CHAPTER-II

RELATION BETWEEN CERTAIN SUMMABILITY METHODS OF AN INFINITE SERIES

2.1. INTRODUCTION:

Hüseyin Bor has established a relation between the $|\overline{N},p_n|_k$ and $|C,1|_k$, $k \geq 1$ summability methods of an infinite series. He pointed out that $|C,1|_k$ summability method can be obtained from $|\overline{N},p_n|_k$ summability method by taking $p_n = 1$ for all values of $n \in N$. He has also remarked that one can find a sequence (p_n) for which the methods $|\overline{N},p_n|_k$ and $|C,1|_k$ are independent from each other. Hence a question arises that, if a series is summable $|C,1|_k$, then what conditions should be imposed on a sequence (p_n) so that the same series becomes summable $|\overline{N},p_n|_k$, $k \geq 1$? . In order to answer this type of question Hüseyin Bor has proved the following two theorems.

THEOREM 1 [10]:

Let (p_n) be a sequence of positive real constants such that as $n \to \infty$

$$np_n = O(P_n), \tag{2.1.1}$$

$$P_n = O(np_n). \tag{2.1.2}$$

If $\sum_{n=0}^{\infty} a_n$ is summable $|C,1|_k$, then it is also summable $|\overline{N},p_n|_k$, $k \geq 1$.

THEOREM 2 [11]:

Let (p_n) be a sequence of positive real constants such that it satisfies the conditions (2.1.1) and (2.1.2). If $\sum_{n=0}^{\infty} a_n$ is summable $|\overline{N},p_n|_k$, then it is also summable $|C,1|_k$, $k \geq 1$.

Further, by putting these two results together, Hüseyin Bor obtained the following theorem.

THEOREM 3 [11]:

Suppose (p_n) is a sequence of nonnegative real constants such that $P_n = \sum_{v=0}^{n} p_v \neq 0$, $P_n \to \infty$ as $n \to \infty$, and that (2.1.1) and (2.1.2) hold. Then summability $|C,1|_k$ is equivalent to summability $|\overline{N},p_n|_k$, $k \geq 1$.

All these theorems of Hüseyin Bor are related to $|C,1|_k$ and $|\overline{N},p_n|_k$ summability methods. Considering more general summability methods such as $|C,1;\delta|_k$ and $|\overline{N},p_n;\delta|_k$, Huseyin Bor generalized his own Theorem 1 by proving the following.

THEOREM 4 [12]:

Let (p_n) be a sequence of positive real constants such that as $n \to \infty$

$$np_n = O(P_n) \tag{2.1.3}$$

$$P_n = O(np_n). \tag{2.1.4}$$

and

$$\sum_{n=v}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}} = O\left[\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right]. \tag{2.1.5}$$

If $\sum_{n=0}^{\infty} a_n$ is summable $|C,1;\delta|_k$, then it is also summable $|\overline{N},p_n;\delta|_k$, $k \geq 1$, $\delta \geq 0$.

On the other hand, Theorem 1 was also generalized by Ö.Cakar and C.Orhan [22], replacing condition (2.1.2) by a weaker condition. Their result is given below:

THEOREM 5 [22] :

Let (p_n) be a sequence of positive real constants for which as $n \to \infty$

$$np_n = O(P_n) \tag{2.1.6}$$

$$\sum_{v=1}^{n}\left(\frac{P_v}{v}\right) = O(P_{n-1}). \tag{2.1.7}$$

If a series $\sum_{n=0}^{\infty} a_n$ is summable $|C,1|_k$, then it is also summable $|\overline{N},p_n|_k$, $k \geq 1$.

It is interesting to note that M.A.Sarigol [51] battered Theorem 1 due to Hüseyin Bor by using only the condition (2.1.1) and dropping condition (2.1.2). He also bettered Theorem 2 by using only the condition (2.1.2) and dropping condition (2.1.1). In fact, his results are as under:

THEOREM 6 [51, Theorem 3.1]:

Let (p_n) be a sequence of positive real constants satisfying condition (2.1.1) . If $\sum_{n=0}^{\infty} a_n$ is summable $|C,1|_k$, then it is summable $|\overline{N},p_n|_k$, $k \geq 1$.

THEOREM 7 [51, Theorem 3.2] :

Let (p_n) be a sequence of positive real constants satisfying condition (2.1.2). If $\sum_{n=0}^{\infty} a_n$ is summable $|\overline{N},p_n|_k$, then it is summable $|C,1|_k$, $k \geq 1$.

Further in this direction, G.Sunouchi and L.S.Bosenquent, proved the following theorem in 1950.

THEOREM 8 [([52],[4])]:

The necessary and sufficient condition for a series $\sum_{n=0}^{\infty} a_n$ to be summable $|\overline{N},q_n|$ whenever it is summable $|\overline{N},p_n|$ is

$$\frac{q_n P_n}{Q_n p_n} = O(1) \tag{2.1.8}$$

as $n \to \infty$.

The sufficiency part of the above Theorem was proved by G.Sunouchi and the necessity part was proved by L.S.Bosanquet.

In 1987, H.Bor and Thorpe [17] proved a more general result in this direction as under:

THEOREM 9 [17]:

Let (p_n) and (q_n) be sequences of positive real constants. If

$$p_n Q_n = O(P_n q_n) \qquad (2.1.9)$$

$$P_n q_n = O(p_n Q_n) \qquad (2.1.10)$$

then the series $\sum_{n=0}^{\infty} a_n$ is summable $\left|\overline{N}, p_n\right|_k$ whenever it is also summable $\left|\overline{N}, q_n\right|_k$, $k \geq 1$.

