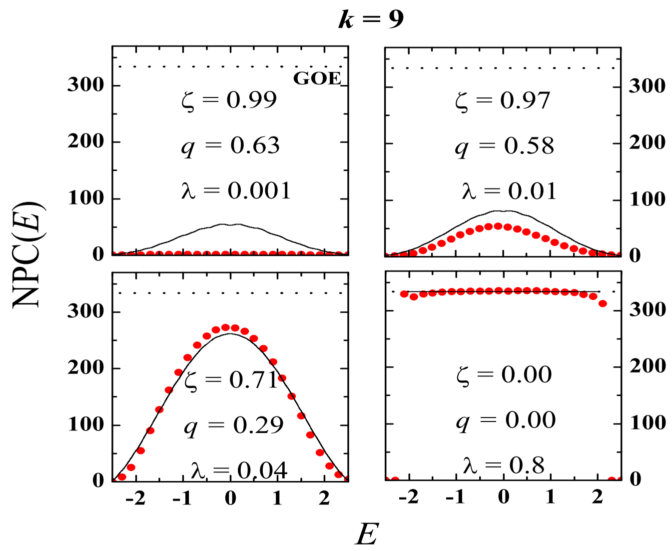

Figure 6.13: Same as Fig. 6.6 but for body rank $k = 9$.

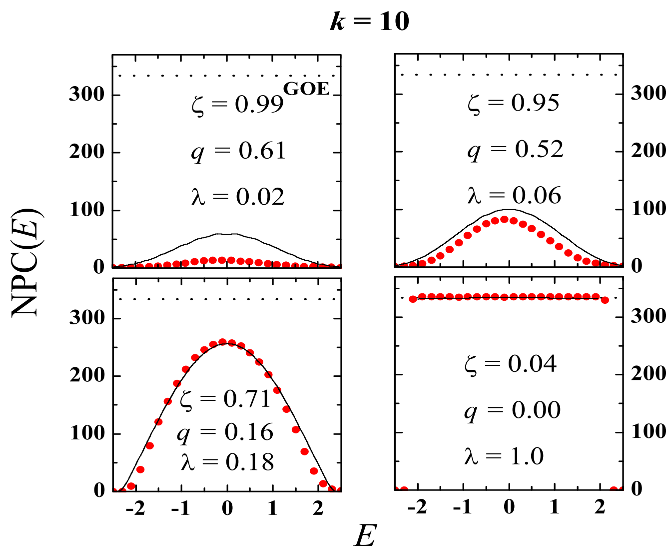

Figure 6.14: Same as Fig. 6.6 but for body rank $k = m = 10$.

ensemble averaged numerical results of EGOE(1+k) and BEGOE(1+k) for both fermion and boson systems respectively. These results are presented in next section.

6.3.1 Results for fermion systems

First let us see the l_H results for EGOE(1+k). We consider a 100 member EGOE($1+k$) ensemble with $m = 6$ fermions distributed in $N = 12$ sp states. Fig. 6.15 represents l_H results for different values of body rank of interaction i.e. $k = 2, 3, 4$ and $k = m = 6$. Here we choose k-body interaction strength $\lambda = 0.8$ so that the system is in chaotic domain corresponding to the thermalization region given by $\zeta^2 \sim 1/2$ [27] and the strength functions $F_\xi(E)$ are well represented by f_{CqN}. In this figure the blue solid circles correspond to the numerical EE results while the continuous black curves are obtained using theoretical expression given by Eq.(6.8). The corresponding ensemble averaged ζ values used in this analysis are given in the figure. Here the q values are obtained from chapter 4. Fig. 6.15 shows a very good agreement between ensemble averaged numerical results and smooth forms given by Eq.(6.8) for all values of k in the bulk of the spectrum. However, small deviations are observed near the tails of the spectrum.