2.2 MAIN RESULTS:

In this chapter, we intend to prove more general results by establishing the relation between $|C,1;\delta|_k$ (see chapter-I, definition 2) and $\left|\overline{N}, p_n;\delta\right|_k$ (see chapter-I, definition 7) summability methods under a weaker condition. Incidentally our results will generalize the results due to Hüseyin Bor (Theorem 1, Theorem 2 and Theorem 4) and Ö.Cakar and C.Orhan (Theorem 5). Our aim in this chapter is also to extend Theorem 6 to Theorem 9 for $X - \left|\overline{N}, p_n\right|_k$ summability. In this regard, we refer the definition of $X - \left|\overline{N}, p_n\right|_k$ summability due to S.M.Mazhar, which is given earlier in chapter-I (see definition 9). S.M.Mazhar has remarked in [39] that the summabilities $|C,1|_k$, $|R,p_n|_k$, $|\overline{N},p_n|$ and $|\overline{N},p_n|_k$ can be obtained from a single summability $X-|\overline{N},p_n|_k$ $k\geq 1$. In fact, we shall prove the following theorems.

THEOREM A [54]:

Let (p_n) be a sequence of positive real constants such that as $n\to\infty$

$$np_n = O(P_n) \tag{2.2.1}$$

$$\sum_{v=1}^{n}\left(\frac{P_v}{v}\right) = O(P_{n-1}) \tag{2.2.2}$$

and

$$\sum_{n=v}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}} = O\left[\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right]. \tag{2.2.3}$$

If $\sum_{n=0}^{\infty} a_n$ is summable $|C,1;\delta|_k$, then it is summable $|\overline{N},p_n;\delta|_k$, $k\geq 1$, $\delta\geq 0$.

Remark 1:

Ö.Cakar and C.Orhan [22] have pointed out that condition (2.1.2) implies condition (2.2.2) but converse is not true. Thus our Theorem A is a generalization of Theorem 4, as we are replacing condition (2.1.4) by a weaker condition (2.2.2).

Remark 2:

It is also interesting to observe that when $\delta=0$, our Theorem A gives Theorem 5.

THEOREM B [54]:

Let (p_n) be a sequence of positive real constants such that as $n \to \infty$

$$P_n = O(np_n) \tag{2.2.4}$$

$$np_n = O(P_n) \tag{2.2.5}$$

and

$$\sum_{n=v}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left[\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}\right]. \tag{2.2.6}$$

If $\sum_{n=0}^{\infty} a_n$ is summable $\left|\overline{N}, p_n; \delta\right|_k$, then it is summable $|C,1;\delta|_k$, $k \geq 1$,

$0 \leq \delta k < 1$, $\delta \geq 0$.

Remark 3 :

It is easy to see that, when $\delta = 0$, Theorem B reduces to Theorem 2 due to Hüseyin Bor.

THEOREM C [55]:

Suppose (p_n), (q_n), (X_n) and (Y_n) are sequences of positive real constants such that as $n \to \infty$

$$P_n q_n = O(p_n Q_n) \tag{2.2.7}$$

$$Q_n = O(q_n X_n) \tag{2.2.8}$$

$$p_n Y_n = O(P_n). \tag{2.2.9}$$

If $\sum_{n=0}^{\infty} a_n$ is summable $X-\left|\overline{N},p_n\right|_k$, then it is summable $Y-\left|\overline{N},q_n\right|_k$, $k \geq 1$.

Remark 4:

It may be observed that if we take $X_n = \frac{P_n}{p_n}$ and $Y_n = \frac{Q_n}{q_n}$ in Theorem C then we get Theorem 9 due to H.Bor and B.Thorpe.

THEOREM D:

Suppose (p_n), (q_n), (X_n) and (Y_n) are sequences of positive real constants such that as $n \to \infty$

$$P_n q_n = O(p_n Q_n) \qquad (2.2.10)$$

$$Y_n q_n = O(Q_n) \qquad (2.2.11)$$

$$P_n = O(X_n p_n) \qquad (2.2.12)$$

If $\sum_{n=0}^{\infty} a_n$ is summable $X-\left|\overline{N},p_n\right|_k$, then it is summable $Y-\left|\overline{N},q_n\right|_k$, $k \geq 1$.

Remark 5 :

It can be observed that if we take $X_n = \frac{P_n}{p_n}$, $Y_n = \frac{Q_n}{q_n}$ and $k=1$ in Theorem D, then we get sufficient part of Theorem 8 due to G.Sunouchi.

Remark 6 :

It is also interesting to observe that, if we take $p_n = 1$ for all $n \in N$ and $Y_n = \frac{Q_n}{q_n}, X_n = \frac{P_n}{p_n}$ in our Theorem D, then we get Theorem 6 due to M.A.Sarigol.

If we interchange the role of the sequences (X_n) and (Y_n) in Theorem D, then we get the following theorem.

THEOREM E:

Suppose (p_n), (q_n), (X_n) and (Y_n) are sequences of positive real constants such that

$$P_n q_n = O(p_n Q_n) \tag{2.2.13}$$

$$X_n q_n = O(Q_n) \tag{2.2.14}$$

$$P_n = O(Y_n p_n) \tag{2.2.15}$$

If $\sum_{n=0}^{\infty} a_n$ is summable $Y - |\overline{N}, q_n|_k$, then it is summable $X - |\overline{N}, p_n|_k$, $k \geq 1$.