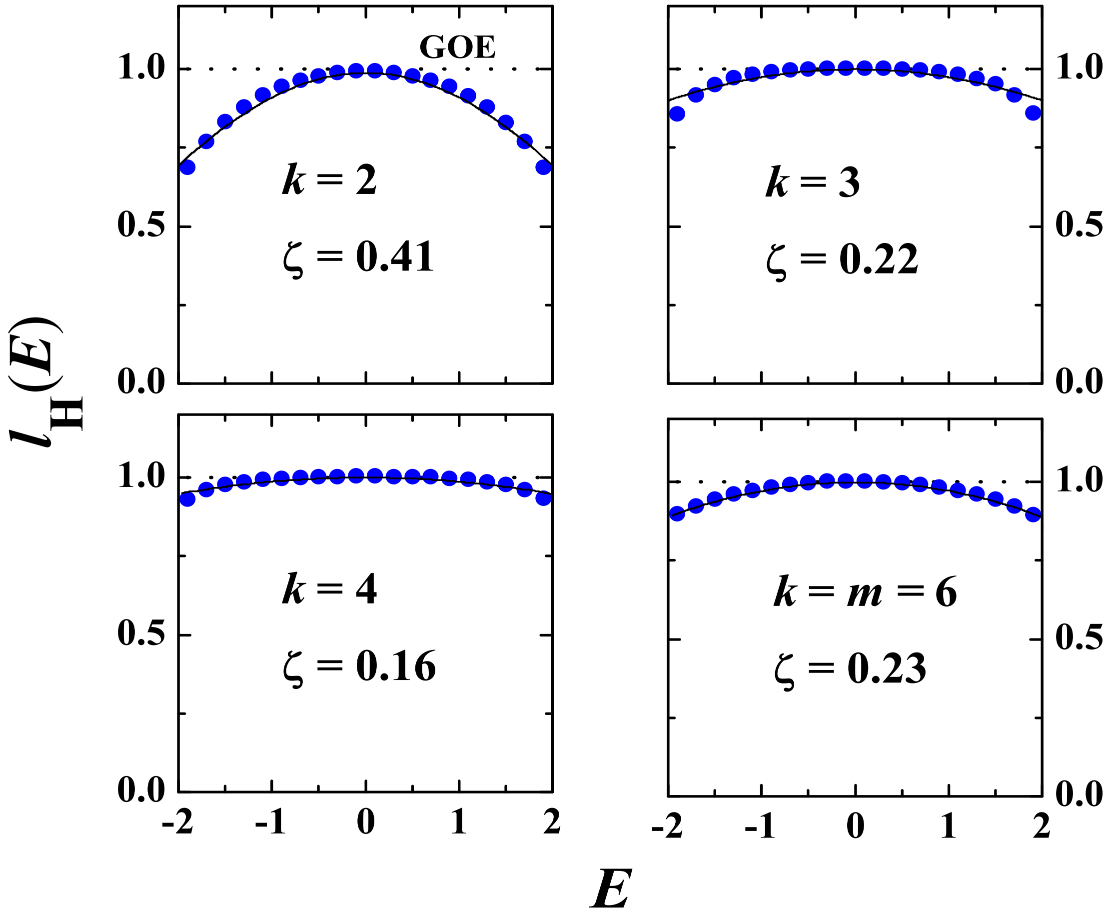

Figure 6.15: Ensemble averaged localization lengths l_H vs. normalized energy E for a 100 member EGOE(1+ k) for a system of $m = 6$ fermions distributed in $N = 12$ sp states. Here, results are shown for various values of body rank k = 2,3,4 and $k = m = 6$ and for $\lambda = 0.8$. Ensemble averaged EGOE(1+ k) results (blue solid circles) are compared with the smooth forms obtained via Eq.(6.8). Dotted lines in each graph represent GOE estimate.