Remark 7 :

It can be observed that if we take $X_n = \frac{P_n}{p_n}$, $Y_n = \frac{Q_n}{q_n}$ and $k = 1$ in Theorem E, then we get necessary part of Theorem 8 due to L.S.Bosenquent.

Remark 8 :

It may noted that if we take $X_n = \frac{P_n}{p_n}$ and $q_n = 1$ for all value of $n \in N$ in Theorem E, then we get Theorem 7 due to M.A.Sarigöl.

Remark 9:

Further, it is interesting to note that if we put $X_n = Y_n = n$ in our Theorems C to Theorem E, then we get a relation between Absolute Reisz summabilities of order k with respect to the sequences (p_n) and (q_n) in the form of the following corollaries:

COROLLARY 1:

Suppose (p_n) and (q_n) are sequences of positive real constants such that as $n \to \infty$

$$P_n q_n = O(p_n Q_n),$$

$$np_n = O(P_n),$$

and

$$Q_n = O(nq_n).$$

If the series $\sum_{n=0}^{\infty} a_n$ is summable $|R, p_n|_k$, then it is also summable $|R, q_n|_k$, $k \geq 1$

COROLLARY 2:

Suppose (p_n) and (q_n) are sequences of positive real constants such that as $n \to \infty$

$$nq_n = O(Q_n),$$

and

$$P_n = O(np_n).$$

If the series $\sum_{n=0}^{\infty} a_n$ is summable $|R, p_n|_k$, then it is also summable $|R, q_n|_k$, $k \geq 1$.

COROLLARY 3:

Suppose (p_n) and (q_n) are sequences of positive real constants such that as $n \to \infty$

$$P_n q_n = O(p_n Q_n)$$

$$np_n = O(P_n)$$

$$Q_n = O(nq_n).$$

If the series $\sum_{n=0}^{\infty} a_n$ is summable $|R, q_n|_k$, then it is also summable $|R, p_n|_k$, $k \geq 1$.

Now we provide one by one, proof of our results from Theorem A to Theorem E.

2.3. PROOF OF THE THEOREMS:

First we establish some general terms. Let (t_n) be sequence of $(\overline{N}, p_n)$ means of the series $\sum_{n=0}^{\infty} a_n$. Then, by definition, we have

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \ ,$$

where

$$s_v = a_0 + a_1 + a_2 + ------ + a_n .$$

Therefore

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{z=0}^{v} a_z$$

$$= \frac{1}{P_n}\left[p_o \sum_{z=0}^{0} a_z + p_1 \sum_{z=0}^{1} a_z + p_2 \sum_{z=0}^{2} a_z + ------ + p_n \sum_{z=0}^{n} a_z \right]$$

$$= \frac{1}{P_n}\left[p_0 a_0 + p_1(a_0 + a_1) + p_2(a_0 + a_1 + a_2) + --- + p_n(a_0 + a_1 + ---- + a_n)\right]$$

$$= \frac{1}{P_n}\left[(p_0 + p_1 + ---- + p_n)a_0 + (p_1 + p_2 + ---- + p_n)a_1 + ------ + p_n a_n \right]$$

$$= \frac{1}{P_n}\left[P_n a_0 + (P_n - p_0)a_1 + (P_n - (p_0 + p_1))a_2 + ------ + p_n a_n \right]$$

$$= \frac{1}{P_n}\left[P_n a_0 + (P_n - P_0)a_1 + (P_n - P_1)a_2 + ------ + (P_n - P_{n-1})a_n \right]$$

$$= \frac{1}{P_n}\sum_{v=0}^{n} (P_n - P_{v-1})a_v \ , n \geq 0 . \qquad (2.3.1)$$

Then for $n \geq 1$, we have

$$t_n - t_{n-1} = \frac{1}{P_n}\sum_{v=1}^{n} (P_n - P_{v-1})a_v - \frac{1}{P_{n-1}}\sum_{v=1}^{n-1} (P_{n-1} - P_{v-1})a_v$$

$$= \frac{1}{P_n}\sum_{v=1}^{n} P_n a_v - \frac{1}{P_n}\sum_{v=1}^{n} P_{v-1} a_v - \frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_{n-1} a_v + \frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_{v-1} a_v$$

$$= \sum_{v=1}^{n} a_v - \frac{1}{P_n}\sum_{v=1}^{n} P_{v-1}a_v - \sum_{v=1}^{n-1} a_v + \frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_{v-1}a_v$$

$$= \frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_{v-1}a_v - \frac{1}{P_n}\sum_{v=1}^{n} P_{v-1}a_v + a_n$$

$$= \frac{1}{P_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v - \frac{1}{P_n}\sum_{v=1}^{n} P_{v-1}a_v$$

$$= \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\sum_{v=1}^{n} P_{v-1}a_v$$

$$= \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v . \qquad (2.3.2)$$

Similarly, if (T_n) denotes a sequence of $(\overline{N}, q_n)$ means of the series $\sum_{n=0}^{\infty} a_n$. Then, by (2.3.1) and (2.3.2), we have

$$T_n = \frac{1}{Q_n}\sum_{v=1}^{n}(Q_v - Q_{v-1})a_v \qquad (2.3.3)$$

and

$$T_n - T_{n-1} = \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n} Q_{v-1}a_v . \qquad (2.3.4)$$

PROOF OF THEOREM A :

Since the series $\sum_{n=0}^{\infty} a_n$ is summable $|C,1;\delta|_k$ it follows that (see chapter-I, definition 2)

$$\sum_{n=0}^{\infty} n^{\delta k-1}|t_n|^k < \infty . \tag{2.3.5}$$

By (2.3.2),we have

$$t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \left(\frac{v P_{v-1}}{v} \right) a_v . \tag{2.3.6}$$

Applying Able's transformation on the right hand side of (2.3.6), we get

$$t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta\left(\frac{P_{v-1}}{v} \right) \sum_{z=1}^{v} z a_z + \left(\frac{P_{n-1}}{n} \right)\left(\frac{p_n}{P_n P_{n-1}} \right) \sum_{z=1}^{n} z a_z$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta\left(\frac{P_{v-1}}{v} \right) \sum_{z=1}^{v} z a_z + \left(\frac{p_n}{P_n} \right) \sum_{z=1}^{n} z a_z .$$

Since

$$\Delta\left(\frac{P_{v-1}}{v} \right) = \frac{1}{v} \Delta(P_{v-1}) - P_v \Delta\left(\frac{1}{v} \right)$$

$$= \frac{1}{v}(P_{v-1} - P_v) - P_v \left(\frac{1}{v} - \frac{1}{v+1} \right)$$

$$= -\frac{p_v}{v} - \frac{P_v}{v(v+1)} .$$

Therefore,

$$t_n - t_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{p_v}{v} \sum_{z=1}^{v} z a_z + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{v(v+1)} \right) \sum_{z=1}^{n} z a_z + \frac{p_n}{n P_n} \sum_{z=1}^{n} z a_z$$

$$= -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{v+1}{v}\right) p_v \left(\frac{1}{v+1} \sum_{z=1}^{v} z a_z\right) + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{v}\right) \left(\frac{1}{v+1} \sum_{z=1}^{n} z a_z\right) +$$

$$\left(\frac{n+1}{n}\right) \frac{p_n}{P_n} \left(\frac{1}{n+1} \sum_{z=1}^{n} z a_z\right).$$

$$= \quad -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{v+1}{v}\right) p_v t_v \; + \; \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{v}\right) t_v \; + \left(\frac{n+1}{n}\right) \frac{p_n}{P_n} t_n .$$

$= \; t_{n,1} + t_{n,2} + t_{n,3}$, say.

Since

$$\left|t_{n,1} + t_{n,2} + t_{n,3}\right|^k \le 4^k \left(\left|t_{n,1}\right|^k + \left|t_{n,2}\right|^k + \left|t_{n,3}\right|^k\right),$$

it follows that,to complete the proof of theorem A , it is enough to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \left|t_{n,i}\right|^k < \infty, \quad \text{for } i = 1,2,3. \tag{2.3.7}$$

Let us apply Hölder's inequality with indices k and k' , where

$\frac{1}{k} + \frac{1}{k'} = 1$, we get

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \left|t_{n,1}\right|^k$$

$$= \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \left|-\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{v+1}{v}\right) p_v t_v\right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left\{\sum_{v=1}^{n-1} p_v |t_v|\right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \left\{\sum_{v=1}^{n-1} p_v |t_v|\right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{\sum_{v=1}^{n-1} p_v |t_v|\right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \left\{\sum_{v=1}^{n-1} p_v |t_v|^k\right\} \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \left\{\sum_{v=1}^{n-1} p_v |t_v|^k\right\}$$

$$= O(1) \sum_{v=1}^{m} p_v |t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} p_v |t_v|^k \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}, \text{ by (2.2.3)}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k-1} |t_v|^k$$

$$= O(1) \sum_{v=1}^{m} v^{\delta k-1} |t_v|^k \text{ , by (2.2.1)}$$

$$= O(1) \text{ as } m \to \infty \text{ , by (2.3.5).}$$

Again , we have

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |t_{n,2}|^k$$

$$= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{v}\right)t_v\right|^k$$

$$\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{v}\right)|t_v|\right\}^k$$

$$= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{v}\right)|t_v|\right\}^k$$

$$\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{v}\right)|t_v|^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{v}\right)\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{v}\right)|t_v|^k\right\} \text{ , by (2.2.2)}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{v}\right)|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{v}\right)|t_v|^k\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}, \text{ by (2.2.3)}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|t_v|^k$$

$$= O(1)\sum_{v=1}^{m}v^{\delta k-1}|t_v|^k \text{ , by (2.2.1)}$$

$$= O(1) \text{ as } m\to\infty \text{ , by (2.3.5).}$$

Finally, we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|t_{n,3}|^k$$

$$= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\left(\frac{n+1}{n}\right)\frac{p_n}{P_n}t_n\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k|t_n|^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|t_n|^k$$

$$= O(1)\sum_{n=1}^{m} n^{\delta k-1}|t_n|^k \quad \text{, by (2.2.1)}$$

$$= O(1) \quad \text{as } m\to\infty \quad \text{, by (2.3.5).}$$

Therefore we get

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|t_{n,i}|^k<\infty, i=1,2,3.$$

This completes the proof of theorem A.