6.3.2 Results for boson systems

Now moving further let us see the l_H results for BEGOE(1+ k). Fig. 6.16, shows results for ensemble averaged l_H vs. normalized energy E for a 100 member BEGOE($1+k$) with $m = 10$ bosons in $N = 5$ sp states for different values of body ranks k. We take the value of $\lambda = 1.0$ so that the system is in thermalization region. Numerical EE results (red solid circles) are compared with theoretical curves (black smooth curves) obtained using Eq. (6.8). The corresponding ensemble averaged ζ values are also shown in the figure. The q values are obtained from chapter 4. A very good agreement between numerical results and theory is obtained for all values of k in the bulk of the spectrum with small deviations near the spectrum tails. This is same as that observed from l_H results for fermions. Hence, in the chaotic domain, the energy variation of l_H, is essentially given by conditional q normal forms for the strength functions in both EGOE(1+k) and BEGOE(1+k).

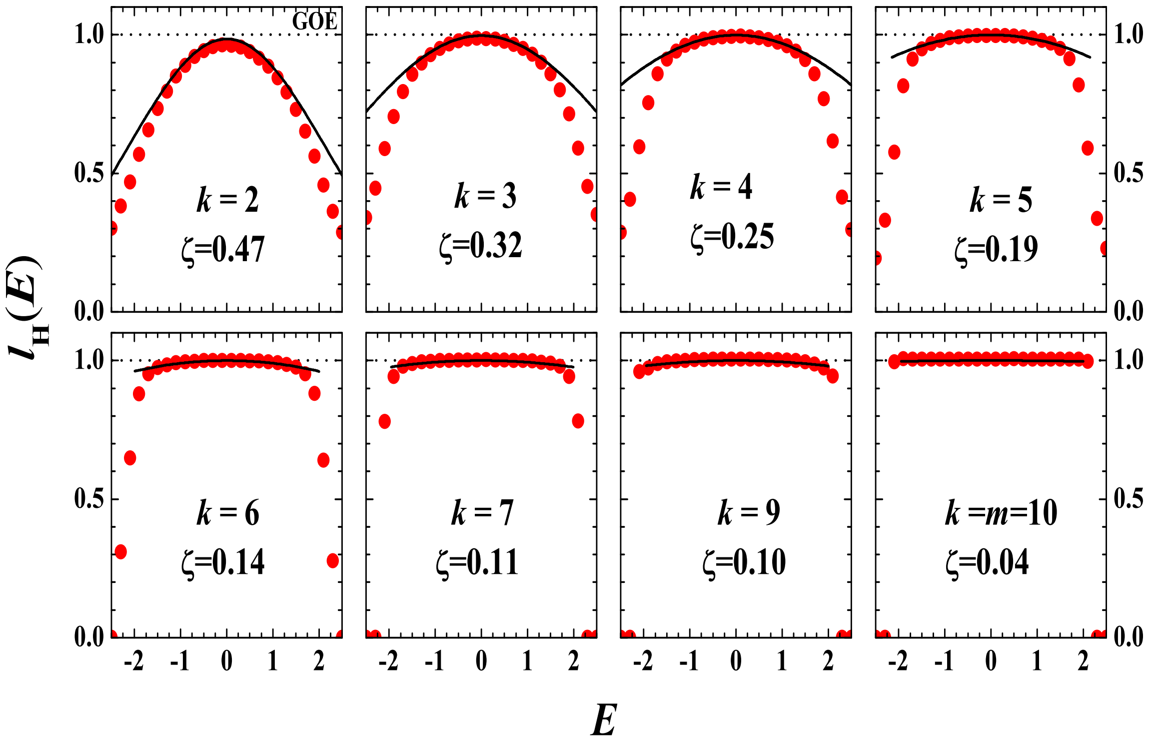

Figure 6.16: Ensemble averaged localization lengths l_H as a function of normalized energy E for a 100 member BEGOE(1+k) with $m = 10$ interacting bosons occupying $N = 5$ sp states. Here, results are shown for various values of body rank k and for $\lambda = 1.0$. Ensemble averaged BEGOE(1+k) results (red solid circles) are compared with the smooth forms obtained via Eq.(6.8). Dotted lines in each graph represent GOE estimate.