PROOF OF THEOREM B:

Since the series $\sum_{n=0}^{\infty}a_n$ is summable $|\overline{N},p_n;\delta|_k$, it follows that (see chapter-I, definition 7)

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|t_n-t_{n-1}|^k<\infty. \qquad (2.3.8)$$

By (2.3.1), we have for $n\geq1$

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v .$$

We write

$$\Delta t_{n-1} = t_{n-1} - t_n = -\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v .$$

Therefore

$$\Delta t_{n-1} = -\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_{v-1}a_v - \frac{p_n}{P_n}a_n$$

i.e. $$\frac{p_n}{P_n}a_n = -\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_{v-1}a_v - \Delta t_{n-1}$$

i.e. $$a_n = -\frac{P_n}{p_n}\Delta t_{n-1} - \frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_{v-1}a_v$$

i.e. $$a_n = -\frac{P_n}{p_n}\Delta t_{n-1} + \frac{P_{n-2}}{p_{n-1}}\Delta t_{n-2} . \qquad (2.3.9)$$

Let (w_n) denotes the nth $(C,1)$ means of the sequence (na_n). Then by definition 1 of chapter-I, we have

$$w_n = \frac{1}{n+1}\sum_{v=1}^{n} v a_v$$

$$= \frac{1}{n+1}\sum_{v=1}^{n} v\left[-\frac{P_v}{p_v}\Delta t_{v-1} + \frac{P_{v-2}}{p_{v-1}}\Delta t_{v-2}\right] , \quad \text{by } (2.3.9)$$

$$= \frac{1}{n+1}\sum_{v=1}^{n-1}(-v)\frac{P_v}{p_v}\Delta t_{v-1} - \frac{nP_n}{(n+1)p_n}\Delta t_{n-1} + \frac{1}{n+1}\sum_{v=1}^{n}(v)\frac{P_{v-2}}{p_{v-1}}\Delta t_{v-2}$$

$$= \frac{1}{n+1}\sum_{v=1}^{n-1}(-v)\frac{P_v}{p_v}\Delta t_{v-1} + \frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)\frac{P_{v-1}}{p_v}\Delta t_v - \frac{nP_n}{(n+1)p_n}\Delta t_{n-1}$$

$$= \frac{1}{n+1}\left\{\sum_{v=1}^{n-1}\frac{\Delta t_{v-1}}{P_v}\left[-vP_v+(v+1)P_{v-1}\right]\right\} - \frac{nP_n}{(n+1)p_n}\Delta t_{n-1}$$

Since

$$\begin{aligned} -vP_v+(v+1)P_{v-1} &= -vP_v+vP_{v-1}+P_{v-1} \\ &= v(P_{v-1}-P_v)+P_{v-1} \\ &= -vp_v+P_{v-1} \\ &= -vp_v+P_v-p_v \\ &= P_V-(v+p_V). \end{aligned}$$

Therefore

$$w_n = \frac{1}{n+1}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)\Delta t_{v-1} - \frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)\Delta t_{v-1} - \frac{nP_n}{(n+1)p_n}\Delta t_{n-1}$$

$$= w_{n,1}+w_{n,2}+w_{n,3}\text{ , say.}$$

Since

$$\left|w_{n,1}+w_{n,2}+w_{n,3}\right|^k \le 4^k\left(\left|w_{n,1}\right|^k+\left|w_{n,2}\right|^k+\left|w_{n,3}\right|^k\right),$$

we see that,to complete the proof of theorem B , it is enough to show that

$$\sum_{n=1}^{\infty} n^{\delta k-1}\left|w_{n,i}\right|^k < \infty \text{ , for } i=1,2,3. \tag{2.3.10}$$

Applying Hölder's inequality with indices k and k', where

$\frac{1}{k}+\frac{1}{k'}=1$, we have

$$\sum_{n=2}^{m+1} n^{\delta k-1}\left|w_{n,1}\right|^k$$

$$= \sum_{n=2}^{m+1} n^{\delta k-1}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)\Delta t_{v-1}\right|^k$$

$$\leq \sum_{n=2}^{m+1} n^{\delta k-1}\left(\frac{1}{n+1}\right)^k\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)\Delta t_{v-1}\right\}^k$$

$$\leq \sum_{n=2}^{m+1}\frac{n^{\delta k-1}}{n^k}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k\left|\Delta t_{v-1}\right|^k\right\}\left\{\sum_{v=1}^{n-1}1\right\}^{k-1}$$

$$= \sum_{n=2}^{m+1}\frac{n^{\delta k-1}}{n}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k\left|\Delta t_{v-1}\right|^k\right\}\left\{\frac{1}{n}\sum_{v=1}^{n-1}1\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{n^{\delta k-1}}{n}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k\left|\Delta t_{v-1}\right|^k\right\}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{2-\delta k}}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k\left|\Delta t_{v-1}\right|^k\right\}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\left|\Delta t_{v-1}\right|^k\sum_{n=v+1}^{m+1}\frac{1}{n^{2-\delta k}}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\left|\Delta t_{v-1}\right|^k\left(\frac{1}{v^{1-\delta k}}\right)$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k+k-1}\left|\Delta t_{v-1}\right|^k \text{ by (2.2.4)}$$

$$= O(1) \text{ as } m\to\infty \text{, by (2.3.8).}$$

Again

$$\sum_{n=2}^{m+1} n^{\delta k-1}\left|w_{n,2}\right|^k$$

$$= \sum_{n=2}^{m+1} n^{\delta k-1}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)\Delta t_{v-1}\right|^k$$

$$\leq \sum_{n=2}^{m+1} n^{\delta k-1}\left(\frac{1}{n+1}\right)^k\left\{\sum_{v=1}^{n-1}\frac{v(v+1)}{v}\left|\Delta t_{v-1}\right|\right\}^k$$