6.4 Fidelity Decay

In the study of relaxation of a complex (chaotic) quantum system after a sudden quench, to an equilibrium state, a quantity called fidelity decay plays a very important role. It is also popularly known in literature as return probability or survival probability or quantum fidelity. Let's say the system is prepared in one of the eigenstates ($\psi(0) = |\kappa\rangle$) of the unperturbed mean-field Hamiltonian $H = h(1)$. Due to a sudden quench at $t = 0$ by

$\lambda V(k)$, the system evolves unitarily with respect to $H \to h(1) + \lambda V(k)$ and the initial state changes after time t to $\psi(t) = |\kappa(t)\rangle = \exp(-iHt)\,|\kappa\rangle$. Then, the probability to find the system in its initial unperturbed state after time t is called fidelity decay which is given by,

$$\begin{aligned} W_0(t) &= |\langle\psi(t)|\psi(0)\rangle|^2 = \left|\sum_E \left[C_k^E\right]^2 \exp -iEt\right|^2 \\ &= \int F_\xi(E) \exp -iEt\; dE \\ &= \int_{s(q)} f_{CqN}(E|\xi;\zeta,q) \exp -iEt\; dE\,. \end{aligned} \tag{6.9}$$

Fidelity decay basically measures the closeness between the initial and final states of the quantum system under consideration. The above equation Eq.(6.9) suggests that fidelity is nothing but the Fourier transform in energy of the strength function; this is valid for times not very short or very long. In the thermalization region, we have two extreme situations : the form of $F_\xi(E)$ is Gaussian for $k = 2$ while it is semi-circle for $k = m$. Recently these two extreme situations are studied analytically in [23] and also numerically in [19,156,157]. The formula for $W_0(t)$ can be given in terms of width of $\lambda V(k)$ scaled by σ_H. We have seen in chapter 5 that in the thermodynamic region, the strength functions can be very well described by f_{CqN}. Following those results, in this chapter we use f_{CqN} to obtain $W_0(t)$ generated by BEGOE(1+k). So far the analytical formula for the Fourier transform of f_{CqN} is not available, therefore we evaluated Eq.(6.9) numerically. The results of $W_0(t)$ are presented in Fig. 6.17. A 100 member BEGOE(1+k) ensemble with $m = 10$, $N = 5$ and $\lambda = 0.5$ is considered for this analysis. The numerical results are obtained for various values of body rank k and are shown by red solid circles in Fig. 6.17. The numerical results are compared with the black smooth curves obtained by taking numerical Fourier transform of expression of $F_\xi(E)$ in terms of f_{CqN} discussed in chapter 5. Here, we have used normalized eigenenergies in the computation of W_0 and therefore the time t is measured in the units of $1/\sigma_H$. The results clearly show that the Fourier transform of f_{CqN} describes the short-time behavior nicely and also captures the positions of the oscillations. The results generated here are consistent with the reported results in [79], obtained using f_{qN} form for $F_\xi(E)$. The statistical properties, related to structure of wavefunction, namely NPC and S^{info} studied in the previous two sections, can be written as integrals involving strength functions over all ξ energies. However, in the study of fidelity decay, strength function with only $\xi = 0$ is involved. It is known that in the strong interaction domain, the decrease in W_0 (for $k = 2$) follows quadratic in time and this Gaussian decrease can last for a quite large time and after that an exponential one emerges [158]. The transition time depends on the ratio of the spectral width and the square of the second moment of strength fucntion (σ_F^2). As here $\lambda >> \lambda_t$, $\zeta^2 \to 0$ giving $\sigma_F^2 \approx 1$, t is in $1/\sigma_H$ units and the spectral width will be in σ_H units. Therefore, the results in Fig. 6.17 describe W_0 nicely for short time and the standard exponential decrease for long time for $k = 2$ seems absent.

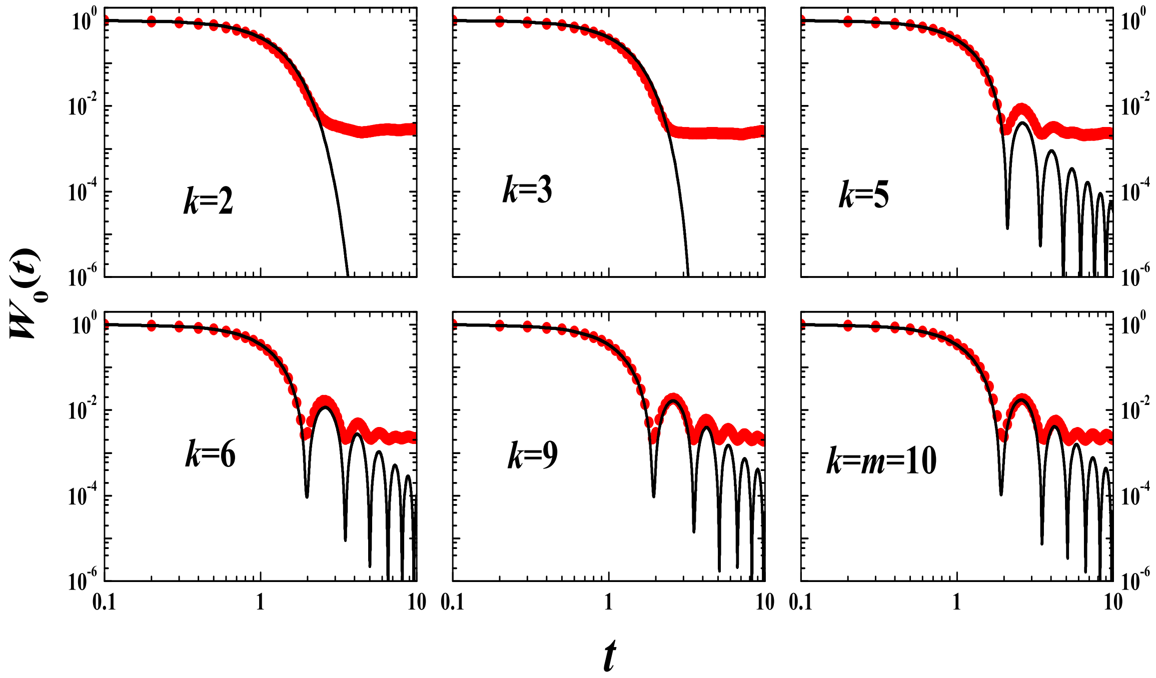

Figure 6.17: Fidelity decay $W_0(t)$ as a function of time for a BEGOE(1+ k) ensemble with $m = 10$ bosons in $N = 5$ sp states. An ensemble of 100 members is considered and the red solid circles represent the numerical results. The continuous black curves are obtained by taking numerical Fourier transform of the strength functions represented by f_{CqN} . Here $\psi(0)$ corresponds to middle states of $h(1)$ spectrum. Here t is measured in the units of σ_H^{-1} .

6.5 Conclusions

In this chapter the conditional q -normal density f_{CqN} is used to describe NPC and a two-parameter (ζ and q) analytical formula for NPC is derived as a function of energy for k -body interaction utilizing the interpolating form f_{CqN} of the strength functions. This formula is tested and found to be in very good agreement with numerical EE results for both EGOE(1+ k) and BEGOE(1+ k) in strong interaction domain. Moving ahead, the localization length l_H related to the information entropy S^{info} is studied numerically using EGOE(1+ k) and BEGOE(1+ k) as a function of energy for k -body interaction utilizing the interpolating form f_{CqN} of the strength functions and results are compared with ensemble averaged EE results. Lastly, the fidelity decay after k -body interaction quench for bosons using BEGOE(1+ k) is described using this interpolating form of the strength functions.