$$= O(1) \sum_{n=2}^{m+1}\frac{n^{\delta k-1}}{n^k}\left\{\sum_{v=1}^{n-1} v^k\left|\Delta t_{v-1}\right|^k\right\}\left\{\sum_{v=1}^{n-1}1\right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1}\frac{1}{n^{2-\delta k}}\left\{\sum_{v=1}^{n-1} v^k\left|\Delta t_{v-1}\right|^k\right\}\left\{\frac{1}{n}\sum_{v=1}^{n-1}1\right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1}\frac{1}{n^{2-\delta k}}\left\{\sum_{v=1}^{n-1} v^k\left|\Delta t_{v-1}\right|^k\right\}$$

$$= O(1) \sum_{v=1}^{m} v^k\left|\Delta t_{v-1}\right|^k\sum_{n=v+1}^{m+1}\frac{1}{n^{2-\delta k}}$$

$$= O(1) \sum_{v=1}^{m} v^{\delta k+k-1}\left|\Delta t_{v-1}\right|^k$$

$$= O(1) \sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k+k-1}\left|\Delta t_{v-1}\right|^k \text{, by (2.2.5)}$$

$$= O(1) \text{ as } m\to\infty \text{, by (2.3.8).}$$

Finally, we have

$$\sum_{n=2}^{m+1} n^{\delta k-1}\left|w_{n,3}\right|^k$$

$$= \sum_{n=2}^{m+1} n^{\delta k-1} \left| \frac{n}{n+1} \frac{P_n}{p_n} \Delta t_{n-1} \right|^k$$

$$= O(1) \sum_{n=2}^{m} n^{\delta k-1} \left(\frac{P_n}{p_n} \right)^k |\Delta t_{n-1}|^k$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^{\delta k+k-1} |\Delta t_{v-1}|^k$$

$= o(1)$ as $m \to \infty$, by (2.3.8).

Hence

$$\sum_{n=1}^{\infty} n^{\delta k-1} |w_{n,i}|^k < \infty \quad , \text{ for } i = 1,2,3.$$

This completes the proof of theorem B.

PROOF OF THEOREM C :

Since the series $\sum_{n=0}^{\infty} a_n$ is summable $X - \left|\overline{N}, p_n\right|_k$, it follows that (see chapter-I, definition 9)

$$\sum_{n=1}^{\infty} X_n^{k-1} |t_n - t_{n-1}|^k < \infty \ . \tag{2.3.11}$$

By (2.3.1), (2.3.2) and (2.3.9) we have

$$T_n = \frac{1}{Q_n} \sum_{v=0}^{n} (Q_n - Q_{v-1}) a_v$$

$$T_n - T_{n-1} = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v \text{ , for } (n \geq 1),$$

and

$$a_n = -\frac{P_n}{p_n}\Delta t_{n-1} + \frac{P_{n-2}}{p_{n-1}}\Delta t_{n-2} .$$

Therefore, we have

$$T_n - T_{n-1} = -\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n} Q_{v-1}\left[-\frac{P_v}{p_v}\Delta t_{v-1} + \frac{P_{v-2}}{p_{v-1}}\Delta t_{v-2}\right]$$

$$= \frac{q_nP_n}{Q_np_n}\Delta t_{n-1} + \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)Q_v\Delta t_{v-1} + \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_{v-1}}{p_v}\right)Q_v\Delta t_{v-1}$$

$$= -\frac{q_nP_n}{Q_np_n}\Delta t_{n-1} + \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{\Delta t_{v-1}}{p_v}\left(Q_{v-1}P_v - Q_vP_{v-1}\right) .$$

But

$$Q_{v-1}P_v - Q_vP_{v-1} = Q_{v-1}P_v - Q_vP_v + Q_vp_v$$

$$= (Q_{v-1} - Q_v)P_v + p_vQ_v$$

$$= -q_vP_v + p_vQ_v .$$

Thus,

$$T_n - T_{n-1} = -\frac{q_nP_n}{Q_np_n}\Delta t_{n-1} + \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{\Delta t_{v-1}}{p_v}\left(-q_vP_v + p_vQ_v\right)$$

$$= -\frac{q_nP_n}{Q_np_n}\Delta t_{n-1} - \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)q_q\Delta t_{v-1} + \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}Q_v\Delta t_{v-1}$$

$$= T_{n,1} + T_{n,2} + T_{n,3} \text{ , say.}$$

Since

$$\left|T_{n,1} + T_{n,2} + T_{n,3}\right|^k \le 4^k\left(\left|T_{n,1}\right|^k + \left|T_{n,2}\right|^k + \left|T_{n,3}\right|^k\right),$$

we see that, to complete the proof of theorem C , it is enough to show that

$$\sum_{n=1}^{\infty} Y_n^{k-1} |T_{n,i}|^k < \infty \quad , \text{ for } i = 1,2,3. \tag{2.3.12}$$

Firstly, we have

$$\sum_{n=1}^{\infty} Y_n^{k-1} |T_{n,1}|^k$$

$$= \sum_{n=1}^{\infty} Y_n^{k-1} \left| \frac{q_n P_n}{Q_n p_n} \Delta t_{n-1} \right|^k$$

$$= \sum_{n=1}^{\infty} Y_n^{k-1} \left(\frac{q_n P_n}{Q_n p_n} \right)^k |\Delta t_{n-1}|^k$$

$$= \sum_{n=1}^{\infty} Y_n^{k-1} \left(\frac{Q_n}{q_n} \right)^k \left(\frac{q_n}{Q_n} \right)^k |\Delta t_{n-1}|^k \quad \text{, by (2.2.7)}$$

$$= O(1) \sum_{n=1}^{\infty} \left(\frac{Q_n}{q_n} \right)^{k-1} |\Delta t_{n-1}|^k$$

$$= O(1) \sum_{n=1}^{\infty} X_n^{k-1} |\Delta t_{n-1}|^k \quad \text{, by (2.2.8)}$$

$$= O(1) \text{ , by (2.3.11).}$$

Another application of Hölder's inequality, gives

$$\sum_{n=1}^{\infty} Y_n^{k-1} |T_{n,2}|^k$$

$$= \sum_{n=2}^{m+1} Y_n^{k-1} \left| -\frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right) q_v \Delta t_{v-1} \right|^k$$

$$\leq \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}}\right)^k \left\{ \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right) q_v |\Delta t_{v-1}| \right\}^k$$

$$\leq \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n}\right)^k \frac{1}{Q_{n-1}} \left\{ \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k q_v |\Delta t_{v-1}|^k \right\} \left\{ \frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n}\right)^k \frac{1}{Q_{n-1}} \left\{ \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k q_v |\Delta t_{v-1}|^k \right\}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k q_v |\Delta t_{v-1}|^k \left\{ \sum_{n=v+1}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n}\right)^k \frac{1}{Q_{n-1}} \right\}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k q_v |\Delta t_{v-1}|^k \left\{ \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\frac{q_n}{Q_n}\right)^k \frac{1}{Q_{n-1}} \right\}, \text{ by (2.2.9)}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k q_v |\Delta t_{v-1}|^k \left\{ \sum_{n=v+1}^{m+1} \left(\frac{Q_n}{q_n}\right)^{k-1} \left(\frac{q_n}{Q_n}\right)^k \frac{1}{Q_{n-1}} \right\}, \text{ by (2.2.7)}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k q_v |\Delta t_{v-1}|^k \left\{ \sum_{n=v+1}^{m+1} \frac{q_n}{Q_n Q_{n-1}} \right\}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k \left(\frac{q_v}{Q_v}\right) |\Delta t_{v-1}|^k$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{Q_v}{q_v}\right)^k \left(\frac{q_v}{Q_v}\right) |\Delta t_{v-1}|^k, \text{ by (2.2.7)}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{Q_v}{q_v}\right)^{k-1} |\Delta t_{v-1}|^k$$

$$= O(1) \sum_{v=1}^{m} X_v^{k-1} |\Delta t_{v-1}|^k \text{ , by (2.2.8)}$$

$$= o(1) \text{ as } m \to \infty \text{ , by (2.3.4).}$$

Finally, we have

$$\sum_{n=1}^{\infty} Y_n^{k-1} |T_{n,3}|^k$$

$$= \sum_{n=2}^{m+1} Y_n^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} Q_v \Delta t_{v-1} \right|^k$$

$$\leq \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k \left\{ \sum_{v=1}^{n-1} Q_v |\Delta t_{v-1}| \right\}^k$$

$$\leq \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n} \right)^k \frac{1}{Q_{n-1}} \left\{ \sum_{v=1}^{n-1} \left(\frac{Q_v}{q_v} \right)^k q_v |\Delta t_{v-1}|^k \right\} \left\{ \frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n} \right)^k \frac{1}{Q_{n-1}} \left\{ \sum_{v=1}^{n-1} \left(\frac{Q_v}{q_v} \right)^k q_v |\Delta t_{v-1}|^k \right\}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{Q_v}{q_v} \right)^k q_v |\Delta t_{v-1}|^k \left\{ \sum_{n=v+1}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n} \right)^k \frac{1}{Q_{n-1}} \right\}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{Q_v}{q_v} \right)^{k-1} |\Delta t_{v-1}|^k \text{ , by ((2.2.7) \&(2.2.9))}$$

$$= O(1) \sum_{v=1}^{m} X_v^{k-1} |\Delta t_{v-1}|^k \text{ ,by (2.2.8)}$$

$$= o(1) \text{ as } m \to \infty \text{ , by (2.3.12).}$$

Therefore, we get

$$\sum_{n=1}^{\infty} Y_n^{k-1} |T_{n,i}|^k < \infty \quad , \text{ for } i = 1,2,3.$$

This completes the proof of theorem C.

PROOF OF THEOREM D :

Since the series $\sum_{n=0}^{\infty} a_n$ is summable $X - |\overline{N}, p_n|_k$, it follows that (see chapter-I, definition 9)

$$\sum_{n=1}^{\infty} X_n^{k-1} |\Delta t_{n-1}|^k < \infty \ . \qquad (2.3.13)$$

Then by (2.3.1), (2.3.2), (2.3.4) and (2.3.9) we have

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v$$

$$t_n - t_{n-1} = \Delta t_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \ ,$$

$$T_n - T_{n-1} = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v \ , \text{ for } (n \geq 1) \ .$$

and

$$a_n = -\frac{P_n}{p_n} \Delta t_{n-1} + \frac{P_{n-2}}{p_{n-1}} \Delta t_{n-2} \ , \qquad (2.3.14)$$

Therefore, we have

$$T_n - T_{n-1} = -\frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} \left[-\frac{P_v}{p_v} \Delta t_{v-1} + \frac{P_{v-2}}{p_{v-1}} \Delta t_{v-2} \right] \ , \text{ by (2.3.14)}$$

$$= \frac{q_n P_n}{Q_n p_n}\Delta t_{n-1} + \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)Q_v \Delta t_{v-1} + \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_{v-1}}{p_v}\right)Q_v \Delta t_{v-1}$$

$$= -\frac{q_n P_n}{Q_n p_n}\Delta t_{n-1} + \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\frac{\Delta t_{v-1}}{p_v}(Q_{v-1}P_v - Q_v P_{v-1}) .$$

But

$$\begin{aligned} Q_{v-1}P_v - Q_v P_{v-1} &= Q_{v-1}P_v - Q_v P_v + Q_v p_v \\ &= (Q_{v-1} - Q_v)P_v + p_v Q_v \\ &= -q_v P_v + p_v Q_v . \end{aligned}$$

Thus

$$\begin{aligned} T_n - T_{n-1} &= -\frac{q_n P_n}{Q_n p_n}\Delta t_{n-1} + \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\frac{\Delta t_{v-1}}{p_v}(-q_v P_v + p_v Q_v) \\ &= -\frac{q_n P_n}{Q_n p_n}\Delta t_{n-1} - \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)q_v \Delta t_{v-1} + \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}Q_v \Delta t_{v-1} \\ &= T_{n,1} + T_{n,2} + T_{n,3} \text{ , say.} \end{aligned}$$

To prove the Theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} Y_n^{k-1}|T_{n,i}|^k < \infty \quad , \text{ for } i = 1,2,3. \qquad (2.3.15)$$

Firstly, we have

$$\sum_{n=1}^{\infty} Y_n^{k-1}|T_{n,1}|^k$$

$$= \sum_{n=1}^{\infty} Y_n^{k-1}\left|\frac{q_n P_n}{Q_n p_n}\Delta t_{n-1}\right|^k$$

$$= \sum_{n=1}^{\infty} Y_n^{k-1} \left(\frac{q_n P_n}{Q_n p_n} \right)^k |\Delta t_{n-1}|^k$$

$$= \sum_{n=1}^{\infty} \left(\frac{Q_N}{q_n} \right)^{k-1} \left(\frac{q_n}{Q_n} \right)^k \left(\frac{P_n}{p_n} \right)^k |\Delta t_{n-1}|^k \text{ , by (2.2.11)}$$

$$= O(1) \sum_{n=1}^{\infty} X_n^k \left(\frac{q_n}{Q_n} \right) |\Delta t_{n-1}|^k \text{ , by (2.2.10)}$$

$$= O(1) \sum_{n=1}^{\infty} X_n^{k-1} |\Delta t_{n-1}|^k \text{ , by (2.2.12)}$$

$$= O(1) \text{ , by (2.3.13).}$$

We now apply Hölder's inequality with indices k and k', where

$\frac{1}{k} + \frac{1}{k'} = 1$. This gives

$$\sum_{n=1}^{\infty} Y_n^{k-1} |T_{n,2}|^k$$

$$= \sum_{n=2}^{m+1} Y_n^{k-1} \left| -\frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right) q_v \Delta t_{v-1} \right|^k$$

$$\leq \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k \left\{ \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right) q_v |\Delta t_{v-1}| \right\}^k$$

$$\leq \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n} \right)^k \frac{1}{Q_{n-1}} \left\{ \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k q_v |\Delta t_{v-1}|^k \right\} \left\{ \frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n} \right)^k \frac{1}{Q_{n-1}} \left\{ \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k q_v |\Delta t_{v-1}|^k \right\}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k q_v |\Delta t_{v-1}|^k \left\{ \sum_{n=v+1}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n}\right)^k \frac{1}{Q_{n-1}} \right\}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k \left(\frac{q_v}{Q_v}\right) |\Delta t_{v-1}|^k \text{ by (2.2.11)}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} |\Delta t_{v-1}|^k$$

$$= O(1) \sum_{v=1}^{m} X_v^{k-1} |\Delta t_{v-1}|^k \text{ , by (2.2.12)}$$

$$= O(1) \text{ as } m \to \infty \text{ , by (2.3.13).}$$

Finally, we have

$$\sum_{n=1}^{\infty} Y_n^{k-1} |T_{n,3}|^k$$

$$= \sum_{n=2}^{m+1} Y_n^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} Q_v \Delta t_{v-1} \right|^k$$

$$\leq \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}}\right)^k \left\{ \sum_{v=1}^{n-1} Q_v |\Delta t_{v-1}| \right\}^k$$

$$\leq \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n}\right)^k \frac{1}{Q_{n-1}} \left\{ \sum_{v=1}^{n-1} \left(\frac{Q_v}{q_v}\right)^k q_v |\Delta t_{v-1}|^k \right\} \left\{ \frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n}\right)^k \frac{1}{Q_{n-1}} \left\{ \sum_{v=1}^{n-1} \left(\frac{Q_v}{q_v}\right)^k q_v |\Delta t_{v-1}|^k \right\}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{Q_v}{q_v}\right)^k q_v |\Delta t_{v-1}|^k \left\{ \sum_{n=v+1}^{m+1} Y_n^{k-1} \left(\frac{q_n}{Q_n}\right)^k \frac{1}{Q_{n-1}} \right\}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{Q_v}{q_v}\right)^{k-1} |\Delta t_{v-1}|^k$$

$$= O(1) \sum_{v=1}^{m} X_v^{k-1} |\Delta t_{v-1}|^k \text{ , by (2.1.10) and (2.2.12)}$$

$$= O(1) \text{ as } m \to \infty \text{ , by (2.3.13).}$$

Therefore, we get

$$\sum_{n=1}^{\infty} Y_n^{k-1} |T_{n,i}|^k < \infty \text{ , for } i = 1,2,3.$$

This completes the proof of theorem D.

PROOF OF THEOREM E :

The proof of Theorem E is similar to Theorem D , which can be obtained by interchanging the role of the sequences (X_n) and (Y_n) in Theorem D.